Signed combinatorial interpretations in algebraic combinatorics

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Combinatorial Structure Constants

Consider a basis $B = \{\xi_\alpha\}$ in a ring R:

$$\xi_\alpha \xi_\beta = \sum_{\alpha \in A} c_{\alpha\beta}^{\gamma} \xi_\gamma$$

Here A is some combinatorial set (partitions, compositions, etc).

Example

Take $R = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}$ and $B = \{s_\lambda : \ell(\lambda) \leq n\}$:

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^{\nu} s_\nu.$$

Here $c_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ are the *Littlewood–Richardson coefficients*.

A Central Motivation

Problem A

Determine a *non–cancellative* combinatorial rule to compute structure coefficients $c_{\alpha\beta}^{\gamma}$.

Why should such a rule exist?

Problem B

Determine a *cancellative* combinatorial rule to compute structure coefficients $c_{\alpha\beta}^{\gamma}$.

What is "combinatorial"?

A Formalization

Informal Definition

Counting X is in $\#\mathbf{P}$ if there exists a set S with $|S| = |X|$, where checking $s \in S$ is poly-time.

Example: LR-coefficients $c^{\nu}_{\lambda,\mu}$

The LR-rule via ballot semistandard Young tableaux gives

$$c^{\nu}_{\lambda,\mu} = \#\text{ballot SSYT}(\nu/\lambda, \mu) \in \#\text{P}$$

A Formalization

Define **GapP** $:= \#\mathrm{P} - \#\mathrm{P}$.

Example: symmetric group characters χ^λ_μ

The Murnaghan–Nakayama rule via border strip tableaux shows

$$\chi^\lambda_\mu = \sum_{T \in \mathrm{BST}(\lambda,\mu)} (-1)^{ht(T)} \in \mathrm{GapP}$$

Problem B, reformulated

Determine a GapP rule for $c^\gamma_{\alpha\beta}$.

Central Takeaway

Combinatorial bases with integral structure constants have signed combinatorial formulas.

(Problem B is easily resolvable in most cases)

Key Ingredient: Möbius Inversion

Let $\mathcal{P} := (A, \prec)$ be a poset.

Hall's Theorem

Take $\eta : A^2 \to \mathbb{Z}$ unitriangular. Then the inverse $\rho : A^2 \to \mathbb{Z}$ of η has the form:

$$\rho(x, y) = \sum_{\ell=0}^{h} \sum_{\substack{\text{chains} \\ x \to z_1 \to \ldots \to z_{\ell-1} \to y}} (-1)^{\ell} \eta(x, z_1) \cdot \eta(z_1, z_2) \cdot \ldots \cdot \eta(z_{\ell-1}, y).$$

Proposition

Suppose $\mathcal{P}$ has polynomial height and the incidence function in $\mathcal{P}$ is poly-time computable. Then η is in GapP implies ρ is in GapP.

Schubert polynomials

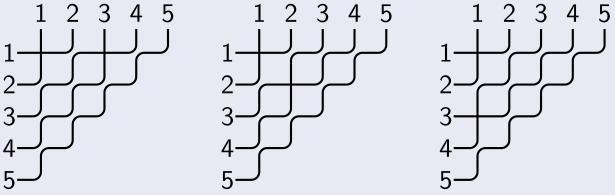
Schubert polynomials give a $\mathbb{Z}$-basis of $R = \mathbb{Z}[x_1, x_2, \ldots, x_n]$:

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{P \in PD(w)} \mathbf{x}^{\mathtt{wt}(P)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} K_{c(w),\alpha} \mathbf{x}^{\alpha}.$$

Here $c(w) \in \mathbb{Z}^n_{\geq 0}$ is the **code** of $w \in S_n$.

Example: $\mathfrak{S}_w$ for $w = 21435$ via Bergeron–Billey '96

PD(w) :



so $\mathfrak{S}_w(x_1, x_2, x_3, x_4, x_5) = x_1^2 + x_1 x_2 + x_1 x_3$.

Setup for $\mathfrak{S}_w$

Take $\mathcal{P} = (V_{n,k}, \trianglelefteq)$ to be the poset on $\alpha \in \mathbb{Z}^n_{\geq 0}$ with $|\alpha| = k$. Here $\trianglelefteq$ is dominance order. Then:

- $\mathcal{P}$ has polynomial height
- $K_{c(w),\alpha} \in \#\mathrm{P}$
- $\mathfrak{S}_w(x_1, \ldots, x_n) = \sum_{c(w) \trianglelefteq \alpha} K_{c(w),\alpha} \mathbf{x}^\alpha$
- diagonal terms $K_{c(w),c(w)} = 1$

Consider the inverse Schubert Kostka coefficients

$$\mathbf{x}^\alpha = \sum_{c(w) \trianglelefteq \alpha} K^{-1}_{c(w),\alpha} \mathfrak{S}_w(x_1, \ldots, x_n).$$

