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Castelnuovo–Mumford regularity for 321-avoiding Kazhdan–Lusztig varieties

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Abstract. We introduce an algorithm to compute the degrees of 321-avoiding unspecialized Grothendieck polynomials. Our result provides an algorithm to compute the Castelnuovo–Mumford regularity of 321-avoiding Kazhdan–Lusztig ideals. This extends the work of an earlier paper of Rajchgot, the author, and Weigandt (2022) which gives a formula in the case of Grassmannian Kazhdan–Lusztig ideals.

Keywords: Kazhdan–Lusztig varieties, Grothendieck polynomials, Castelnuovo–Mumford regularity, excited Young diagrams

1 Introduction

A. Woo and A. Yong [10] introduced *Kazhdan–Lusztig varieties* to study singularities of Schubert varieties. Kazhdan–Lusztig varieties are generalized determinantal varieties which include *Matrix Schubert varieties* [4] as special cases. Another well-studied class of these Kazhdan–Lusztig varieties is the *ladder determinantal varieties*, introduced by S. S. Abhyankar [1].

The *Castelnuovo–Mumford regularity* of a graded module is an invariant used to measure its complexity. In general, this regularity may be computed using the minimal free resolution of the module. Using the fact that Kazhdan–Lusztig varieties are Cohen–Macaulay, one may instead compute their regularities combinatorially in terms of degrees of *unspecialized Grothendieck polynomials*.

Our main results Theorems 2.8 and 3.3 extend the work of [8] to provide an algorithm which computes the Castelnuovo–Mumford regularity for Kazhdan–Lusztig varieties indexed by a pair of 321-avoiding permutations. These results continue the work of J. Rajchgot, the author, and A. Weigandt [8] which provides a combinatorial formula to compute the regularity for Kazhdan–Lusztig varieties indexed by a pair of grassmannian permutations. This is an extended abstract of [9].

Due to a correspondence with matrix Schubert varieties in this case, this result in [8] may be recovered using the results of O. Pechenik–D. Speyer–A. Weigandt [7]. The work

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in [7] uses different techniques to compute the regularity of arbitrary matrix Schubert varieties. Our paper extends the techniques used in [8] to compute the regularities of certain Kazhdan–Lusztig varieties which are *not* isomorphic to matrix Schubert varieties. That is, the results of this abstract cannot, in general, be recovered using [7].

2 Combinatorial Background

In this section we define the underlying combinatorial objects used for our algorithm.

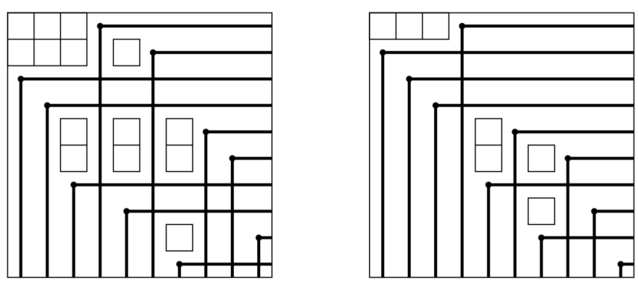
2.1 Pipe complexes

Let S_n denote the symmetric group on n letters. The **Rothe diagram** of $u \in S_n$ is the subset

$$D(u) = \{(i,j) \in [n] \times [n] \mid u_i > j \text{ and } u_j^{-1} > i\}.$$

We illustrate $D(u)$ as cells remaining in the $n \times n$ grid after placing points in cells (i, u_i) for each $i \in [n]$ and drawing a line through cells which appear weakly south or weakly east of each (i, u_i). Let $\ell(u) := \#D(u)$ denote the **Coxeter length** of u.

Example 2.1. Below are $D(v)$ and $D(w)$ for $v = 46128935(10)7$ and $w = 412368597(10)$.



Here $\ell(w) = \#D(w) = 7$.

Define an algebra over $\mathbb{Z}$ generated by $\{e_u \mid u \in S_n\}$ with multiplication such that

$$e_u e_{s_i} = \begin{cases} e_{us_i} & \text{if } \ell(us_i) > \ell(u) \\ e_u & \text{otherwise.} \end{cases}$$

Here s_i is the simple transposition permuting elements i and $i+1$.

