SATURATION PROPERTY FAILS FOR SCHUBERT COEFFICIENTS

IGOR PAK* AND COLLEEN ROBICHAUX*

Abstract. The saturation property for Littlewood–Richardson coefficients was established by Knutson and Tao in 1999. In 2004, Kirillov conjectured that the saturation property extends to Schubert coefficients. We disprove this conjecture in a strong form, by showing that it fails for a large family of instances. We also refute the saturation property for Schubert coefficients under *bit scaling* and discuss computational complexity implications.

1. Introduction

1.1. Saturation property. The *saturation conjecture* (now *saturation theorem*) was proven by Knutson and Tao [KT99] and is one of the most celebrated results in Algebraic Combinatorics. Despite its apparent simplicity, it became the last piece of the puzzle resolving *Horn's problem*, which describes possible spectra of three Hermitian matrices satisfying the equation $A+B=C$. See e.g. [Buch00, Ful00] for overviews of different aspects of this remarkable story and [BVW17, Kum14] for some later developments.

The *saturation theorem* states that for all partitions λ, μ, ν with $|\lambda| = |\mu| + |\nu|$, we have:

$$c^{\lambda}_{\mu\nu} > 0 \quad \Longleftrightarrow \quad c^{N\lambda}_{N\mu,N\nu} > 0 \;\; \textit{for any} \;\; N \geq 1, \tag{1.1}$$

where $c^{\lambda}_{\mu\nu}$ denote the *Littlewood–Richardson* (LR) *coefficients*, see e.g. [Mac95, §1.9]. We refer to (1.1) as the *saturation property.* Note that $\Rightarrow$ is the easy direction, which follows directly from several combinatorial interpretations of LR coefficients. These include interpretations in terms of the number of *lattice tableaux*, see e.g. [Sta99, §A1.3], or in terms of *Gelfand–Tsetlin patterns*, see e.g. [Zel99]. On the other hand, the direction $\Leftarrow$ is quite difficult and involves technical combinatorial [KT99], algebraic [DW00, KM08], or algebro-geometric arguments [Bel06].

The LR coefficients play a central role in Algebraic Combinatorics and related areas, so in the aftermath of the saturation theorem, a number of generalizations of (1.1) have been proposed, see e.g. a large collection in [Kir04]. Unfortunately, in the quarter century since the original paper, very few saturation type properties have been established, all of them remarkable. These include the quantum version by Belkale [Bel08, §4.1], the general reductive group version (up to a factor of 2) by Kapovich–Millson [KM08] (see also [BK10]), and the equivariant version by Anderson–Richmond–Yong [ARY13] (see also [ARY19]). Most recently, an unexpected Möbius strip version by Min [Min24] resolved the *Gao–Orelowitz–Yong saturation conjecture* for the *Newell–Littlewood numbers* [GOY21].

Among positive results, let us also mention *Fulton's conjecture* resolved by Knutson, Tao, and Woodward [KTW04], which can be viewed as a variation on (1.1) :

$$c^{\lambda}_{\mu\nu} = 1 \quad \Longleftrightarrow \quad c^{N\lambda}_{N\mu,N\nu} = 1 \;\; \textit{for any} \;\; N \geq 1. \tag{1.2}$$

Here the direction $\Leftarrow$ follows easily from the saturation property, while the direction $\Rightarrow$ requires further work. This *uniqueness property* was also generalized in several ways, notably in [BKR12].

In the negative direction, there is a large number of counterexamples to the saturation property for various extensions of LR coefficients that are scattered across the literature. For example, it was shown by Èlashvili [Èla92] that the saturation fails for root system B (explaining the factor

January 7, 2026.

*Department of Mathematics, UCLA, Los Angeles, CA 90095, USA. Email: {pak,robichaux}@math.ucla.edu.

of 2 mentioned above), see also [Zel99, p. 340] and [DM06, §4.2]. Similarly, Buch observed that the saturation property easily fails for the K-theoretic generalization [Buch02, p. 71]. Most recently, Yadav, Yong and the second author noted that the saturation fails for Schur P-polynomials [RYY22, Remark 7.7]; see also [CR23, §7.4] for a larger example in the geometric context.

For *Kronecker coefficients*, which famously generalize the LR coefficients, the saturation property fails already for the two row partitions, see e.g. [Ros01, Ex. 2]. Because of their crucial role in *Geometric Complexity Theory* (GCT), Mulmuley stated a weak version of the property [Mul09, §1.6]. Soon after, Briand, Orellana and Rosas [BOR09], disproved Mulmuley's weak version. Curiously, a version of the uniqueness property (1.2) continues to hold for Kronecker coefficients, see [SS16] and references therein.

Finally, for *reduced Kronecker coefficients*, the saturation property was conjectured independently by Kirillov [Kir04, Conj. 2.33] and Klyachko [Kly04, Conj. 6.2.4]. These constants occupy an intermediate place between LR and Kronecker coefficients, as they generalize the former and are a special case of the latter. Only recently, Panova and the first author constructed a large family of counterexamples in this case [PP20]; see also §5.2 for the computational complexity context.

1.2. **Schubert saturation.** *Schubert coefficients* $\{c^w_{u,v} \, : \, u, v, w \in S_\infty\}$ can be defined as structure constants for *Schubert polynomials*:

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \; = \; \sum_{w \in S_\infty} c^w_{u,v} \, \mathfrak{S}_w \, . \tag{1.3}$$

Here S_∞ consists of permutations with all but finitely many fixed points, and $\mathfrak{S}_u \in \mathbb{N}[x_1, x_2, \ldots]$.

It is known that $c^w_{u,v}$ are nonnegative integers, as they count certain intersection numbers. These coefficients play a major role in *Schubert calculus*, a rapidly developing area of algebraic geometry, motivated in part by rich connections with representation theory and algebraic combinatorics, see e.g. [AF24, Knu22].

We note that *Grassmannian permutations* w (permutations with at most one descent) correspond to integer partitions $\lambda = \lambda(w)$. Famously, the corresponding Schubert polynomials are symmetric and coincide with Schur polynomials: $\mathfrak{S}_w = s_{\lambda(w)}$. Thus, Schubert coefficients can be viewed as advanced generalizations of LR coefficients, see e.g. [Man01, §2.6.4].

It is then natural to ask if Schubert coefficients also satisfy a saturation property extending (1.1). In [Kir04], Kirillov formulated this as a conjecture (see below), which remained open until now. Motivated by GCT and with complexity applications as a motivation (cf. §5.1), Mulmuley also speculated that saturation might hold in this case [Mul09, §3.7].

