CM regularity and Kazhdan-Lusztig varieties

Colleen Robichaux
UCLA

joint work with Jenna Rajchgot and Anna Weigandt
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Schubert varieties

Let $\mathcal{F}l_n(\mathbb{C})$, the **complete flag variety**, be the set of complete flags

$$0 = V_0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V_n = \mathbb{C}^n, \quad \text{where } \dim V_i = i.$$

We can identify $\mathcal{F}l_n(\mathbb{C})$ with $B_- \backslash \mathrm{GL}_n(\mathbb{C})$, where $B_- \subset \mathrm{GL}_n(\mathbb{C})$ is the opposite Borel subgroup.

$\mathrm{B}_- \times \mathrm{B}$ acts on $\mathcal{F}l_n(\mathbb{C})$ with finitely many orbits X_w° called **Schubert cells**. The **Schubert varieties** X_w are closures of these orbits. Moreover, Bruhat order $\geq$ on permutations gives

$$X_w = \coprod_{v \geq w} X_v^\circ.$$

Kazhdan–Lusztig varieties of Woo–Yong '06

Let $e_v = \mathrm{B}_-\backslash\mathrm{B}_- v$ denote the fixed points of the left action of a maximal torus $T \subset B$ on X_w, where $v \geq w \in S_n$. Let $\Omega_v^\circ = e_v B_-$ denote the opposite Schubert cell. Then an affine neighborhood of e_v is simply $v\Omega_{id}^\circ$, so one may restrict to studying $X_w \cap v\Omega_{id}^\circ$.

Theorem [Kazhdan–Lusztig '79]

$$X_w \cap v\Omega_{id}^\circ \cong (X_w \cap \Omega_v^\circ) \times \mathbb{A}^{\ell(v)}$$

Of particular interest is the **Kazhdan–Lusztig variety**

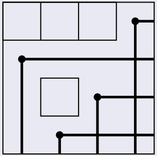
$$\mathcal{N}_{v,w} = X_w \cap \Omega_v^\circ.$$

Kazhdan–Lusztig varieties of Woo–Yong '06

Kazhdan–Lusztig variety $\mathcal{N}_{v,w}$ has defining ideal

$$I_{v,w} = \langle r_w(i,j) + 1 \text{ minors of } \mathbf{z}_{i\times j}(v)\rangle \subset \mathbb{C}[z_{ij}\,|\,(i,j) \in D(v)].$$

Example: $w = 4132$, $v = 4231$



$$\xrightarrow{r_w} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix} \xrightarrow{\mathbf{z}(v)} \begin{pmatrix} z_{11} & z_{12} & z_{13} & 1 \\ z_{21} & 1 & 0 & 0 \\ z_{31} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

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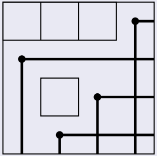

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$$I_{v,w} = \langle z_{11},\ z_{12}, z_{13},\ z_{11} - z_{12}z_{21},\ -z_{12}z_{31},\ -z_{31} \rangle$$

Matrix Schubert varieties $\overline{X}_w$ and classical determinantal varieties are all examples of KL varieties.

Minimal free resolution

Consider the coordinate ring S/I. The **minimal free resolution**

$$0 \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{l,j}} \to \cdots \to \bigoplus_{j\in\mathbb{Z}} S(-j)^{\beta_{0,j}} \to S/I \to 0.$$

The K-**polynomial** of S/I

$$\mathcal{K}(S/I;\mathbf{t}) := \sum_{j\in\mathbb{Z}, i\geq 0} (-1)^i \beta_{i,j} t^j.$$

The **Castelnuovo–Mumford regularity** of S/I

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j} \neq 0\}.$$

Proposition

For Cohen–Macaulay S/I

$$\operatorname{reg}(S/I) = \deg \mathcal{K}(S/I;\mathbf{t}) - \operatorname{codim}_S I.$$

Matrix Schubert varieties

Matrix Schubert varieties $\overline{X}_w$ are special cases of $\mathcal{N}_{v,w'}$.

Combining results of Fulton '92, Buch '02, and Knutson–Miller '05:

Theorem

$$\mathrm{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w(x_1, \ldots, x_n)) - \ell(w),$$

where $\mathfrak{G}_w(x_1, \ldots, x_n)$ is the Grothendieck polynomial and $\ell(w)$ is the Coxeter length of w.