Label the boxes of $D(u)$ along rows so that kth westmost box in row i is assigned the label $i + k - 1$. Given $P \subseteq D(u)$ let $\mathsf{word}(P)$ in $D(u)$ be the sequence formed by reading the labels of P in $D(u)$, moving east to west across rows, starting with the northmost row and progressing south. Define $I(u) := \mathsf{word}(D(u))$ in $D(u)$. The **Demazure product** of $\mathsf{word}(P)$, denoted $\delta(P)$, is the permutation determined by

$$e_{s_{i_1}} \cdots e_{s_{i_k}} = e_{\delta(P)},$$

where $\mathsf{word}(P) = (i_1, i_2, \ldots, i_k)$.

Take $v \geqslant w \in S_n$, where $\geqslant$ denotes Bruhat order on S_n. Define

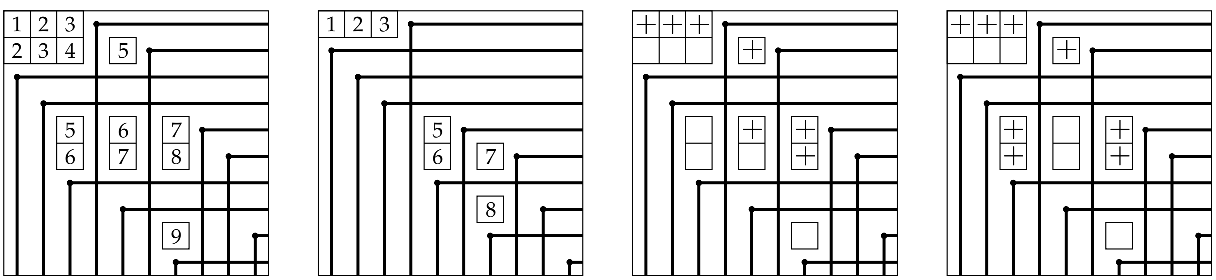
$$\mathsf{Pipes}(v, w) = \{P \subseteq D(v) \mid \mathsf{word}(P) = (i_1, i_2, \ldots, i_{\ell(w)}) \text{ in } D(v) \text{ and } \delta(P) = w\}.$$

Similarly, let

$$\overline{\mathsf{Pipes}}(v, w) = \{P \subseteq D(v) \mid \delta(P) = w\}.$$

We illustrate $P \subseteq D(v)$ by marking $(i, j) \in D(v)$ with a $+$ whenever $(i, j) \in P$. Lastly, let $D^{NE}(v, w) \subseteq D(v)$ be the boxes corresponding to the earliest subsequence of $\mathsf{word}(D(v))$ that forms $I(w)$. Since $v \geqslant w$, $D^{NE}(v, w)$ exists.

Example 2.2. The left two diagrams are $D(v)$ and $D(w)$ for w, v as in Example 2.1 with $I(v)$ and $I(w)$ labeled. This gives $I(v) = (3, 2, 1, 5, 7, 6, 8)$ in $D(v)$. The third diagram is $D^{NE}(v, w) \in \mathsf{Pipes}(v, w) \subseteq \overline{\mathsf{Pipes}}(v, w)$ and the fourth is another $P \in \overline{\mathsf{Pipes}}(v, w)$.



As defined by Woo–Yong [11], the **unspecialized Grothendieck polynomial** is

$$\mathfrak{G}_{v,w}(\mathbf{t}) = \sum_{P \in \overline{\mathsf{Pipes}}(v,w)} (-1)^{\#P - \ell(w)} \prod_{(i,j) \in P} t_{ij}. \tag{2.1}$$

By setting $v = w_0 \in S_n$ and specializing variables t_{ij}, these unspecialized Grothendieck polynomials recover the *double Grothendieck polynomials* of [6]. Note that we follow the conventions of [8] for $\mathfrak{G}_{v,w}(\mathbf{t})$, which differ from those in [11].