For a permutation $w \in S_n$, the *Lehmer code*, also called the *inversion index*, is defined as

$$\mathsf{code}(w) \; := \; (c_1, \ldots, c_n) \in \mathbb{N}^n, \quad \text{where} \quad c_i \; := \; |\{j \, : \, j > i, \, w(j) < i\}|.$$

Clearly, $c_1 + \ldots + c_n = \ell(w)$ is the *number of inversions*. We can now define the operation of *code scaling* as follows:

$$N * w \; := \; \mathsf{code}^{-1}(Nc_1, Nc_2, \ldots, Nc_n, 0, \ldots, 0) \in S_{Nn} \, .$$

It is easy to see that the code of a Grassmannian permutation w is a partition $\lambda(w)$ written in reverse, so this code scaling corresponds to the usual multiplication of partitions by a constant N. Therefore, the following conjecture is a natural generalization of the saturation theorem:

Conjecture 1.1 (Kirillov [Kir04, Conj. 6.28])**.** *For every $u, v, w \in S_n$, we have:*

$$c^w_{u,v} > 0 \quad \Longleftrightarrow \quad c^{N*w}_{N*u,\, N*v} > 0 \;\; \text{for any} \;\; N \geq 1 \, . \tag{1.4}$$

We disprove Kirillov's conjecture in a strong form, by constructing a large family of triples of permutations for which the saturation property fails:

Theorem 1.2. *Let $u \in S_n$ and let $j - i \geq 2$, such that:*

$$u \lessdot ut_{ij} \qquad \text{and} \qquad u(i) < u(j) - 1,$$

where $t_{ij} = (i, j)$ is a transposition and $x \lessdot y$ is the cover relation in the strong Bruhat order of S_n. Finally, let $v = (i, i+1)$ be a simple transposition, and let $w = ut_{ij}$. Then:

$$c^w_{u,v} = 1 \qquad \text{and} \qquad c^{N*w}_{N*u,N*v} = 0 \quad \text{for all} \quad N > 1. \tag{1.5}$$

We illustrate the theorem by an explicit sequence of counterexamples:

Corollary 1.3. *For all $n \geq 4$, let $v := (n-2, n-3)$ be a simple transposition S_n, and let*

$$u := (2, 3, \ldots, n-2, 1, n, n-1), \qquad w := (2, 3, \ldots, n, 1, n-2, n-1).$$

Then:

$$c^w_{u,v} = 1 \qquad \text{and} \qquad c^{N*w}_{N*u,N*v} = 0 \quad \text{for all} \quad N > 1.$$

In particular, this shows that Kirillov's Conjecture 1.1 fails for all $n \geq 4$ and all $N \geq 2$. Note also that in Theorem 1.2, we always have v is a simple transposition. In this case, Schubert coefficients have a simple combinatorial interpretation given by *Monk's rule* (Proposition 2.1). The proof of Theorem 1.2 (see §3.3) uses this combinatorial rule for the first part of (1.5), and the *St. Dizier–Yong vanishing condition* (Lemma 3.2) for the second part.

Let us emphasize that although (1.4) is a direct generalization of the saturation property (1.1), it fails for what was originally an "easy direction" $\Rightarrow$. Also, observe that the permutations in Corollary 1.3 have at most 2 descents. In this case, Schubert coefficients have two different combinatorial interpretations [Cos09, BKPT16], cf. §5.1.

Finally, note that there is more than one way to define an operation on permutations to produce a saturation property. For completeness, we obtain similar results for the folklore *bit scaling* operation (see §4.1), which can be viewed as a partial tensor product with the identity permutation. The saturation property in this case also generalizes the saturation property for LR coefficients, so it is natural to ask whether this property holds (cf. §5.3). Again, we refute this possibility in a strong sense, see Theorem 4.4 and Corollary 4.6. The proof in this case uses Monk's rule (Proposition 2.1) and the *dimension condition* (Proposition 2.2).

1.3. **Structure of the paper.** We start with the algebraic combinatorics background in Section 2, where we include standard definitions, notation and basic results in the area. In Section 3, we prove Theorem 1.2, thus giving counterexamples to Kirillov's Conjecture 1.1. Then, in Section 4 we introduce bit scaling and discuss analogous results for the corresponding saturation property. We conclude with final remarks and open problems in Section 5. Notably, we discuss the failure of the saturation property (1.4) in the context of the complexity of Schubert vanishing.

2. Background

2.1. **Basic notation.** We use $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $[n] = \{1, 2, \ldots, n\}$. To simplify the notation, for a set A and elements x, y, we write $A + x$ to denote $A \cup \{x\}$, and $A - y$ to denote $A - \{y\}$.

We use $\mathbb{K}$ to denote the set of infinite sequences with entries in $\mathbb{N}$ with finite support. When writing such sequences, we omit the infinite tail of zeros, and write only the prefix with the support of the sequence, so e.g. $30120000\ldots$ is written as 3012.

We use S_n to denote the symmetric group, which we view as the group of permutations of $[n]$. Denote by ι the inclusion $\iota : S_n \hookrightarrow S_{n+1}$ defined by $w(1)\cdots w(n) \mapsto w(1) \cdots w(n)\, n+1$. As above, let $S_\infty = \bigcup_{n \geq 1} S_n$ denote the group of permutations of $\mathbb{N}_{\geq 1} = \{1, 2, \ldots\}$ with all but finitely many fixed points, where the inclusion is given by ι. By analogy with infinite sequences, when writing such permutations we omit the tail of fixed points, so e.g. $34125678\ldots$ is written as 3412.

For convenience, we use both the sequence and word notation for permutations, so for example $(3, 1, 4, 2)$ and 3142 correspond to the same permutation in S_4. We also use $s_i = (i, i+1)$ to denote simple transpositions swapping i and $i+1$, and $t_{ij} = (i, j)$ to denote general transpositions swapping i and j, where $i < j$. We hope this does not lead to confusion.

For a permutation $w \in S_n$, let $\ell(w) := |\{(i,j) : 1 \le i < j \le n,\, w(i) > w(j)\}|$ denote the *number of inversions* in w. For permutations $u, v \in S_n$, we write $u \lessdot v$ if $v = ut_{ij}$ for some $i < j$, and $\ell(v) = \ell(u) + 1$. This is the *cover relation* for the *strong Bruhat order*, which can now be defined by transitivity.

Finally, let $\mathrm{Des}(w) := \{i : w(i) > w(i+1)\}$ denote the *set of descents* in w. A permutation $w \in S_n$ is *Grassmannian* if it has at most one descent.

2.2. **Schubert coefficients.** Below we give a brief reminder of few basic results on Schubert polynomials. We refer to [Knu16, Mac91, Man01] for standard introductions to combinatorial aspects, and to [AF24, Ful97] for geometric aspects.