Problem

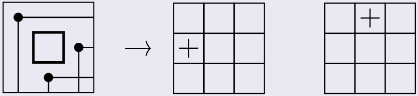
Give an easily computable formula for $\deg(\mathfrak{G}_w(x_1, \ldots, x_n))$, where $w \in S_n$.

Schubert polynomials via reduced pipe dreams

By Bergeron–Billey '93 and Fomin–Kirillov '94,

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{P \in rPD(w)} x^{wt(P)}$$

Example: $\mathfrak{S}_w$ for $w = 132$.



so $\mathfrak{S}_w(x_1, x_2, x_3) = x_1 + x_2$.

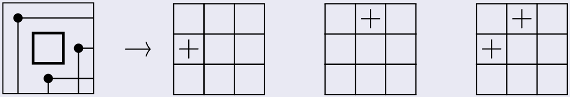
In general, $\deg(\mathfrak{S}_w) = \ell(w)$.

Grothendieck polynomials via pipe dreams

By Fomin–Kirillov '94,

$$\mathfrak{G}_w(x_1, \ldots, x_n) = \sum_{P \in PD(w)} (-1)^{(\#+'s) - \ell(w)} x^{wt(P)}$$

Example: $\mathfrak{G}_w$ for $w = 132$.



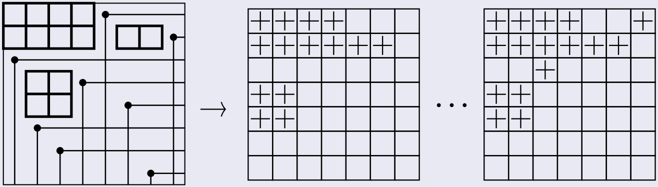
so $\mathfrak{G}_w(x_1, x_2, x_3) = x_1 + x_2 - x_1 x_2$.

Thus $\deg(\mathfrak{G}_w) = \max\{\#P \mid P \in PD(w)\}$, and $\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg(\mathfrak{G}_w) - \deg(\mathfrak{S}_w)$.

Finding the degree of Grothendieck polynomials

Let's take a look at a larger example:

Example: $w = 58146237$



In general, how can we more easily compute $\deg(\mathfrak{G}_w)$?

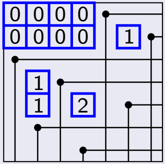
Finding the degree of $\mathfrak{G}_v$ vexillary

Theorem [Rajchgot–R.–Weigandt '22]

Suppose $v \in S_n$ vexillary. Then

$$\deg(\mathfrak{G}_v) = \ell(v) + \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



$\rightsquigarrow$

0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\deg(\mathfrak{G}_v) = \ell(v) + ((2+1)+(1)) = 12 + 4 = 16.$

Finding the degree of $\mathfrak{G}_v$ vexillary

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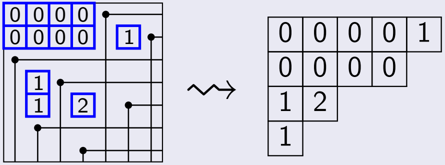


gives $\deg(\mathfrak{G}_v) = \ell(v) + ((2+1)+(1)) = 12 + 4 = 16.$

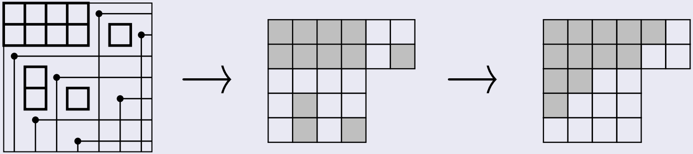
Pechenik–Speyer–Weigandt '21 give a result for general $w \in S_n$.

Intuition via Excited Young Diagrams

Using Knutson–Miller–Yong '09,

$$\mathfrak{G}_v(\mathbf{x};\mathbf{y}) = \sum_{D \in \mathrm{KExcitedYD}(\mu(v),\lambda(v))} (-1)^{\#D - |\lambda(v)|} wt(D).$$

Example: $v = 5713624 \mapsto (\mu(v), \lambda(v))$

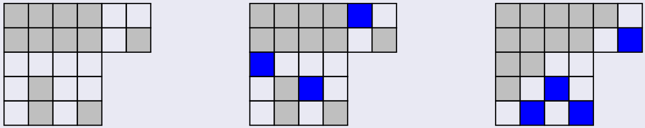


So $\deg(\mathfrak{G}_v) = \max\{\#D \mid D \in \mathrm{KExcitedYD}(\mu(v),\lambda(v))\}$.

