Planetary Orbits

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1 Planetary Orbits

We will derive some of the basic formulas for planetary orbits from first principles. Let r be the distance of the satellite from the primary and θ measure the angle relative to the primary. Assume that the satellite has negligible mass compared to the primary.

2 Derivatives in Polar Coordinates

Let us start out with the formula for x in polar coordinates and take a couple of derivatives with respect to time:

$$x = r\cos(\theta) \tag{2.1a}$$

$$x' = r'\cos(\theta) - r\sin(\theta)\theta' \tag{2.1b}$$

$$x'' = r''\cos(\theta) - 2r'\sin(\theta)\theta' - r\cos(\theta)\theta'^2 - r\sin(\theta)\theta'' \tag{2.1c}$$

If $\theta = 0$, that is, the coordinates are rotated so that the y component of the position is 0, then x'' represents the radial acceleration of the satellite:

$$\frac{-GM}{r^2} = r'' - r\theta'^2 \tag{2.2}$$

If $\theta = \frac{\pi}{2}$, that is, the system is rotated so that the x component of the position is 0, then x'' is the acceleration perpindicular to the radius:

$$0 = 2r'\theta' + r\theta'' \tag{2.3}$$

3 Kepler's Second Law (Equal Area in Equal Time)

Multiply the right hand side of (2.3) by $\frac{1}{2}r$ and you have the derivative of $\frac{1}{2}r^2\theta'$, which is the area swept out by the satellite per unit time. Equation (2.3) says that $\frac{1}{2}r^2\theta'$ is a constant

$$\frac{1}{2}r^2\theta' = k_1 \tag{3.1}$$

Equation (3.1) is precisely Kepler's Second Law of Planetary Motion; a planet sweeps out equal areas in equal times.

4 Conservation of Energy

Let us look at the square of the velocity, which is proportional to the kinetic energy of the satellite

$$v^2 = r'^2 + r^2\theta'^2 \tag{4.1}$$

Take half the time derivative of (4.1), then apply (2.3) and then (2.2):

$$vv' = r'r'' + rr'\theta'^2 + r^2\theta'\theta'' \tag{4.2a}$$

$$= r'r'' - rr'\theta'^2 \tag{4.2b}$$

$$= r' \ \frac{-GM}{r^2} \tag{4.2c}$$

If we integrate (4.2) and rearrange, we get

$$\frac{1}{2}v^2 - \frac{GM}{r} = k_2 \tag{4.3}$$

When (4.3) is multiplied by the mass of the satellite, it becomes the law of conservation of energy. $\frac{1}{2}mv^2$ is the kinetic energy and $\frac{-GMm}{r}$ is the potential energy of the satellite.

If k_2 is positive, the satellite has enough kinetic energy to escape the primary's gravity. If k_2 is negative, the satellite will stay in orbit around the primary.

5 Using Initial Information

So now we have two constants. Both can be computed from the initial position, p, and velocity, v, of the satellite relative to the primary:

$$k_1 = \frac{1}{2}\left|p \times v\right| \tag{5.1}$$

$$k_2 = \frac{1}{2}\left|v\right|^2 - \frac{GM}{|p|} \tag{5.2}$$

6 Kepler's First Law (Elliptical Orbits)

Dividing (4.1) by θ'^2 and rearranging, then using (3.1) and (4.3) to supply θ' and v in terms of r, gives:

$$\left(\frac{r'}{\theta'}\right)^2 = \left(\frac{v}{\theta'}\right)^2 - r^2 \tag{6.1a}$$

$$= \frac{2\left(k_2 + \frac{GM}{r}\right)}{\left(\frac{2k_1}{r^2}\right)^2} \tag{6.1b}$$

$$= \frac{k_2}{2k_1^2}r^4 + \frac{GM}{2k_1^2}r^3 - r^2 \tag{6.1c}$$

If we divide (6.1) by r^4 and change variables to $s = 1/r$, we get

$$\left(\frac{s'}{\theta'}\right)^2 = -s^2 + \frac{GM}{2k_1^2}s + \frac{k_2}{2k_1^2} \tag{6.2a}$$