Schubert polynomials give a graded $\mathbb{Z}$-linear basis of polynomials $\mathbb{Z}[x_1, x_2, \ldots]$, which we define recursively. Let $w_\circ = (n, n-1, \ldots, 1) \in S_n$. Define

$$\mathfrak{S}_{w_\circ}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}\,, \quad \text{and}$$
$$\mathfrak{S}_{w}(x_1, \ldots, x_n) := \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n) \quad \text{if} \quad w(i) < w(i+1),$$

where

$$\partial_i f \ := \ \frac{f - s_i f}{x_i - x_{i+1}}\,.$$

Note that under the inclusion $\iota : S_n \hookrightarrow S_{n+1}$ we have $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. This allows us to consider $\mathfrak{S}_w$ for each $w \in S_\infty$.

As Schubert polynomials $\{\mathfrak{S}_w\}$ form a polynomial basis, *Schubert coefficients* (also called *Schubert structure coefficients*) $\{c^w_{u,v}\}$ are defined by (1.3). Although $c^w_{u,v} \in \mathbb{N}$, they have no known combinatorial interpretation in full generality. However, such an interpretation is known when v is a simple transposition:

Proposition 2.1 (*Monk's rule* [Man01, §2.7.1])**.** *Let $u \in S_n$ and let $1 \le k \le n-1$. Then:*

$$\mathfrak{S}_u \cdot \mathfrak{S}_{s_k} \ = \sum_{\substack{i \le k < j \\ u \lessdot ut_{ij}}} \mathfrak{S}_{ut_{ij}}\,.$$

By their definition, Schubert polynomials $\mathfrak{S}_\rho$ are homogeneous of degree $\ell(\rho)$. Thus we have:

Proposition 2.2 (*dimension condition*)**.** *Suppose $\ell(w) \neq \ell(u) + \ell(v)$. Then $c^w_{u,v} = 0$.*

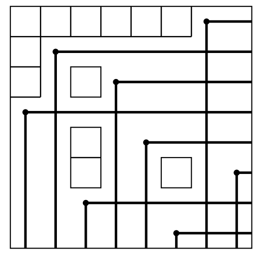
2.3. **Rothe diagrams.** For a permutation $w \in S_n$, the *Rothe diagram* is defined as

$$D(w) \ := \ \big\{(i,j) : 1 \le i, j \le n,\, j < w(i),\, i < w^{-1}(j)\big\} \ \subset \mathbb{N}^2.$$

Note that $|D(w)| = \ell(w)$ is the number of inversions in w. Depending on the context, we refer to elements of $D(w)$ as *squares* or *boxes*.

For a permutation $w \in S_\infty$, its *Lehmer code* is the vector $\mathsf{code}(w) = (c_1, c_2, \ldots)$, where c_i is the number of boxes in row i of $D(w)$. We shorten this to "code" when the context is clear. For the identity permutation $\mathsf{id} \in S_\infty$, the code is the all zero sequence. Note that $\mathsf{code} : S_\infty \to \mathbb{K}$ is a bijection, i.e. $\mathsf{code}(w)$ uniquely determines $w \in S_\infty$ and code^{-1} is well defined.

Example 2.3. For a permutation $w = 72415836 \in S_8$, we have $\mathsf{code}(w) = (6, 1, 2, 0, 1, 2)$ and $\ell(w) = 12$. In this case $n = 8$, and the corresponding Rothe diagram $D(w)$ is shown below. Here the squares are in positions $(i,j) \in D(w)$, and the dots are in positions $(i, w(i))$, $1 \le i \le n$.



3. Code scaling

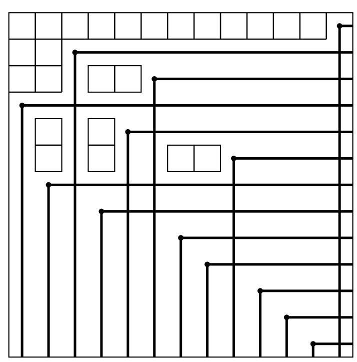
3.1. **Preliminaries.** Let $w \in S_n$ be a permutation where $\mathsf{code}(w) = (c_1, c_2, \dots, c_n)$. Recall that for a given integer $N \geq 2$, the *code scaling* $N * w$ is the unique permutation with the code $(Nc_1, Nc_2, \dots, Nc_n)$.

Observe that $\ell(N * w) = N\ell(w)$. Thus, $\ell(u) + \ell(v) = \ell(w)$ if and only if $\ell(N * u) + \ell(N * v) = \ell(N * w)$ for any $N > 1$. Additionally, it is clear by construction that code scaling preserves the underlying descent sets of the permutations: $\mathrm{Des}(N * w) = \mathrm{Des}(w)$.

Example 3.1. Take $w = (7, 2, 4, 1, 5, 8, 3, 6)$ with $\mathsf{code}(w) = (6, 1, 2, 0, 1, 2)$ as in Example 2.3. Let $N = 2$. By definition,

$$2 * w \ = \ \mathsf{code}^{-1}(12, 2, 4, 0, 2, 4) \ = \ (13, 3, 6, 1, 5, 9, 2, 4, 7, 8, 10, 11, 12).$$

We then have $\ell(2 * w) = 2\ell(w) = 24$ and $\mathrm{Des}(2 * w) = \mathrm{Des}(w) = \{1, 3, 6\}$. The corresponding Rothe diagram $D(2 * w)$ is the following:



3.2. **St. Dizier–Yong vanishing condition.** For permutations $u, v, w \in S_n$, let $\mathsf{code}(w) = (c_1, \dots, c_n)$. Consider integer fillings of boxes in $D(u) \cup D(v)$ with entries in $[n]$. We view $D(u) \cup D(v)$ as a subset of $[n] \times [2n]$, where $D(v)$ is shifted right horizontally by n units to the right of $D(u)$.

In [ARY21] Adve, Yong, and the second author define an *indicator tableau* to be an integer filling $T : D(u) \cup D(v) \to [n]$, such that

(i) the number of i's in T is equal to $c_i = \mathsf{code}(w)_i$, for each $i \in [n]$,
(ii) each column of T strictly increases from top to bottom, and
(iii) if an entry m appears in row r of T, then $m \leq r$.

Denote by $\mathsf{Tab}^w_{u,v}$ to be the set of such indicator tableaux. The *St. Dizier–Yong vanishing condition* is the following:

Lemma 3.2 (St. Dizier–Yong [SY22, Thm B, §4.3])**.** *We have:*

$$\mathsf{Tab}^w_{u,v} \ = \ \varnothing \quad \Longrightarrow \quad c^w_{u,v} \ = \ 0.$$

3.3. **Proof of Theorem 1.2.** For the first part, the equality $c^w_{u,v} = 1$ follows directly from Monk's rule (Proposition 2.1) and the construction of permutations u, v, w.