2.2 Skew Excited Young Diagrams

A permutation $u \in S_n$ is 321-**avoiding** if there does not exist a 321 *pattern* in u, i.e., indices $i < j < k$ such that $u_k < u_j < u_i$. For example, $u = 1\underline{7}2\underline{5}83\underline{4}6$ is not 321-avoiding; we underlined the positions of a 321 pattern. Let $S_n^{321-\mathrm{av}} := \{u \in S_n \mid u \text{ is 321-avoiding}\}$.

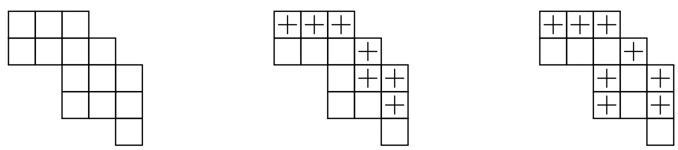
For this subsection assume $v \geqslant w$ where $v, w \in S_n^{321-\mathrm{av}}$. Let

$$\phi_v : \{P \subseteq D(v)\} \to \{S \subset [n] \times [n]\}$$

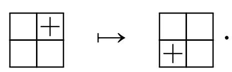
be the map which deletes empty rows and columns of $D(v)$ from $P \subset D(v)$. The shape $\mathcal{R}_v := \phi_v(D(v))$ is a skew Young diagram, *i.e.*, λ/μ for some partitions $\mu \subseteq \lambda$. Our conventions for drawing Young diagrams reflect the diagrams in English notation across the y-axis. Define $D_{\mathtt{top}}(v, w) := \phi_v(D^{NE}(v, w))$.

We visualize $D \subseteq \mathcal{R}_v$ by marking $(i, j) \in [n] \times [n]$ with a $+$ when $(i, j) \in D$. In general, we call a collection of $+$'s inside $\mathcal{R}_v$ a **diagram** in $\mathcal{R}_v$.

Example 2.3. For v, w as in Example 2.2, the left picture is $\mathcal{R}_v$, the middle is $D_{\mathtt{top}}(v, w)$, and the rightmost diagram is $\phi_v(P)$ for the rightmost P in Example 2.2.

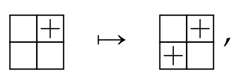


An **excited move** of a diagram D in $\mathcal{R}_v$ is the operation on a 2×2 subsquare of D such that

 (2.2)

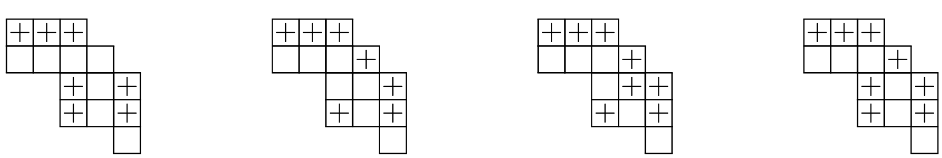
For this move to occur, the subsquare must be contained in $\mathcal{R}_v$. We let $\mathsf{SEYD}(v, w)$ denote the set of $D \subseteq \mathcal{R}_v$ which can be computed through sequential applications of excited moves to $D_{\mathtt{top}}(v, w)$. We call $D \in \mathsf{SEYD}(v, w)$ a **skew excited Young diagram**.

We also may apply **K-theoretic excited moves** to diagrams in $\mathcal{R}_v$

 (2.3)

again, where all cells pictured are contained in $\mathcal{R}_v$. Write $\overline{\mathsf{SEYD}}(v, w)$ for the set of diagrams which can be obtained by sequential applications of excited and K-theoretic excited moves on $D_{\mathtt{top}}(v, w)$ in $\mathcal{R}_v$. We say $D \in \overline{\mathsf{SEYD}}(v, w)$ is a **K-theoretic skew excited Young diagram**. Let $\#D$ denote the number of pluses in D. We say $D \in \overline{\mathsf{SEYD}}(v, w)$ define to be **maximal** if $D' \in \overline{\mathsf{SEYD}}(v, w)$ implies $\#D' \leqslant \#D$.

Example 2.4. Continuing Example 2.3, the left two diagrams are in $\mathsf{SEYD}(v, w)$. The right two diagrams are both maximal diagrams in $\overline{\mathsf{SEYD}}(v, w)$.



