Castelnuovo-Mumford regularity and Kazhdan-Lusztig varieties

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Kazhdan–Lusztig varieties of Woo–Yong '06

$B_- \times B$ acts on $\mathcal{F}l_n(\mathbb{C})$ with finitely many orbits X_w° called **Schubert cells**. The **Schubert varieties** X_w are closures of X_w°.

Let $e_v = B_-\backslash B_- v$ denote the fixed points of the left action of a maximal torus $T \subset B$ on X_w, where $v \geq w \in S_n$. Let $\Omega_v^\circ = e_v B_-$ denote the **opposite Schubert cell**. Then an affine neighborhood of e_v is simply $v\Omega_{id}^\circ$, so one may restrict to studying $X_w \cap v\Omega_{id}^\circ$.

Theorem [Kazhdan–Lusztig '79]

$$X_w \cap v\Omega_{id}^\circ \cong (X_w \cap \Omega_v^\circ) \times \mathbb{A}^{\ell(v)}$$

Of particular interest is the **Kazhdan–Lusztig variety**

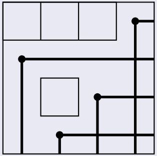
$$\mathcal{N}_{v,w} = X_w \cap \Omega_v^\circ.$$

Kazhdan–Lusztig varieties of Woo–Yong '06

Kazhdan–Lusztig variety $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i \times j}(v) \rangle \subset \mathbb{C}[z_{ij} \,|\, (i,j) \in D(v)].$$

Example: $w = 4132$, $v = 4231$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31} \rangle$$

Matrix Schubert varieties $\overline{X}_w$ and classical determinantal varieties are all examples of KL varieties.

Minimal free resolution

Consider the coordinate ring S/I. The **minimal free resolution**

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{l,j}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}} \to S/I \to 0.$$

The K-**polynomial** of S/I

$$\mathcal{K}(S/I;\mathbf{t}) := \sum_{j\in\mathbb{Z}, i\geq 0} (-1)^i \beta_{i,j} t^j.$$

The **Castelnuovo–Mumford regularity** of S/I

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Proposition

For Cohen–Macaulay S/I

$$\operatorname{reg}(S/I) = \deg \mathcal{K}(S/I;\mathbf{t}) - \operatorname{codim}_S I.$$

Matrix Schubert varieties

Matrix Schubert varieties $\overline{X}_w$ are special cases of $\mathcal{N}_{v,w'}$.

Combining results of Fulton '92, Knutson–Miller '05, and Buch '02:

Theorem

$$\text{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w(x_1, \ldots, x_n)) - \ell(w),$$

where $\mathfrak{G}_w(x_1, \ldots, x_n)$ is the Grothendieck polynomial and $\ell(w)$ is the Coxeter length of w.

Problem

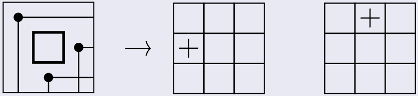
Give an easily computable formula for $\deg(\mathfrak{G}_w(x_1, \ldots, x_n))$, where $w \in S_n$.

Schubert polynomials via reduced pipe dreams

By Bergeron–Billey '93 and Fomin–Kirillov '94,

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{P \in rPD(w)} x^{wt(P)}$$

Example: $\mathfrak{S}_w$ for $w = 132$.



so $\mathfrak{S}_w(x_1, x_2, x_3) = x_1 + x_2$.

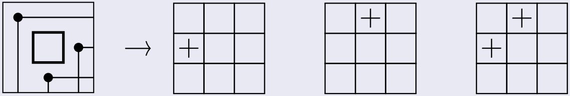
In general, $\deg(\mathfrak{S}_w) = \ell(w)$.

Grothendieck polynomials via pipe dreams

By Fomin–Kirillov '94,

$$\mathfrak{G}_w(x_1,\dots,x_n) = \sum_{P\in PD(w)} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}$$

Example: $\mathfrak{G}_w$ for $w = 132$.