Intuition via Excited Young Diagrams

$$\mathfrak{G}_v(\mathbf{x};\mathbf{y}) = \sum_{D \in \mathrm{KExcitedYD}(\mu(v),\lambda(v))} (-1)^{\#D - |\lambda(v)|}\, wt(D).$$

Example: Maximizing D for $v = 5713624$



so maximizing D depends on certain antidiagonals of $\lambda(v)$.

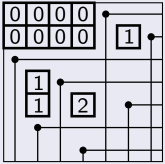
Finding the regularity of $\overline{X}_v$ vexillary

Theorem [Rajchgot–R.–Weigandt '22]

Suppose $v \in S_n$ vexillary. Then

$$\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \deg(\mathfrak{G}_v) - \ell(v) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $v = 5713624$



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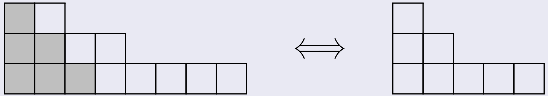
0	0	0	0	1
0	0	0	0	
1	2			
1				

gives $\mathrm{reg}(\mathbb{C}[\overline{X}_v]) = ((2+1)+(1)) = 4$.

Grassmannian permutations

A permutation $w \in S_n$ is Grassmannian if it has a unique descent k, i.e. if $i \neq k$, then $w_i < w_{i+1}$. To each Grassmannian permutation $w \in S_n$, we can uniquely associate a partition λ with k parts.

Example: $w = 24813567$ and $\lambda = (5, 2, 1)$



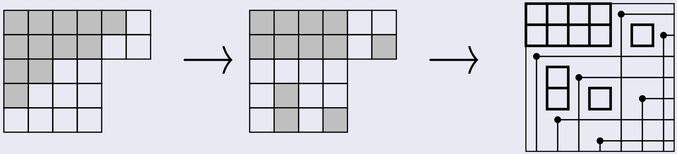
Computing CM-regularity of certain KL varieties

Theorem [Rajchgot–R.–Weigandt '22]

For $u_\rho, w_\gamma \in S_n$ Grassmannian with descent k, $(u_\rho, w_\gamma) \mapsto v$ vexillary such that

$$\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}}).$$

Example: $u_{(5,4,2,1,0)}, w_{(6,6,4,4,4)} \mapsto v = 5713624$



gives $\mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho,w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_v]) = 4$.

Application I: KLSS Conjecture

Fix $k \in [n]$. Let Y denote the space of $n \times n$ matrices of the form

$$\begin{bmatrix} A & I_k \\ I_{n-k} & 0 \end{bmatrix}, \text{ where } A \in M_{k\times(n-k)}(\mathbb{C}).$$

The map

$$\pi : GL_n(\mathbb{C}) \to Gr(k, n)$$

induces an isomorphism from Y onto an affine open subvariety U of $Gr(k, n)$. Let $Y_w := \pi|_Y^{-1}(X_w \cap U)$.

Kummini-Lakshmibai-Sastry-Seshadri conjectured the regularities of Y_{u_ρ}. They consider ρ such that $\rho_k = 0$ and ρ_i not 'too big'.

Application I: KLSS Conjecture

Conjecture [Kummini-Lakshmibai-Sastry-Seshadri '15]

For certain $u_\rho \in S_n$ Grassmannian with descent k,

$$\mathrm{reg}(\mathbb{C}[Y_{u_\rho}]) = \sum_{i=1}^{k-1} i(\rho_i - \rho_{i+1}).$$

But these Y_{u_ρ} are just KL varieties!

Corollary [Rajchgot–R.–Weigandt '22]

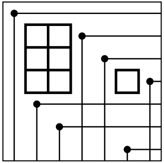
For $u_\rho \in S_n$ as in KLSS and $w_\gamma = (\mathrm{Id}_k + n - k) \times (\mathrm{Id}_{n-k})$

$$\mathrm{reg}(\mathbb{C}[Y_{u_\rho}]) = \mathrm{reg}(\mathbb{C}[\mathcal{N}_{u_\rho, w_\gamma}]) = \mathrm{reg}(\mathbb{C}[\overline{X}_{u_\rho}]).$$

How does our formula compare to the KLSS conjecture?