Corollary

$K^{-1}_{c(w),\alpha} \in \mathrm{GapP}$

Argument for $\mathfrak{S}_w$

$$
\begin{aligned}
\mathfrak{S}_u \cdot \mathfrak{S}_v &= \Big(\sum_{c(u) \trianglelefteq \alpha} K_{c(u),\alpha}\, \mathbf{x}^\alpha \Big) \cdot \Big(\sum_{c(v) \trianglelefteq \beta} K_{c(v),\beta}\, \mathbf{x}^\beta \Big) \\
&= \sum_{c(u) \trianglelefteq \alpha} \sum_{c(v) \trianglelefteq \beta} \Big(K_{c(u),\alpha}\, K_{c(v),\beta} \Big)\, \mathbf{x}^{\alpha+\beta} \\
&= \sum_{c(u) \trianglelefteq \alpha} \sum_{c(v) \trianglelefteq \beta} \sum_{\alpha+\beta \trianglelefteq c(w)} \Big(K_{c(u),\alpha}\, K_{c(v),\beta}\, K^{-1}_{c(w),\alpha+\beta} \Big) \mathfrak{S}_w \\
&= \sum_w c^w_{uv} \mathfrak{S}_w .
\end{aligned}
$$

Corollary

Schubert structure coefficients $c^w_{uv} \in \mathrm{GapP}$

Main Theorem

Theorem [Pak–R. '24]

Structure constants for the following families are in GapP:

- Symmetric Bases
  - s_λ, m_λ, e_λ, h_λ, p_λ
- Quasiymmetric Bases
  - M_α, F_α, $\mathfrak{S}^*_\alpha$, $\mathcal{S}_\alpha$
- Polynomial Bases
  - $\mathfrak{M}_\alpha$, $\mathfrak{F}_\alpha$, atom_α, κ_α, $\mathfrak{S}_w$, $\mathcal{L}_\alpha$, $\mathfrak{G}_w$

Using the same techniques, plethysm coefficients $a^\nu_{\lambda\mu}$ for $f_\lambda[g_\mu] = \sum_\nu a^\nu_{\lambda\mu} s_\nu$ are also in GapP for symmetric bases f_λ, g_μ.

Main Theorem – Extended

Theorem [Pak–R. '24]

Structure constants for the following are in GapP (or GapP/FP):

- Symmetric Bases
  - s_λ, m_λ, e_λ, h_λ, p_λ
  - $P_\lambda(x;\alpha)$, $P_\lambda(x;t)$, $P_\lambda(x;q,t)$ (GapP/FP)
- Quasiymmetric Bases
  - M_α, F_α, $\mathfrak{S}^*_\alpha$, $\mathcal{S}_\alpha$
  - Ψ_α, Φ_α, $\mathfrak{p}_\alpha$ (GapP/FP)
- Polynomial Bases
  - $\mathfrak{M}_\alpha$, $\mathfrak{F}_\alpha$, atom_α, κ_α, $\mathfrak{S}_w$, $\mathcal{L}_\alpha$, $\mathfrak{G}_w$

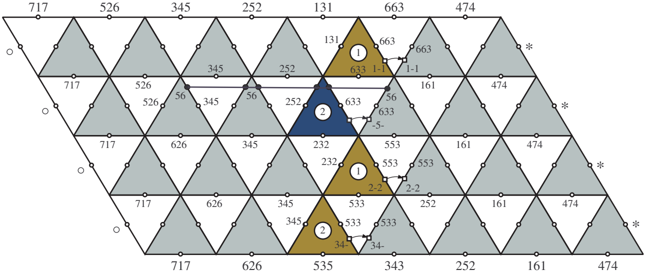
Here GapP/FP are functions f/g where $f \in$ GapP and g is poly-time computable (i.e. in FP).

What next? Improved signed formulas!

We construct $O(n^9)$ puzzle pieces $\mathcal{T}_n$. For $u, v, w \in S_n$, we build a parallelogram region $\Gamma(u, v, w)$. Using Knutson's recurrence '03:

Theorem (Pak–R. '25)

The number of signed puzzles of $\Gamma(u, v, w)$ using $\mathcal{T}_n$ is $c_{u,v}^{w}$.



Application

Using this signed puzzle rule, we find:

Corollary (Pak–R. '25)

Fix k, and let

$$\gamma_k(n) := \sum_{u,v,w \in S_n : \mathrm{inv}(w)=k} c_{u,v}^{w}.$$

Then γ_k is a polynomial in n.

This follows from inequalities imposed by our rule + Ehrhart theory.

Conclusion

- Integral structure constants are in GapP
- Plethysm coefficients are in GapP
- Develop improved GapP formulas for your favorite combinatorial basis
- Use GapP formulas to illuminate existing structure, develop asymptotics, and approach #P formulas

Thank you!