For the second part, let us show that the set of indicator tableaux $\mathsf{Tab}^{N*w}_{N*u,N*v}$ is empty for each $N > 1$. The result follows immediately by Lemma 3.2.

Since $u \lessdot ut_{ij} = w$, by the definition of the Rothe diagram, we have:

$$D(w) \ = \ D(u) \ - \ \big\{(j,c) \ : \ c \in [u(i)+1, u(j)-1]\big\} \ \cup \ \big\{(i,c) \ : \ c \in [u(i), u(j)-1]\big\}.$$

In particular, we have

$$\mathsf{code}(w) \ = \ \mathsf{code}(u) + (u(j) - u(i))\mathsf{e}_i \ - \ (u(j) - u(i) - 1)\mathsf{e}_j \,,$$

where $\mathsf{e}_k = (0, \ldots, 1, \ldots, 0)$ is the k-th standard basis vector. By definition of the code scaling, we then have:

$$\mathsf{code}(N * w) \ = \ \mathsf{code}(N * u) \ + \ N(u(j) - u(i))\mathsf{e}_i \ - \ N(u(j) - u(i) - 1)\mathsf{e}_j \,. \tag{3.1}$$

Let $D := D(N * u) \cup D(N * v)$. Suppose there exists an indicator tableau

$$T \in \mathsf{Tab}^{N*w}_{N*u,N*v} \tag{3.2}$$

which by definition is a filling of D. Note that for every $r \in [n]$, tableau T contains $\mathsf{code}(N * w)_r$ many r's by assumption. Note also that every such entry of r must appear in row r or below.

Claim: For all $(r, c) \in D$, we have:

$$T(r, c) \in \{i, j\} \quad \text{if} \quad r = j, \quad \text{and} \quad T(r, c) = r \quad \text{otherwise.}$$

Proof. By (3.1), we have $\mathsf{code}(N * w)_r = \mathsf{code}(N * u)_r$ for all $r < i$. This forces $T(r, c) = r$ for all $(r, c) \in D$. Similarly, for $r > j$, we have $\mathsf{code}(N * w)_r = \mathsf{code}(N * u)_r$. Thus again, we have $T(r, c) = r$ for all $(r, c) \in D$.

For the case of $r = i$, Equation (3.1) gives:

$$\mathsf{code}(N * w)_i \ = \ \mathsf{code}(N * u)_i \ + \ N(u(j) - u(i)).$$

Similarly, this implies that $T(i, c) = i$ for all $(i, c) \in D$. Note that $D(N * v)$ contains N boxes in row i and no boxes elsewhere. This leaves

$$\big(N\mathsf{code}(u)_i + N(u(j) - u(i))\big) \ - \ \big(N\mathsf{code}(u)_i + N\mathsf{code}(v)_i\big) \ = \ N(u(j) - u(i) - 1) \ > \ 0$$

many i's left to place in T.

For the case $i < r < j$, we again have $\mathsf{code}(N * w)_r = \mathsf{code}(N * u)_r$. Since $u \lessdot ut_{ij} = w$, it follows that any $(r, c) \in D$ lies below of some square $(i, c) \in D$. Since there can be no repetition of entries in columns of T, this gives $T(r, c) \neq i$. Then similarly as in the previous cases, this forces $T(r, c) = r$ for $(r, c) \in D$. Finally, for the case $r = j$, this leaves $T(j, c) \in \{i, j\}$ for all $(j, c) \in D$. □

Now we focus on the entries in row j of T. By the Claim, this row must contain the remaining $N(u(j) - u(i) - 1)$ entries i. Since the entries in columns of T must be distinct, and row i contains all i's, these $N(u(j) - u(i) - 1)$ many i's cannot appear below a square from row i.

Recall that row j of D contains $\mathsf{code}(N * u)_j = N\mathsf{code}(u)_j$ many squares. Additionally, note that since $u \lessdot ut_{ij} = w$, we know that if $i < m < j$, then $u(m) \notin [u(i), u(j)]$. Combining these facts with the definition of the Rothe diagram, it follows that D has

$$\mathsf{code}(N * u)_i - (j - i - 1) \ = \ N\mathsf{code}(u)_i - (j - i - 1)$$

many squares in row j of D directly below boxes in row i of D. Summarizing, since T must have distinct column entries, we have:

$$N\mathsf{code}(u)_j \ - \ (N\mathsf{code}(u)_i - (j - i - 1)) \ \geq \ N(u(j) - u(i) - 1). \tag{3.3}$$

By (3.1) and the fact that $u \lessdot ut_{ij} = w$, we have:

$$\mathsf{code}(u)_j \;=\; (\mathsf{code}(u)_i - (j-i-1)) \,+\, (u(j) - u(i) - 1).$$

Applying this to the left-hand side of Equation (3.3) gives

$$N\mathsf{code}(u)_j \,-\, \big(N\mathsf{code}(u)_i - (j-i-1)\big) \;=\; \big(N(u(j)-u(i)-1)\big) \,-\, (N-1)(j-i-1). \tag{3.4}$$

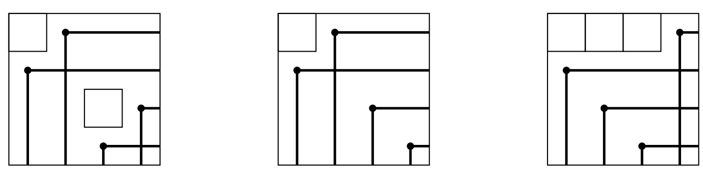
Combining Equations (3.3) and (3.4), we conclude:

$$0 \;\geq\; (N-1)(j-i-1),$$

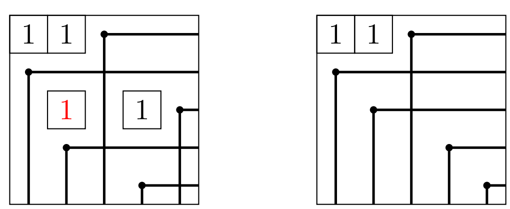
a contradiction with assumptions $j - i \geq 2$ and $N > 1$. We conclude that there is no T as in (3.2), i.e. $\mathsf{Tab}^{N*w}_{N*u,N*v} = \varnothing$. The result now follows by Lemma 3.2. □

3.4. **Examples and special cases.** We start with the simplest example which both illustrates the proof above and motivates other special cases.

Example 3.3. Take $u = 2143$, $v = s_1 = 2134$ and $w = ut_{13} = 4123$. Note that $\mathsf{code}(u) = (1,0,1)$, $\mathsf{code}(v) = (1)$, $\mathsf{code}(w) = (3)$, and that $u \lessdot w$. By Monk's rule, we have $c^w_{u,v} = 1$.