Using [3] and the fact that $v, w \in S_n^{321-\mathrm{av}}$ we obtain the following:

Proposition 2.5. *For $v \geqslant w$ where $v, w \in S_n^{321-av}$, the map ϕ_v restricted to $\overline{\mathsf{Pipes}}(v,w)$ gives a bijection*

$$\widetilde{\phi}_v : \overline{\mathsf{Pipes}}(v,w) \to \overline{\mathsf{SEYD}}(v,w) \tag{2.4}$$

such that for $P \in \overline{\mathsf{Pipes}}(v,w)$, $\#P = \#\widetilde{\phi}_v(P)$.

Combining Proposition 2.5 with Equation (2.4) produces the following result.

Corollary 2.6. *Suppose $v, w \in S_n$. Then*

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \max\{\#D \mid D \in \overline{\mathsf{SEYD}}(v,w)\}.$$

Example 2.7. Since the rightmost diagram in Example 2.4 is maximal, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = 8$ by Corollary 2.6.

In Section 4.1 we give an algorithm to compute statistics $\Delta_{v,w}(q)$ from $D_{\mathtt{top}}(v,w)$ for certain $q \in \mathbb{Z}_{>0}$. Using Corollary 2.6, we prove the following.

Theorem 2.8. *Suppose $v \geqslant w$, where $v, w \in S_n^{321-av}$. Then if $D_{\mathtt{top}}(v,w) = \bigcup_{q\in[s]} C_q$ where C_q are the connected components of $D_{\mathtt{top}}(v,w)$,*

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \#D(w) + \sum_{q\in[s]} \Delta_{v,w}(q).$$

A proof sketch for Theorem 2.8 appears in Section 4.2.

3 Castelnuovo–Mumford Regularity of Kazhdan–Lusztig varieties

In this section, we define Castelnuovo–Mumford regularity and Kazhdan–Lusztig varieties. We then recall results of [8] which provide combinatorial interpretations of the Castelnuovo–Mumford regularity of Kazhdan–Lusztig varieties.

3.1 Castelnuovo–Mumford Regularity

Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring with the standard grading and let $I \subseteq S$ be a homogeneous ideal. The **Hilbert series** of S/I is a formal power series

$$H(S/I;t) = \sum_{k\in\mathbb{Z}} \dim_{\mathbb{C}}((S/I)_k)t^k = \frac{K(S/I;t)}{(1-t)^n}.$$

The numerator of the Hilbert series $K(S/I;t) \in \mathbb{C}[t^{\pm 1}]$ is the **K-polynomial** of S/I. A minimal free resolution of S/I is the complex

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}(S/I)} \to \bigoplus_j S(-j)^{\beta_{l-1,j}(S/I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(S/I)} \to S/I \to 0,$$

where $l \leqslant n$ and $S(-j)$ is the free S-module shifted by j in degree. The **Castelnuovo-Mumford regularity** of S/I, written $\mathrm{reg}(S/I)$, is the statistic

$$\mathrm{reg}(S/I) := \max\{j - i \mid \beta_{i,j}(S/I) \neq 0\}.$$

In cases where S/I is Cohen-Macaulay,

$$\mathrm{reg}(S/I) = \deg K(S/I;t) - \mathrm{ht}_S I, \tag{3.1}$$

where $\mathrm{ht}_S I$ denotes the height of the ideal I. For more context, consult [2, Lemma 2.5].

3.2 Kazhdan–Lusztig varieties

For $v \in S_n$, define $M^{(v)} = (m_{ij})$ to be the matrix such that for $i, j \in [n]$,

$$m_{ij} = \begin{cases} 1 & \text{if } j = v_i, \\ z_{ij} & \text{if } (i,j) \in D(v), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{C}[\mathbf{z}^v] := \mathbb{C}[z_{ij} \mid (i,j) \in D(v)]$. For $v \geqslant w \in S_n$ the **Kazhdan–Lusztig ideal** $J_{v,w} \subseteq \mathbb{C}[\mathbf{z}^v]$ is defined by

$$J_{v,w} = \langle r_w(i,j) + 1 - \text{minors in } M^{(v)}_{[i],[j]} \mid (i,j) \in D(w) \rangle,$$

where $M_{I,J}$ denote the submatrix of M with row indices in I and column indices in J for $I, J \subseteq [n]$. As noted in [8] when $v \in S_n^{321-\mathrm{av}}$, $J_{v,w}$ is homogeneous with respect to the standard grading.

