Kepler's Equation

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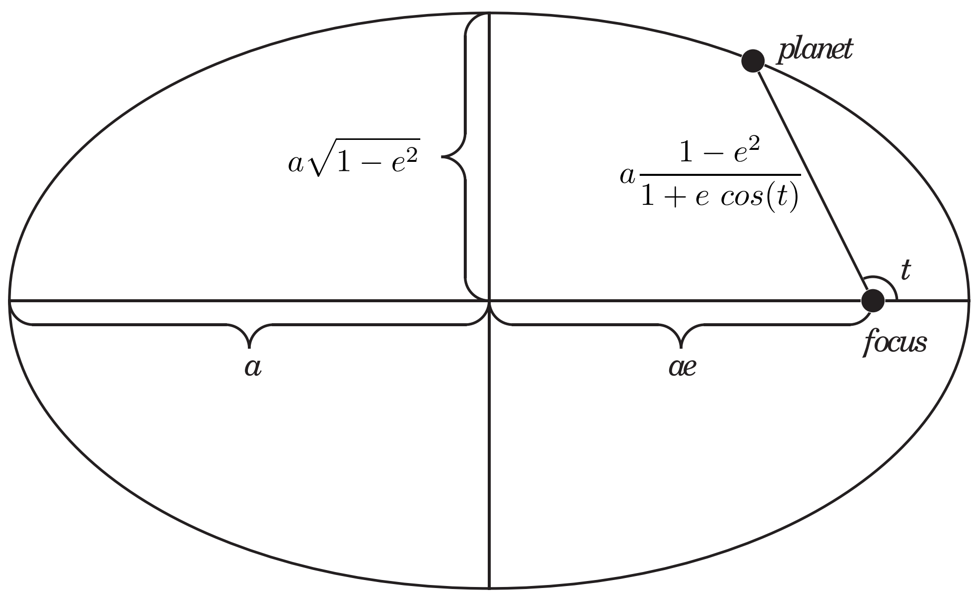
1 Kepler's Equation

As derived in Planetary Orbits, Kepler's Second Law of Planetary Motion says that a planet sweeps out equal areas in equal times. To use this law to compute the position of a planet at a given time, we must know the area of a wedge of an ellipse. We will do this by unflattening the ellipse to a circle, computing the area in the circle, then flattening the circle back to the ellipse.



Offset the ellipse so that the center is at the origin:

$$(ae + r\cos(t), r\sin(t))$$

$$= a\left(e + \frac{1-e^2}{1+e\cos(t)}\cos(t), \frac{1-e^2}{1+e\cos(t)}\sin(t)\right) \tag{1.1a}$$

$$= \frac{a}{1+e\cos(t)}\left(e(1+e\cos(t)) + (1-e^2)\cos(t), (1-e^2)\sin(t)\right) \tag{1.1b}$$

$$= \frac{a}{1+e\cos(t)}\left(e+\cos(t), (1-e^2)\sin(t)\right) \tag{1.1c}$$

Unflatten the ellipse to a circle by stretching in the y direction by $1/\sqrt{1-e^2}$:

$$\frac{a}{1+e\cos(t)}\left(e+\cos(t),\, \sqrt{1-e^2}\sin(t)\right) \tag{1.2}$$

We can verify that (1.2) is a circle by computing its absolute value:

$$\frac{a}{1+e\cos(t)}\sqrt{e^2+2e\cos(t)+\cos^2(t)+(1-e^2)\sin^2(t)}$$

$$= \frac{a}{1+e\cos(t)}\sqrt{1+2e\cos(t)+e^2\cos^2(t)} \tag{1.3a}$$

$$= \frac{a}{1+e\cos(t)}(1+e\cos(t)) \tag{1.3b}$$

$$= a \tag{1.3c}$$

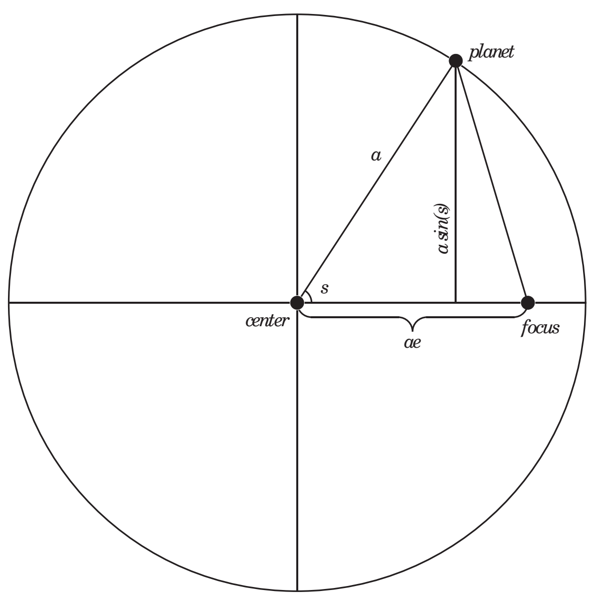
The first trick is one that is used pretty commonly with integration substitutions which use $z = \tan(s/2)$; compute the tangent of half the angle of (1.2) using $\tan(s/2) = y/(a+x)$, where s is the angle relative to the origin:

$$\tan\left(\frac{s}{2}\right) = \frac{\sqrt{1-e^2}\sin(t)}{1+e\cos(t)+e+\cos(t)} \tag{1.4a}$$

$$= \frac{\sqrt{1-e^2}}{1+e}\,\frac{\sin(t)}{1+\cos(t)} \tag{1.4b}$$

$$= \sqrt{\frac{1-e}{1+e}}\tan\left(\frac{t}{2}\right) \tag{1.4c}$$

Thus, the position on the circle is simply $a(\cos(s), \sin(s))$. The angle s is called the eccentric anomaly.



The area we are looking for is the area swept out by the radius of the circle ($a^2s/2$), minus the triangle formed by the center of the circle, the focus of the ellipse, and the planet ($a^2e\sin(s)/2$), all scaled by $\sqrt{1-e^2}$. That is, the area swept out by a line through the focus of an ellipse from angle 0 to angle t is

$$A = \frac{a^2\sqrt{1-e^2}}{2}(s - e\sin(s)) \tag{1.5}$$

where s is computed from t as in (1.4). The quantity $s - e\ sin(s)$ is called the mean anomaly since it has a constant rate of change with respect to time for planetary motion. Equation (1.5) is called Kepler's Equation.

2 Converting Between s and t

The next trick is one that was inspired by David Cantrell on sci.math; it is better to compute $t-s$ than blindly to use *atan* on (1.4).

$$\tan\left(\frac{t-s}{2}\right) = \frac{\tan(t/2) - \tan(s/2)}{1 + \tan(t/2)\tan(s/2)} \tag{2.1a}$$

$$= \frac{\left(\sqrt{1+e} - \sqrt{1-e}\right)\tan(t/2)}{\sqrt{1+e} + \sqrt{1-e}\tan^2(t/2)} \tag{2.1b}$$

$$= \frac{\left(\sqrt{1+e} - \sqrt{1-e}\right)\sin(t)(1+\cos(t))}{\sqrt{1+e}(1+\cos(t))^2 + \sqrt{1-e}\ \sin^2(t)} \tag{2.1c}$$

$$= \frac{\left(\sqrt{1+e} - \sqrt{1-e}\right)\sin(t)}{\sqrt{1+e}(1+\cos(t)) + \sqrt{1-e}(1-\cos(t))} \tag{2.1d}$$

$$= \frac{e\sin(t)}{1 + \sqrt{1-e^2} + e\cos(t)} \tag{2.1e}$$

where (2.1e) follows from

$$\frac{\sqrt{1+e} + \sqrt{1-e}}{\sqrt{1+e} - \sqrt{1-e}} = \frac{1 + \sqrt{1-e^2}}{e} \tag{2.2}$$

Now we can compute s using

$$s = t - 2\arctan\left(\frac{e\sin(t)}{1 + \sqrt{1-e^2} + e\cos(t)}\right) \tag{2.3}$$

Equations (1.5) and (2.3) allow us to compute time from position.

By an almost identical argument, we get

$$t = s + 2\arctan\left(\frac{e\sin(s)}{1 + \sqrt{1-e^2} - e\cos(s)}\right) \tag{2.4}$$

Inverting (1.5) and using (2.4) allows us to compute position from time.