Observe that $2*u = 31524$, $2*v = 21345 = v$, and $2*w = 7123456$. Since $\mathsf{code}(2*w) = (6)$, there is a unique way to fill $D = D(2*u) \cup D(2*v)$ with 1's.



By the definition of indicator tableaux, they must be increasing in columns, ruling out the filling above. Thus $\mathsf{Tab}^{2*w}_{2*u,2*v} = \varnothing$. By Lemma 3.2, we have $c^{2*w}_{2*u,2*v} = 0$, giving a counterexample to the saturation property (1.4).

The following corollary follows immediately from Theorem 1.2 by taking $j = i+2$.

Corollary 3.4. *Let $u \in S_n$ such that $u(i+1) < u(i) < u(i+2) - 1$. Let $v = (i, i+1)$ and let $w = u \cdot (i, i+2)$. Then:*

$$c^w_{u,v} \;=\; 1 \qquad \textit{and} \qquad c^{N*w}_{N*u,N*v} \;=\; 0 \;\; \textit{for all} \;\; N > 1. \tag{3.5}$$

Note that Corollary 1.3 is a special case of Corollary 3.4. Note also that Example 3.3 is a special case of Corollary 1.3 when $n = 4$.

4. Bit scaling

4.1. **Preliminaries.** We now introduce a new scaling operation on permutations, which we define in terms of Rothe diagrams.

Let $w \in S_n$ be a permutation with maximal descent d where $d \le k$ for fixed k. For $j \in [n]$ let $C_j \subset D(w)$ denote the jth column of the Rothe diagram. Then take $S(w)$ to be the set of pairs (i, j) such that $i > k$ and $(i, j) \in C_j$. Denote by $J(w) \subset [n]$ the set of indices j of such columns C_j. We call these *shaded columns.*

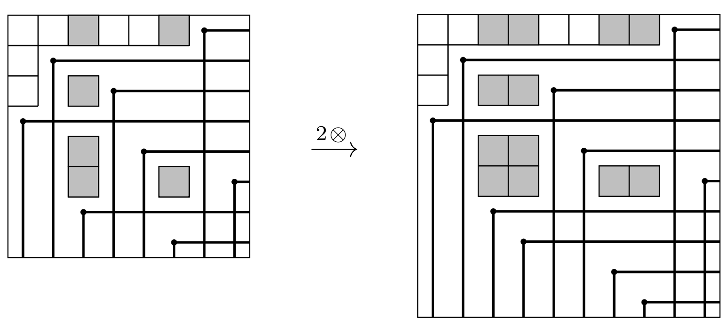
For an integer $N \ge 2$, define $N \otimes D(w)$ to be the set of squares where each shaded column C_j, $j \in J(w)$, is replaced with N copies of C_j. Denote by $N \otimes w$ the unique permutation in S_∞ with $D(N \otimes w) = N \otimes D(w)$. Permutation $N \otimes w$ is called the *bit scaling* of w with respect to k.

Alternatively, one can define $N \otimes w$ by taking the prefix $w(1) \cdots w(k)$ and meshing it with the *Kronecker product* of the suffix $w(k+1) \cdots w(n)$ with the identity permutation of size N, see Remark 4.2 below.

Finally, we note that bit scaling preserves the set of descents, i.e. $\mathrm{Des}(N \otimes w) = \mathrm{Des}(w)$. Moreover, for a Grassmannian permutation w corresponding to the partition $\lambda = \lambda(w)$, the permutation $N \otimes w$ is also Grassmannian, corresponding to the partition $N\lambda(w)$; we omit the details.

Example 4.1. Take $w = (7, 2, 4, 1, 5, 8, 3, 6)$ as in Example 2.3, with $n = 8$, $\mathrm{Des}(w) = \{1, 3, 6\}$ and maximal descent $d = 6$. Set $k = d$. We have $S(w) = \{(7, 3), (8, 6)\}$, so $J(w) = \{3, 6\}$. Below the shaded columns are $C_3 = \{(1, 3), (3, 3), (5, 3), (6, 3)\}$ and $C_6 = \{(1, 6), (6, 6)\}$.

Now let $N = 2$. Doubling the shaded columns gives the Rothe diagram $D(2 \otimes w) = 2 \otimes D(w)$ shown below. Thus we have in this case $2 \otimes w = (9, 2, 5, 1, 6, 10, 3, 4, 7, 8)$.



Remark 4.2. In the *Kronecker product of permutations*, one permutation is viewed as a block matrix, with all blocks given by the second permutation. For example, the Kronecker product of $(1, 3, 2, 4)$ and $(1, 2, 3)$ is given by $(1, 2, 3, 7, 8, 9, 4, 5, 6, 10, 11, 12)$. The bit scaling can be viewed as a similar operation, where each element in $S(w)$ is replaced with an identity permutation of size N.

4.2. **Underlying motivation for this interpretation.** Suppose $w \in S_n$ such that $\mathrm{Des}(w) \subset \{k_1 < \ldots < k_\ell\}$. Set $k_0 := 0$ and $k_{\ell+1} := n$. Let $\delta = (\delta_1, \ldots, \delta_n)$, such that $\delta_i = m$ when $k_m < i \le k_{m+1}$. Then we may equivalently encode w as a sequence $\tau = (\tau_1, \ldots, \tau_n) \in \{0, \ldots, \ell\}^n$, where $\tau_i := \delta_{w(i)}$. We denote this as $\mathsf{seq}(w)$, given with respect to $\{k_1 < \ldots < k_\ell\}$. Note that viewing $w \in S_\infty$ will result in infinitely many trailing ℓ's in $\mathsf{seq}(w)$. Then $\mathsf{seq}(N \otimes w)$ can be constructed from $\mathsf{seq}(w)$ by replacing each ℓ with N copies of ℓ.

Example 4.3. Return to $w = (7, 2, 4, 1, 5, 8, 3, 6)$ as in Example 4.1. Then $\mathrm{Des}(w) = \{1 < 3 < 6\}$, so $\ell = 3$. In this case, since $n = 8$ we have $\delta = 01122233$. Then $\mathsf{seq}(w) = 21312302$, which gives $\mathsf{seq}(2 \otimes w) = 2133123302$. This corresponds to $2 \otimes w = (9, 2, 5, 1, 6, 10, 3, 4, 7, 8)$, just as above.

4.3. **Saturation property under bit scaling is false.** Once the bit scaling operation is defined, it is natural to ask if it satisfies the saturation property:

$$c^{w}_{u,v} > 0 \quad\Longleftrightarrow\quad c^{N\otimes w}_{N\otimes u,\, N\otimes v} > 0 \ \ \textit{for any} \ \ N \geq 1\,. \tag{4.1}$$

If true, this would give a far-reaching extension of the usual saturation property (1.1) for LR coefficients to general Schubert coefficients. The following result shows that (4.1) fails on a large family of examples.