Let $B_+, B_- \subset \mathrm{GL}_n(\mathbb{C})$ denote the Borel and opposite Borel subgroups, respectively. As defined in [10], the **Kazhdan–Lusztig variety** is the intersection of the Schubert variety $B_-\backslash\overline{B_- w B_+} \subseteq B_-\backslash\mathrm{GL}_n(\mathbb{C})$ with the opposite Schubert cell $B_-\backslash B_- v B_-$. The coordinate ring of this Kazhdan–Lusztig variety is precisely $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$. Through this fact, $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ is Cohen-Macaulay. Again we follow the conventions used in [8] rather than those in [10]. For additional context concerning Kazhdan–Lusztig varieties, see the survey [12].

Reformulating [11, Theorem 4.5] for the case $v, w \in S_n^{321-\mathrm{av}}$,

Lemma 3.1. *[8, Lemma 6.3] Let $v, w \in S_n^{321-av}$ where $w \leqslant v$. Then*

$$K(\mathbb{C}[\mathbf{z}^v]/J_{v,w};t) = \sum_{P \in \overline{\mathsf{Pipes}}(v,w)} (-1)^{\#P - \ell(w)}(1-t)^{\#P}.$$

We apply this Lemma along with Equation (3.1) for the following proposition.

Proposition 3.2. *[8, Proposition 6.4] Let $v, w \in S_v^{321-av}$ where $w \leqslant v$. Then,*

$$\deg K(\mathbb{C}[\mathbf{z}^v]/J_{v,w}; t) = \deg \mathfrak{G}_{v,w}(\mathbf{t}).$$

Furthermore, the Castelnuovo-Mumford regularity of $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ is given by

$$\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \deg \mathfrak{G}_{v,w}(\mathbf{t}) - \#D(w).$$

By combining Proposition 3.2 and Theorem 2.8,

Theorem 3.3. *Suppose $v \geqslant w$, where $v, w \in S_n^{321-av}$. Then if $D_{\mathtt{top}}(v, w) = \bigcup_{q \in [s]} C_q$ where C_q are the connected components of $D_{\mathtt{top}}(v, w)$,*

$$\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \sum_{q \in [s]} \Delta_{v,w}(q).$$

In [5] S. R. Ghorpade–C. Krattenthaler give an algorithm to compute a related invariant called the a-invariant of certain two-sided ladder determinantal varieties. Two-sided ladder determinantal varieties are Kazhdan–Lusztig varieties indexed by particular $v, w \in S_n^{321-\mathrm{av}}$. In this setting, the a-invariant is easily computed from the Castelnuovo–Mumford regularity. As we show in the full version of this abstract, Theorem 3.3 may be applied to generalize [5, Lemma 14].

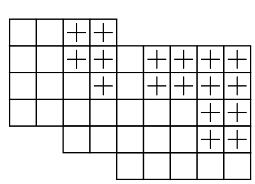
4 Construction and Recurrence for Theorem 2.8

Assume $v \geqslant w \in S_n^{321-\mathrm{av}}$. In Section 4.1 we describe how to compute the statistics $\Delta_{v,w}(q)$ used in Theorem 2.8. In Section 4.2 we sketch the proof of Theorem 2.8.

4.1 Construction for Theorem 2.8

We index $\mathcal{R}_v$ using matrix indexing, identifying the northwest most box in $\mathcal{R}_v$ with $(1, 1)$. Suppose $D_{\mathtt{top}}(v, w) = \bigcup_{q \in [s]} C_q$ where C_q are the connected components of $D_{\mathtt{top}}(v, w)$. Order C_q such that the indices increase when viewing components from northwest to southeast.

Example 4.1. Consider $D = D_{\mathtt{top}}(v, w)$ below for some $v, w \in S_{15}^{321-\mathrm{av}}$.



Then D has connected components C_1 and C_2, where $C_1 = \{(1,3),(1,4),(2,3),(2,4),(3,3)\}$ and $C_2 = \{(2,6),(2,7),(2,8),(2,9),(3,6),(3,7),(3,8),(3,9),(4,8),(4,9),(5,8),(5,9)\}$.