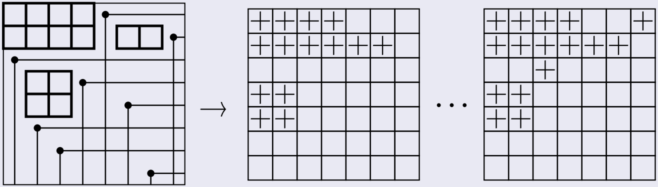
so $\mathfrak{G}_w(x_1,x_2,x_3) = x_1 + x_2 - x_1x_2$.

Thus $\deg(\mathfrak{G}_w) = \max\{\#P \mid P \in PD(w)\}$, and
$\mathrm{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w) - \deg(\mathfrak{S}_w)$.

Finding the degree of Grothendieck polynomials

Let's take a look at a larger example:

Example: $w = 58146237$



In general, how can we more easily compute $\deg(\mathfrak{G}_w)$?

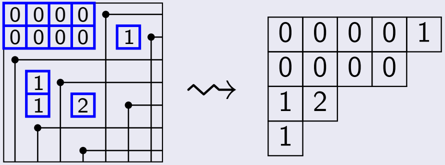
Finding the degree of $\mathfrak{G}_v$ vexillary

Theorem [Rajchgot–R.–Weigandt '23]

Suppose $v \in S_n$ vexillary. Then

$$\deg(\mathfrak{G}_v) = \ell(v) + \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



gives $\deg(\mathfrak{G}_v) = \ell(v) + ((2+1)+(1)) = 12 + 4 = 16.$

Pechenik–Speyer–Weigandt '21 give a result for general $w \in S_n$.

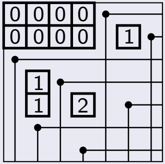
Finding the regularity of $\overline{X}_v$ vexillary

Theorem [Rajchgot–R.–Weigandt '23]

Suppose $v \in S_n$ vexillary. Then

$$\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



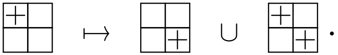
$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = ((2+1)+(1)) = 4.$

K-Excited Young Diagrams

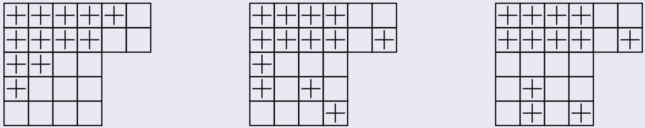
A K-excited move on a diagram D in μ is the local operation on a 2×2 subsquare of D such that



A **K-excited Young diagram** of λ in μ is a diagram obtainable by applying K-excited moves to the Young diagram of λ in μ.

Example: For $\mu = (6, 6, 4, 4, 4)$, $\lambda = (5, 4, 2, 1, 0)$:

Below are some K-excited Young diagrams of λ in μ:

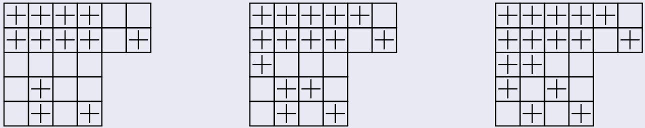


Intuition via Excited Young Diagrams

Using Knutson–Miller–Yong '09,

$$\mathfrak{G}_v(\mathbf{x};\mathbf{y}) = \sum_{D \in \mathrm{KEYD}(\mu(v),\lambda(v))} (-1)^{\#D - |\lambda(v)|}\, wt(D).$$

Example: Maximizing D for $v = 5713624$



so maximizing D depends on certain antidiagonals of $\lambda(v)$.

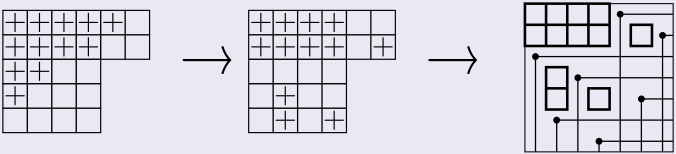
Computing CM-regularity of certain KL varieties

Theorem [Rajchgot–R.–Weigandt '23]

For $u_\rho, w_\gamma \in S_n$ Grassmannian with descent k, $(u_\rho, w_\gamma) \mapsto v$ vexillary such that

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $u_{(5,4,2,1,0)}, w_{(6,6,4,4,4)} \mapsto v = 5713624$

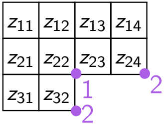


gives $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = 4$.