Theorem 4.4. *Let $u \in S_n$ and let $1 \leq i < d < j \leq n$ where d is the maximal descent in u. Suppose also that $u \lessdot ut_{ij}$. Finally, let $v = s_i$ and $w = ut_{ij}$. Set $k = d$. Then:*

$$c^{w}_{u,v} = 1 \qquad \textit{and} \qquad c^{N\otimes w}_{N\otimes u,\, N\otimes v} = 0 \quad \textit{for all} \ \ N > 1.$$

In particular, for $d = n-1$ and $j = n$, we have:

Corollary 4.5. *Let $u \in S_n$ be a permutation with $u(i) = n-2$, $u(n-1) = n$, $u(n) = n-1$, where $i < n-2$. Let $v = (i, i+1)$ and $w = u \cdot (i, n)$. Then:*

$$c^{w}_{u,v} = 1 \qquad \textit{and} \qquad c^{N\otimes w}_{N\otimes u,\, N\otimes v} = 0 \quad \textit{for all} \ \ N > 1.$$

Even more explicitly, by analogy with Corollary 1.3, we have:

Corollary 4.6. *For all $n \geq 4$, let $v := (n-2, n-3)$ be a simple transposition S_n, and let*

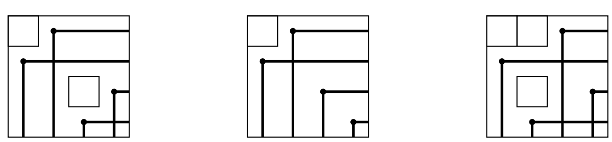
$$u := (2, 3, \ldots, n-2, 1, n, n-1)\,, \qquad w := (2, 3, \ldots, n-1, 1, n, n-2).$$

Then:

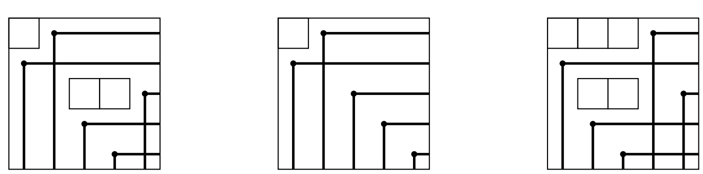
$$c^{w}_{u,v} = 1 \qquad \textit{and} \qquad c^{N\otimes w}_{N\otimes u,\, N\otimes v} = 0 \quad \textit{for all} \ \ N > 1.$$

Similar to the Corollary 1.3, this shows that the bit scaling saturation (4.1) fails for all $n \geq 4$ and all $N \geq 2$. Moreover, it fails already for permutations with at most two descents. We conclude with a small example which both illustrates the proof of Theorem 4.4 and motivates other special cases.

Example 4.7. Take $u = 2143$, $v = s_1 = 2134$ and $w = ut_{14} = 3142$. Note that $u \lessdot w$. By Monk's rule, we have $c^{w}_{u,v} = 1$. Below we show Rothe diagrams of u, v, w, from left to right:



We then have $\mathrm{Des}(v) \subset \mathrm{Des}(u) = \mathrm{Des}(w) = \{1, 3\}$, and maximal descent $d = 3$. Then we have $2 \otimes u = 21534$, $2 \otimes v = 21345 = v$, and $2 \otimes w = 41523$. Below we show Rothe diagrams of $2 \otimes u$, $2 \otimes v$, $2 \otimes w$, from left to right:



Counting squares in these Rothe diagrams, observe that $\ell(2\otimes u) = 3$, $\ell(2\otimes v) = 1$ and $\ell(2\otimes w) = 5$. Thus, by Proposition 2.2, we have $c^{2\otimes w}_{2\otimes u, 2\otimes v} = 0$, giving a counterexample to the saturation property (4.1).

4.4. **Proof of Theorem 4.4.** For the first part, the equality $c^w_{u,v} = 1$ follows directly from Monk's rule (Proposition 2.1) and the construction of permutations u, v, w.

For the second part, let us show that

$$\ell(N \otimes u) \, + \, \ell(N \otimes u) \, < \, \ell(N \otimes w) \quad \text{for all} \quad N \geq 2. \tag{4.2}$$

By Proposition 2.2, the result follows.

Since $v = s_i$, we have $D(v) = \{(i,i)\}$. As illustrated by the Example 4.7 above, we have $N \otimes v = v$. In particular, we have $\ell(N \otimes v) = \ell(v) = 1$.

Since $u \lessdot ut_{ij} = w$, by the definition of the Rothe diagram, we have:

$$D(w) \, = \, D(u) \, - \, \big\{(r, u(j)) \, : \, i < r\big\} \cup \big\{(r, u(i)) \, : \, (r, u(j)) \in D(u)\big\} \, + \, (i, u(i)).$$

Denote by $(C_1, \ldots, C_n)$ and $(C'_1, \ldots, C'_n)$ the columns of $D(u)$ and $D(w)$, respectively. Then we have:

$$C'_h = \begin{cases} C_{u(j)} + (i,h) & \text{if} \quad h = u(i), \\ C_{u(i)} & \text{if} \quad h = u(j), \\ C_h & \text{otherwise.} \end{cases} \tag{4.3}$$

By assumption, $u(j) \in J(u)$ and $u(i) \notin J(u)$. Then we have $J(w) = J(u) - u(j) + u(i)$. By (4.3) and the definition of bit scaling, we have:

$$\begin{aligned}
|D(N \otimes w)| \, &= \, \sum_{h \notin J(w)} |C'_h| \, + \, \sum_{h \in J(w)} N \cdot |C'_h| \\
&= \, |C'_{u(j)}| \, + \, N \cdot |C'_{u(i)}| \, + \, \sum_{h \notin J(w) - u(j)} |C'_h| \, + \, \sum_{h \in J(w) - u(i)} N \cdot |C'_h| \\
&= \, |C_{u(i)}| \, + \, N \cdot \big(|C_{u(j)}| + 1\big) \, + \, \sum_{h \notin J(w) - u(j)} |C_h| \, + \, \sum_{h \in J(w) - u(i)} N \cdot |C_h| \\
&= \, |C_{u(i)}| \, + \, N \cdot \big(|C_{u(j)}| + 1\big) \, + \, \sum_{h \notin J(u) - u(i)} |C_h| \, + \, \sum_{h \in J(u) - u(j)} N \cdot |C_h| \\
&= \, N \, + \, \sum_{h \notin J(u)} |C_h| \, + \, \sum_{h \in J(u)} N \cdot |C_h| \\
&= \, N \, + \, |D(N \otimes u)|.
\end{aligned}$$