For each $q \in [s]$, define $\mathsf{Diag}_{v,w}(C_q) = \{\mathsf{b}^q_k\}_{k\in[\ell_q]}$ to be the westmost then southmost diagonal of boxes in C_q of maximal length ℓ_q. Boxes in $\mathsf{Diag}_{v,w}(C_q)$ are ordered increasingly northwest to southeast.

For $q \in [s]$ in decreasing order, compute $\mathtt{md}(C_q) = \{\mathsf{d}^q_k\}_{k\in[\ell_q]} \subseteq C_q$, such that $\mathtt{md}(C_q)$ is the westmost then southmost diagonal of length ℓ_q that minimizes

$$\#\big([\|\psi_E(\mathsf{d}^q_{\ell_q})\| + 1] \cap \{\|\mathsf{d}^{q'}_{k'}\|\}_{q'>q,k'\in[\ell_{q'}]}\big).$$

Here $\|\mathsf{b}\| := \mathsf{b}(1) + \mathsf{b}(2)$ for $\mathsf{b} = (\mathsf{b}(1),\mathsf{b}(2)) \in D_{\mathtt{top}}(v,w)$. Boxes in $\mathtt{md}(C_q)$ are ordered increasingly northwest to southeast.

Set $D^{(0)}_{\mathtt{zip}}(v,w) := D_{\mathtt{top}}(v,w)$. For $q \in [s]$, we define $D^{(q)}_{\mathtt{zip}}(v,w)$ iteratively by applying exited moves to certain pluses in $C_q \subseteq D^{(q-1)}_{\mathtt{zip}}(v,w)$. In $D^{(q-1)}_{\mathtt{zip}}(v,w)$, set

$$S = \{\mathsf{b} \in C_q - \mathtt{md}(C_q) \text{ weakly southwest of } \mathtt{md}(C_q)\}.$$

To each in $\mathsf{b} \in S$, working west to east and south to north, let b' be the new position of b after applying as many excited moves as possible to b. Let

$$D^{(q)}_{\mathtt{zip}}(v,w) := D^{(q-1)}_{\mathtt{zip}}(v,w) - S \cup \{\mathsf{b}' \mid \mathsf{b} \in S\}.$$

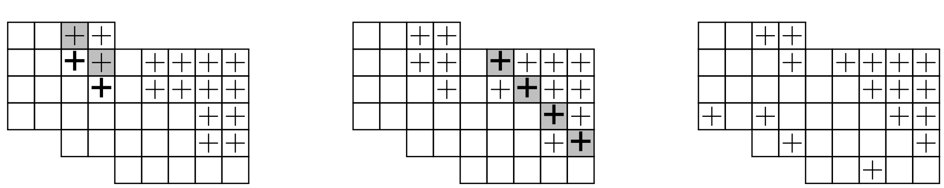
Define $D_{\mathtt{zip}}(v,w) := D^{(s)}_{\mathtt{zip}}(v,w)$. For $\mathsf{b} \in \mathtt{md}(C_q)$, define $\mathsf{trail}_{v,w}(\mathsf{b})$ such that

$$\mathsf{trail}_{v,w}(\mathsf{b}) := \max\{k \in \{0,1,\ldots,n\} \mid \mathsf{b} + (k',-k'), \mathsf{b} + (k',1-k'), \\ \mathsf{b} + (k'-1,-k') \in \mathcal{R}_v - D_{\mathtt{zip}}(v,w) \text{ for each } k' \in [k]\}.$$

We define the statistic

$$\Delta_{v,w}(q) := \sum_{k\in[\ell_q]} \mathsf{trail}_{v,w}(\mathsf{d}^q_k).$$

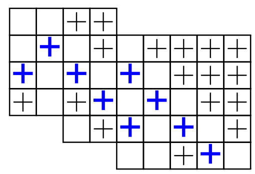
Example 4.2. We continue with v,w as in Example 4.1. Left to right, the diagrams below are $D^{(0)}_{\mathtt{zip}}(v,w)$, $D^{(1)}_{\mathtt{zip}}(v,w)$, and $D^{(2)}_{\mathtt{zip}}(v,w)$, respectively. In $D^{(0)}_{\mathtt{zip}}(v,w)$, $\mathsf{Diag}_{v,w}(C_1)$ is bolded, and $\mathtt{md}(C_1)$ is shaded. In $D^{(1)}_{\mathtt{zip}}(v,w)$, $\mathsf{Diag}_{v,w}(C_2)$ is bolded, and $\mathtt{md}(C_2)$ is shaded.