Application: one-sided mixed ladder determinantal ideals

A ladder L is a Young diagram filled with indeterminates z_{ij}. The ideal $I_L \subseteq \mathbb{C}[L]$ is generated by NW minors of L determined by marked points on its boundary. This defines the one-sided mixed ladder determinantal variety $\mathbb{C}[L]/I_L$.

For example, we can take L:



These are Grassmannian KL-varieties

$$\mathbb{C}[L]/I_L \cong \mathbb{C}[\mathcal{N}_{u_\rho, w_\gamma}] \cong \mathbb{C}[\overline{X}_v].$$

Corollary [Rajchgot–R.–Weigandt '23]

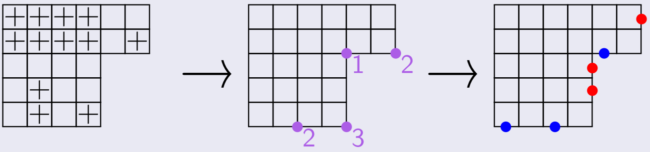
$$\operatorname{reg}(\mathbb{C}[L]/I_L) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

One-sided ladders and lattice paths

To each one-sided ladder, we can associate families of non-intersecting NE-oriented lattice paths.

The marked points on horizontal edges determine starting points of the paths and the marked points on vertical edges determine ending points of the paths.

Example: Constructing a ladder L and lattice paths from v.



CM-regularity of one-sided ladders and lattice paths

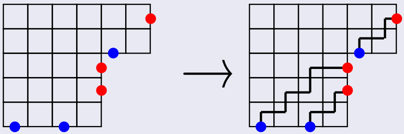
Following work of Krattenhaler–Prohaska '99 and Ghorpade '02 combined with Knutson–Miller–Yong '09 and Matsumura '17, $\mathcal{K}(\mathbb{C}[L]/I_L;\mathbf{t})$ is determinantal in terms of these lattice paths.

Theorem [Krattenhaler–Ghorpade '15, Rajchgot–R.–Weigandt '23]

For a one-sided ladder L

$$\operatorname{reg}(\mathbb{C}[L]/I_L) = \max_{P\in NILP(L)} \#\{\text{elbows } \lrcorner \text{ in } P\}.$$

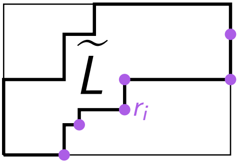
Example: $P \in NILP(L)$ with maximal number of elbows



gives $\operatorname{reg}(\mathbb{C}[L]/I_L) = 4 = \operatorname{reg}(\mathbb{C}[\overline{X}_v])$.

Two-sided mixed ladder determinantal ideals

A two-sided ladder $\widetilde{L}$ is a skew-Young diagram filled with z_{ij}'s. $I_{\widetilde{L}}$ is the ideal generated by the NW r_i minors of $\widetilde{L}$. This defines the two-sided mixed ladder determinantal variety $\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}$.



Theorem [Escobar-Fink-Rajchgot-Woo ('23+)]

For particular $u, w \in S_n$ 321-avoiding

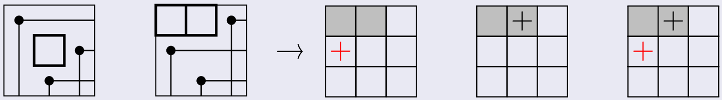
$$\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}} \cong \mathbb{C}[\mathcal{N}_{u,w}].$$

Generalizing Grassmannian KL formula

Woo–Yong '12 introduce unspecialized Grothendiecks $\mathfrak{G}_{u,w}$, the K-polynomials of $\mathcal{N}_{u,w}$:

$$\mathfrak{G}_{u,w}(x_1,\ldots,x_n) = \sum_{P\in PD(w)\cap R_u} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}.$$

Example: $\mathfrak{G}_{u,w}$ for $w = 132$, $u = 312$.