$$= \left(\frac{GM}{4k_1^2}\right)^2 - \left(s - \frac{GM}{4k_1^2}\right)^2 + \frac{k_2}{2k_1^2} \tag{6.2b}$$

$$= c^2 - (s-b)^2 \tag{6.2c}$$

where $b = \frac{GM}{4k_1^2}$ and $c^2 = b^2 + \frac{k_2}{2k_1^2}$. Thus, rearranging the terms in (6.2), and taking square roots, we get

$$\theta' = \frac{(s-b)'}{\sqrt{c^2 - (s-b)^2}} \tag{6.3a}$$

$$= \frac{u'}{\sqrt{1-u^2}} \tag{6.3b}$$

where $u = \frac{s-b}{c}$. The right hand side of (6.3) is the derivative of $\cos^{-1}(u)$. Thus, if we integrate (6.3) and solve for r, we get

$$r = \frac{1}{b + c\cos(\theta - k_3)} \tag{6.4}$$

Equation (6.4) is the polar equation for an ellipse with a focus at the origin. k_3 is the angle of periapsis, usually taken to be 0. Since we know that the minimum distance is $a(1-e)$ and the maximum distance is $a(1+e)$, it follows that

$$r = \frac{a(1-e^2)}{1 + e\cos(\theta)} \tag{6.5}$$

where $a(1-e^2) = 1/b$ and $e = c/b$. Equation (6.5) is Kepler's First Law of Planetary Motion: the orbit of a planet is an ellipse with the Sun at one focus.

In terms of the basic orbital constants from (5.1) and (5.2),

$$a = \frac{-GM}{2k_2} \tag{6.6}$$

and

$$e = \sqrt{1 + 8k_2\left(\frac{k_1}{GM}\right)^2} \tag{6.7}$$

Solving for θ yields

$$\cos(\theta) = \frac{1}{e}\left(\frac{4k_1^2}{GMr} - 1\right) \tag{6.8}$$

7 Vis-Viva Equation

Equation (4.1) says that when $r' = 0$, $v = r\theta'$. Equation (3.1) says that $r\theta' = \frac{2k_1}{r}$. Thus, when $r' = 0$,

$$v = \frac{2k_1}{r} \tag{7.1}$$

Combining (6.8) with (4.3), we get that when $r' = 0$,

$$\frac{2k_1^2}{r^2} - \frac{GM}{r} = k_2 \tag{7.2}$$

multiplying (7.2) by r^2 and rearranging, we get that when r is maximum or minimum,

$$k_2 r^2 + GMr - 2k_1^2 = 0 \tag{7.3}$$

The sum of the roots of (7.3) is $\frac{-GM}{k_2}$. Thus, the semimajor axis, the average of the minimum and maximum distance of an elliptical orbit, is $a = \frac{-GM}{2k_2}$. Therefore, we get

$$k_2 = \frac{-GM}{2a} \tag{7.4}$$

Using (7.4) in (4.3), we get the Vis-Viva equation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{7.5}$$

8 Kepler's Third Law (Period-Distance Relation)

The product of the roots of (7.3) is $\frac{-2k_1^2}{k_2}$ while the product of the minimum and maximum distance of an elliptical orbit is $a^2(1-e^2)$. With (7.4) we get that

$$k_1^2 = -\frac{k_2}{2}a^2(1-e^2) = \frac{GM}{4a}a^2(1-e^2) = \frac{GMa}{4}(1-e^2) \tag{8.1}$$

We also know that k_1 is the constant rate that area is swept out along the elliptical orbit. Over one whole period P, the area swept out is $\pi a^2 sqrt(1-e^2)$, thus

$$k_1 P = \pi a^2 \sqrt{1-e^2} \tag{8.2}$$

Thus, squaring (8.2), using (8.1), and cancelling, we get

$$GM\ P^2 = 4\pi^2 a^3 \tag{8.3}$$

Equation (8.3) verifies Kepler's Third Law of Planetary Motion; the square of the period of a planet is proportional to the cube of the semimajor axis of its orbit.

9 Time Dependence

Combining (6.8) and (8.3) and using the results from Kepler's Equation, we can compute the position in orbit at any time.