Using $D^{(0)}_{\texttt{zip}}(v,w)$ and $\mathsf{Diag}_{v,w}(C_1)$, we find $\Delta_{v,w}(1) = (2+1)$. Similarly using $D^{(1)}_{\texttt{zip}}(v,w)$ and $\mathsf{Diag}_{v,w}(C_2)$, we compute $\Delta_{v,w}(2) = (2+2+1+1)$. Therefore, Theorem 2.8 determines

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \#D(w) + \Delta_{v,w}(1) + \Delta_{v,w}(2) = 17 + 3 + 6 = 26.$$

By Corollary 2.6, there exists $D \in \overline{\mathsf{SEYD}}(v,w)$ where $\#D = 26$. We have drawn such a diagram below. This is computed by applying $\mathsf{trail}_{v,w}(\mathsf{d}^q_k)$-many K-theoretic excited moves along the antidiagonals of $\mathsf{d}^q_k \in \mathtt{md}(C_q)$ for each $k \in [\ell_q], q \in [s]$. The pluses that result from these K-theoretic excited moves are drawn in blue.



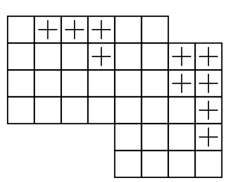
Theorem 3.3 gives $\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \Delta_{v,w}(1) + \Delta_{v,w}(2) = 9$. This corresponds precisely with the number of blue pluses in the diagram above.

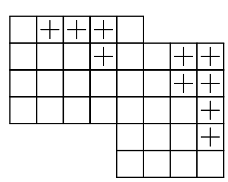
4.2 Proof Sketch for Theorem 2.8

In proving Theorem 2.8, we utilize a particular recurrence on $\mathfrak{G}_{v,w}(\mathbf{t})$. Let (a,b) be the northmost then eastmost plus in $D_{\texttt{top}}(v,w)$. Take (a',b') to be the northmost then eastmost box in $\mathcal{R}_v$. Set $i = \mathsf{word}(\phi_v^{-1}(\{(a,b)\}))$ and $i' = \mathsf{word}(\phi_v^{-1}(\{(a',b')\}))$ in $D(v)$. Define $v_P = s_{i'}v$, which gives $\mathcal{R}_{v_P} = \mathcal{R}_v - \{(a',b')\}$. Define $w_P := s_i w$, $w_C := w$, and $v_C := v_P$.

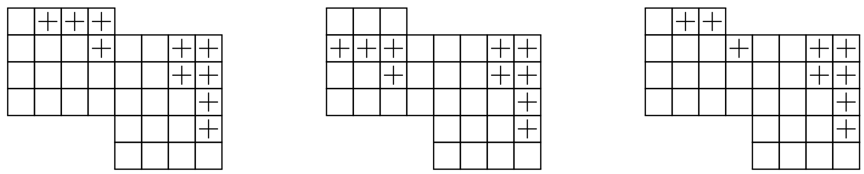
To proceed, we first establish $v_P, w_P \in S_n^{321-\mathrm{av}}$. This follows from the definition of (a,b) and (a',b') as northeast most choices along with the graphical definition of 321-avoiding. That is, $u \in S_n^{321-\mathrm{av}}$ if and only if $\mathcal{R}_u$ is a skew Young diagram.

Example 4.3. Below we have $D_{\texttt{top}}(v,w)$ on the left and $D_{\texttt{top}}(v_C,w_C)$ on the right for particular $v,w \in S_{15}^{321-\mathrm{av}}$. In this case, $(a,b) \neq (a',b')$.