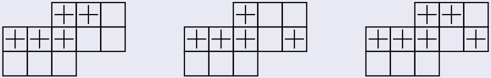
so $\mathfrak{G}_{u,w}(x_1,x_2,x_3) = x_1$.

In current work, we provide an algorithm to compute $\operatorname{reg}(\mathbb{C}[\mathcal{N}_{u,w}])$ for $u, w \in S_n$ 321-avoiding.

K-skew excited Young diagrams

Let $u \geq w \in S_n$ be 321-avoiding. Define $\mathcal{R}_u$ as the skew-Young diagram associated to u. Mark positions in $\mathcal{R}_u$ with $+$'s that correspond to the earliest subword of w in u.

Example: Below are the diagrams corresponding to $u = 47128356$ and $w = 14273568$.



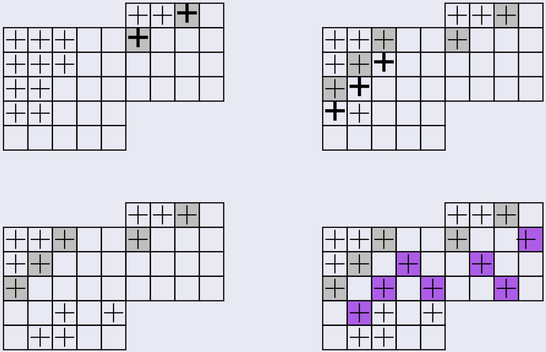
A **K-skew excited Young diagram** of w in u is a diagram obtainable by applying K-excited moves to this initial diagram.

Proposition [R. '23+]

For $u \geq w \in S_n$ 321-avoiding, $PD(w) \cap R_u$ bijects with K-skew excited Young diagrams of w in u.

Algorithm example computing $\operatorname{reg}(\mathbb{C}[\mathcal{N}_{u,w}])$

Example: Constructing a maximal skew-excited Young diagram given by certain 321-avoiding $u, w \in S_{15}$.



gives $\operatorname{reg}(\mathbb{C}[\mathcal{N}_{u,w}]) = 7$.

CM-regularity of two-sided ladders and lattice paths

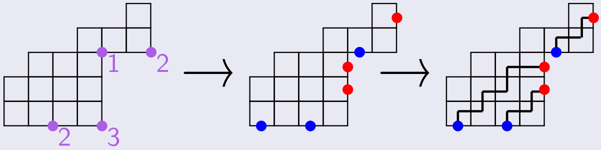
Generalizing work of Krattenhaler–Ghorpade '15 combined with Woo–Yong '12, we can compute $\text{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}})$ using lattice paths:

Theorem [Krattenhaler–Ghorpade '15, R. '23+]

For a two-sided ladder $\widetilde{L}$

$$\text{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}) = \max_{P \in NILP(\widetilde{L})} \#\{\text{unforced elbows } \square \text{ in } P\}.$$

Example: $P \in NILP(\widetilde{L})$ with maximal number of elbows



gives $\text{reg}(\mathbb{C}[\widetilde{L}]/I_{\widetilde{L}}) = 3$.

Conclusions

- We can express $\text{reg}(\mathbb{C}[\overline{X}_w])$ in terms of the degree of the K-polynomial and the codimension of I_w.
- Use that $\text{reg}(\mathbb{C}[\overline{X}_w]) = \deg \mathfrak{G}_w - \ell(w)$.
- For v vexillary, we obtain an easily computable formula for $\deg \mathfrak{G}_v$, and thus for $\text{reg}(\mathbb{C}[\overline{X}_v])$.
- By relating $\mathcal{N}_{u_\rho, w_\nu}$ to $\overline{X}_v$, we obtain formulas for regularities of one-sided ladders.
- We connect our formulas to the combinatorics of lattice paths.
- We generalize our formulas to an algorithm for 321-avoiding KL-varieties. This specializes to an algorithm for two-sided ladders.