Computing CM-regularity of certain KL varieties

Take $u_\rho = 1457236$, so $k = 4$. Our theorem computes



$\rightsquigarrow$

1	1	3
1	1	
1	1	

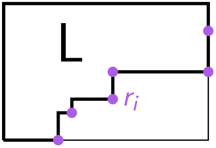
$$\operatorname{reg}(\mathbb{C}[Y_{u_\rho}]) = 3 + 1 + 1 = 5.$$

Here $\rho = (3, 2, 2, 0)$, so KLSS conjecture gives

$$\begin{aligned}\operatorname{reg}(\mathbb{C}[Y_{u_\rho}]) &= \sum_{i=1}^{3} i(\rho_i - \rho_{i+1}) \\ &= 1(3-2) + 2(2-2) + 3(2-0) = 7.\end{aligned}$$

Application II: one-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the one-sided mixed ladder determinantal variety $X(L)$.



Further, these are KL-varieties

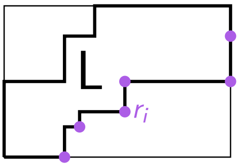
$$X(L) \cong \mathcal{N}_{u_\rho, w_\gamma} \cong \overline{X}_v.$$

Corollary [Rajchgot–R.–Weigandt '22]

$$\mathrm{reg}(\mathbb{C}[X(L)]) = \sum_{i=1}^{n} \#\mathrm{ad}(\lambda(\mathrm{v})|_{\geq \mathrm{i}})$$

Ongoing work: two-sided mixed ladder determinantal ideals

Consider a matrix $X = (x_{ij})$ of indeterminates. Let L denote the submatrix of X defined by choosing SE and NW corners. $I(L)$ is the ideal generated by the NW r_i minors of L. This defines the two-sided mixed ladder determinantal variety $\tilde{X}(L)$.



Further, for any such variety, we can construct $u, w \in S_n$ 321-avoiding such that

$$\tilde{X}(L) \cong \mathcal{N}_{u,w}.$$

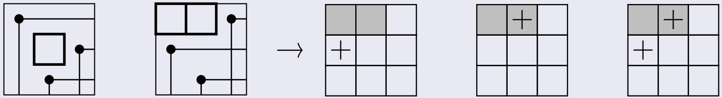
Krattenhaler-Ghorpade '15 give constructions for the related a-invariant for certain $\tilde{X}(L)$.

Ongoing work: two-sided mixed ladder determinantal ideals

For $\mathcal{N}_{u,w}$, the K-polynomials are the unspecialized Grothendiecks $\mathfrak{G}_{v,w}$ of Woo–Yong, where

$$\mathfrak{G}_{v,w}(x_1,\ldots,x_n) = \sum_{P \in PD(w) \cap R_v} (-1)^{(\#+'s)-\ell(w)} x^{wt(P)}$$

Example: $\mathfrak{G}_{v,w}$ for $w = 132$, $v = 312$.



Ongoing work: two-sided mixed ladder determinantal ideals

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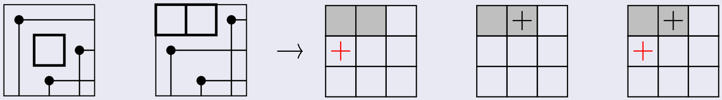

so $\mathfrak{G}_{v,w}(x_1, x_2, x_3) = x_1$.

Currently, we are working towards formulas for the degree of $\mathfrak{G}_{v,w}$ when v, w are 321-avoiding.

Conclusions

- We can express $\operatorname{reg}(\mathbb{C}[\overline{X}_w])$ in terms of the degree of the K-polynomial and the codimension of I_w.
- Use that $\operatorname{reg}(\mathbb{C}[\overline{X}_w]) = \deg \mathfrak{G}_w - \ell(w)$.
- For v vexillary, we obtain an easily computable formula for $\deg \mathfrak{G}_v$, and thus for $\operatorname{reg}(\mathbb{C}[\overline{X}_v])$.
- By relating $\mathcal{N}_{u_\rho, w_\nu}$ to $\overline{X}_v$, we correct a conjecture of KLSS and obtain formulas for regularities of one-sided ladders.