Since $N > 1$ and $\ell(N \otimes v) = 1$, we conclude:

$$\begin{aligned}
\ell(N \otimes w) \, &= \, |D(N \otimes w)| \, = \, |D(N \otimes u)| \, + \, N \, = \, \ell(N \otimes u) \, + \, N \\
&> \, \ell(N \otimes u) \, + \, 1 \, = \, \ell(N \otimes u) \, + \, \ell(N \otimes v).
\end{aligned}$$

This gives (4.2) and completes the proof. □

5. Final remarks

5.1. Our own motivation to study the saturation property for Schubert coefficients lies in connection to the *Schubert vanishing problem* $\big[c^w_{u,v} =^? 0\big]$. The idea here is to extend the approach in [DM06, MNS12] to the *LR vanishing problem* $\big[c^\lambda_{\mu\nu} =^? 0\big]$. There, the authors independently observed[1] that the saturation property (1.1) implies that the vanishing of LR–coefficients can be solved by a linear program (LP). This gives a deterministic poly-time algorithm for deciding LR vanishing.

There are two main ingredients in the approach above, both nontrivial. First, one needs a combinatorial interpretation of the LR coefficient $c^\lambda_{\mu\nu}$ to show that it counts the number of integer points in a convex polytope $Q_{\lambda\mu\nu}$ defined by integer constraints. Second, one needs a saturation

[1] The original preprints appeared on the arXiv in January 2005, within a day of each other.

property to reduce the LR vanishing problem to $Q_{\lambda\mu\nu}$ containing a *rational* point. One then applies known results that LP is in P to conclude the same for the LR vanishing.

In our most recent paper [PR25] capping a series of weaker results, we show that the vanishing of Schubert coefficients $\big[c^u_{vw} =^? 0\big]$ is in $\mathsf{coRP} \subseteq \mathsf{BPP}$, i.e. can be decided in probabilistic polynomial time with a one-sided error (in the case of a positive answer). This is so low in the polynomial hierarchy that it suggests a possibility that Schubert vanishing might be in P, at least in some cases.

Since the reduction to LP is really the only approach that we know (short of derandomization), we would need the two ingredients described above. As mentioned in the introduction, there are at least two different combinatorial interpretations of 2-*step Schubert coefficients*, which correspond to permutations with at most two descents [Cos09, BKPT16]. While we are not aware how to restate either of these combinatorial interpretations in terms of the number of integer points in polytopes, this case is a natural place to start.

Question 5.1. *Can the vanishing problem for* 2*-step Schubert coefficients be decided in poly-time?*

Unfortunately, our Corollaries 1.3 and 4.6 imply that in the 2-step case, the natural saturation properties fail. The question remains wide open, unlikely to be resolved by the existing tools.

5.2. In connection to Question 5.1, it is worth comparing how the failure of saturation properties for other algebraic combinatorics constants discussed in the introduction relate to the complexity of the corresponding vanishing problems.

For example, it remains open whether the vanishing of the *Clebsch–Gordan* (CG) *coefficients* (particular generalizations of LR coefficients to other root systems) can be decided in poly-time. The problem is resolved for the even weights by a combination of results in [DM06] and [KM08]. It is known that the Ehrhart positivity conjectures in [KTT04] would imply that *CG vanishing* is in P in full generality. Unfortunately, these conjectures remain wide open even in the special case of Kostka numbers, see e.g. [Ale19].

In some cases, there are known computational complexity obstacles for the vanishing problem, immediately invalidating a saturation property approach as above. Notably, it is known that the vanishing problem for the Kronecker coefficients is NP-hard [IMW17]. Later, in [PP20], it was shown that the vanishing of reduced Kronecker coefficients is also NP-hard. By itself, the NP-hardness of vanishing does not automatically imply that the saturation property fails, but it does suggest a more involved underlying structure in the problem.

5.3. There are other possible scaling operations which can be defined on permutations, i.e. by replacing each box of the Rothe diagram with an $N \times N$ square of boxes. Unfortunately, code scaling and bit scaling are the only operations we know that preserve the descents. This is a necessary condition to ensure that the operation applied to Grassmannian permutations corresponds to the usual partition multiplication.

Acknowledgements. We thank Sara Billey, Jesús De Loera, Pavel Galashin, Oliver Pechenik, Frank Sottile, Alex Yong, Sylvester Zhang and Paul Zinn-Justin for interesting discussions and helpful comments. The first author was partially supported by the NSF grant CCF-2302173. The second author was partially supported by the NSF MSPRF grant DMS-2302279.

References

[ARY19] Anshul Adve, Colleen Robichaux and Alexander Yong, Vanishing of Littlewood–Richardson polynomials is in P, *Comput. Complexity* **28** (2019), 241–257.

[ARY21] Anshul Adve, Colleen Robichaux and Alexander Yong, An efficient algorithm for deciding vanishing of Schubert polynomial coefficients, *Adv. Math.* **383** (2021), Paper No. 107669, 38 pp.; extended abstract in *Proc. 31st FPSAC* (2020), Art. 52, 12 pp.

[Ale19] Per Alexandersson, Polytopes and large counterexamples, *Exp. Math.* **28** (2019), 115–120.

[AF24] David Anderson and William Fulton, *Equivariant cohomology in algebraic geometry*, Cambridge Univ. Press, Cambridge, UK, 2024, 446 pp.

[ARY13] David Anderson, Edward Richmond and Alexander Yong, Eigenvalues of Hermitian matrices and equivariant cohomology of Grassmannians, *Compos. Math.* **149** (2013), 1569–1582.

[Bel06] Prakash Belkale, Geometric proofs of Horn and saturation conjectures, *J. Algebraic Geom.* **15** (2006), 133–173.

[Bel08] Prakash Belkale, Quantum generalization of the Horn conjecture, *Jour. AMS* **21** (2008), 365–408.

[BK10] Prakash Belkale and Shrawan Kumar, Eigencone, saturation and Horn problems for symplectic and odd orthogonal groups, *J. Algebraic Geom.* **19** (2010), 199–242.

[BKR12] Prakash Belkale, Shrawan Kumar and Nicolas Ressayre, A generalization of Fulton's conjecture for arbitrary groups, *Math. Ann.* **354** (2012), 401–425.

[BVW17] Nicole Berline, Michèle Vergne and Michael Walter, The Horn inequalities from a geometric point of view, *Enseign. Math.* **63** (2017), 403–470.

[BOR09] Emmanuel Briand, Rosa Orellana and Mercedes Rosas, Reduced Kronecker coefficients and counterexamples to Mulmuley's strong saturation conjecture SH (with an appendix by K. Mulmuley), *Comp. Complexity* **18** (2009), 577–600.