Below are $D_{\texttt{top}}(v',w')$, $D_{\texttt{top}}(v'_C,w'_C)$, and $D_{\texttt{top}}(v'_P,w'_P)$, listed from left to right, for particular $v',w' \in S_{15}^{321-\mathrm{av}}$. In this case, $(a,b) = (a',b')$.



We give a general correspondence of these K-theoretic skew excited Young diagrams.

Lemma 4.4. *For $v \geqslant w$ and $v, w \in S_n^{321-av}$, the following hold:*

1. *If $(a,b) \neq (a',b')$,*
$$\overline{\mathsf{SEYD}}(v,w) = \overline{\mathsf{SEYD}}(v_C, w_C).$$
2. *If $(a,b) = (a',b')$,*
$$\overline{\mathsf{SEYD}}(v,w) = \overline{\mathsf{SEYD}}(v_C, w_C) \bigsqcup \left\{ D \cup (a,b) \mid D \in \overline{\mathsf{SEYD}}(v_C, w_C) \cup \overline{\mathsf{SEYD}}(v_P, w_P) \right\}.$$

Combining Corollary 2.6 with Lemma 4.4,

Corollary 4.5. *For $v \geqslant w$ and $v, w \in S_n^{321-av}$, the following hold:*

1. *if $(a,b) \neq (a',b')$, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \deg(\mathfrak{G}_{v_C,w_C}(\mathbf{t}))$.*
2. *If $(a,b) = (a',b')$, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \max(\deg(\mathfrak{G}_{v_P,w_P}(\mathbf{t})), \deg(\mathfrak{G}_{v_C,w_C}(\mathbf{t}))) + 1$.*

With this recurrence established, we now sketch the proof.

Proof sketch of Theorem 2.8:

We proceed by induction on $\ell(v)$. For $\ell(v) = 0$, the statement is trivial since in this case $\mathsf{SEYD}(v,w) = \varnothing$. Suppose the statement holds for v such that $\ell(v) = k-1$ for $k \geqslant 1$. Consider v such that $\ell(v) = k$. For brevity let $d(u_1,u_2) = \sum_{q \in [s]} \Delta_{u_1,u_2}(q)$ where $D_{\mathtt{top}}(u_1,u_2)$ has s components.

If $(a,b) \neq (a',b')$, by Corollary 4.5, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \deg(\mathfrak{G}_{v_C,w_C}(\mathbf{t}))$. Using Lemma 4.4, we determine $D_{\mathtt{zip}}(v,w) = D_{\mathtt{zip}}(v_C,w_C)$, so $d(v,w) = d(v_C,w_C)$. Thus the result follows by the inductive assumption.

Now assume $(a,b) = (a',b')$. Then by Corollary 4.5,
$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \max(\deg(\mathfrak{G}_{v_P,w_P}(\mathbf{t})), \deg(\mathfrak{G}_{v_C,w_C}(\mathbf{t}))) + 1.$$
Since $\#D(w) = \#D(w_C)$ and $\#D(w) = \#D(w_P) + 1$, from the inductive assumption,
$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \#D(w) + \max(d(v_P,w_P), d(v_C,w_C) + 1).$$
Thus it suffices to prove
$$d(v,w) = \max(d(v_P,w_P), d(v_C,w_C) + 1). \tag{4.1}$$

To establish Equation (4.1), we perform a careful case analysis on the position of $(a,b) \in C_q$ in relation to $\mathtt{md}(C_q)$. The following claim is useful in this examination.

Claim 4.6. *Suppose* $(a, b) = (a', b') \in C_q$ *where* C_q *is a connected component in* $D_{\mathtt{top}}(v, w)$*. Let* $R_q = \{\mathsf{d} \in C_q \mid \mathsf{d}$ *lies weakly southwest of* $(a, b)\}$*. Then*

1. $D_{\mathtt{top}}(v_P, w_P) = D_{\mathtt{top}}(v, w) - (a, b)$*, and*
2. $D_{\mathtt{top}}(v_C, w_C) = D_{\mathtt{top}}(v, w) - R_q \cup R'_q$*,*

where $R'_q = \{\mathsf{d} + (1, -1) \mid \mathsf{d} \in R_q\}$*.*

With Equation (4.1) proven, Theorem 2.8 follows. □

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