[Buch00] Anders S. Buch, The saturation conjecture (after A. Knutson and T. Tao). With an appendix by William Fulton, *Enseign. Math.* **46** (2000), 43–60.

[Buch02] Anders S. Buch, A Littlewood–Richardson rule for the K-theory of Grassmannians, *Acta Math.* **189** (2002), 37–78.

[BKPT16] Anders S. Buch, Andrew Kresch, Kevin Purbhoo and Harry Tamvakis, The puzzle conjecture for the cohomology of two-step flag manifolds, *J. Algebraic Combin.* **44** (2016), 973–1007.

[CR23] Pierre-Emmanuel Chaput and Nicolas Ressayre, Reduction for branching multiplicities, *Int. Math. Res. Not. IMRN* **2023** (2023), 15207–15265.

[Cos09] Izzet Coskun, A Littlewood–Richardson rule for two-step flag varieties, *Invent. Math.* **176** (2009), 325–395.

[DM06] Jesús A. De Loera and Tyrrell B. McAllister, On the computation of Clebsch–Gordan coefficients and the dilation effect, *Experiment. Math.* **15** (2006), 7–19.

[DW00] Harm Derksen and Jerzy Weyman, Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients, *Jour. AMS* **13** (2000), 467–479.

[Èla92] Alexander G. Èlashvili, Invariant algebras, in *Lie groups, their discrete subgroups, and invariant theory*, AMS, Providence, RI, 1992, 57–64.

[Ful97] William Fulton, *Young tableaux*, Cambridge Univ. Press, Cambridge, UK, 1997, 260 pp.

[Ful00] William Fulton, Eigenvalues, invariant factors, highest weights, and Schubert calculus, *Bull. AMS* **37** (2000), 209–249.

[GOY21] Shiliang Gao, Gidon Orelowitz and Alexander Yong, Newell–Littlewood numbers, *Trans. AMS* **374** (2021), 6331–6366.

[IMW17] Christian Ikenmeyer, Ketan D. Mulmuley and Michael Walter, On vanishing of Kronecker coefficients, *Comp. Complexity* **26** (2017), 949–992.

[KM08] Michael Kapovich and John J. Millson, A path model for geodesics in Euclidean buildings and its applications to representation theory, *Groups Geom. Dyn.* **2** (2008), 405–480.

[KTT04] Ronald C. King, Christophe Tollu and Frédéric Toumazet, Stretched Littlewood–Richardson and Kostka coefficients, in *Symmetry in physics*, AMS, Providence, RI, 2004, 99–112.

[Kir04] Anatol N. Kirillov, An invitation to the generalized saturation conjecture, *Publ. RIMS* **40** (2004), 1147–1239.

[Kly04] Alexander Klyachko, Quantum marginal problem and representations of the symmetric group, preprint (2004), 47 pp.; `arXiv:quant-ph/0409113`.

[Knu16] Allen Knutson, Schubert calculus and puzzles, in *Adv. Stud. Pure Math.* **71**, Math. Soc. Japan, Tokyo, 2016, 185–209.

[Knu22] Allen Knutson, Schubert calculus and quiver varieties, in *Proc. ICM* (2022, virtual), Vol. VI, EMS Press, 4582–4605.

[KT99] Allen Knutson and Terence Tao, The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products I: Proof of the saturation conjecture, *Jour. AMS* **12** (1999), 1055–1090.

[KT03] Allen Knutson and Terence Tao, Puzzles and (equivariant) cohomology of Grassmannians, *Duke Math. J.* **119** (2003), 221–260.

[KTW04] Allen Knutson, Terence Tao, and Christopher Woodward. The honeycomb model of $GL_n(\mathbb{C})$ tensor products II: Puzzles determine facets of the Littlewood-Richardson cone. *Jour. AMS*, 17(1), (2004) 19–48.

[Kum14] Shrawan Kumar, A survey of the additive eigenvalue problem. With an appendix by M. Kapovich, *Transform. Groups* **19** (2014), 1051–1148.

[Mac91] Ian G. Macdonald, *Notes on Schubert polynomials*, Publ. LaCIM, UQAM, Montreal, 1991, 116 pp.; available at tinyurl.com/382f7an7

[Mac95] Ian G. Macdonald, *Symmetric functions and Hall polynomials* (Second ed.), Oxford U. Press, New York, 1995, 475 pp.

[Man01] Laurent Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*, SMF/AMS, Providence, RI, 2001, 167 pp.

[Min24] Jaewon Min, Proof of the Newell–Littlewood saturation conjecture, preprint (2024), 50 pp.; `arXiv:2409.00233`.

[Mul09] Ketan D. Mulmuley, Geometric Complexity Theory VI: the flip via saturated and positive integer programming in representation theory and algebraic geometry, preprint (2009, v4), 139 pp.; `arXiv: 0704.0229`.

[MNS12] Ketan D. Mulmuley, Hariharan Narayanan and Milind Sohoni, Geometric complexity theory III. On deciding nonvanishing of a Littlewood–Richardson coefficient, *J. Algebraic Combin.* **36** (2012), 103–110.

[PP20] Igor Pak and Greta Panova, Breaking down the reduced Kronecker coefficients, *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 4, 463–468.

[PR25] Igor Pak and Colleen Robichaux, Vanishing of Schubert coefficients in probabilistic polynomial time, preprint (2025), 15 pp.; `arXiv:2509.16467`.

[RYY22] Colleen Robichaux, Harshit Yadav and Alexander Yong, Equivariant cohomology, Schubert calculus, and edge labeled tableaux, in *Facets of Algebraic Geometry*, Vol. II, Cambridge Univ. Press, Cambridge, UK, 2022, 284–335.

[Ros01] Mercedes H. Rosas, The Kronecker product of Schur functions indexed by two-row shapes or hook shapes, *J. Algebraic Combin.* **14** (2001), 153–173.

[SS16] Steven V. Sam and Andrew Snowden, Proof of Stembridge's conjecture on stability of Kronecker coefficients, *J. Algebraic Combin.* **43** (2016), 1–10.

[SY22] Avery St. Dizier and Alexander Yong, Generalized permutahedra and Schubert calculus, *Arnold Math. J.* **8** (2022), 517–533.

[Sta99] Richard P. Stanley, *Enumerative Combinatorics*, vol. 1 (Second ed.) and vol. 2, Cambridge Univ. Press, 2012 and 1999, 626 pp. and 581 pp.

[Zel99] Andrei Zelevinsky, Littlewood–Richardson semigroups, in *New perspectives in algebraic combinatorics*, Cambridge Univ. Press, Cambridge, UK, 1999, 337–345.