arXiv:1810.10361v1 [math.CO] 24 Oct 2018

COMPLEXITY, COMBINATORIAL POSITIVITY, AND NEWTON POLYTOPES

ANSHUL ADVE, COLLEEN ROBICHAUX, AND ALEXANDER YONG

ABSTRACT. The nonvanishing problem asks if a coefficient of a polynomial is nonzero. Many families of polynomials in algebraic combinatorics admit combinatorial counting rules and simultaneously enjoy having *saturated Newton polytopes* (SNP). Thereby, in amenable cases, nonvanishing is in the complexity class NP $\cap$ coNP of problems with "good characterizations". This suggests a new algebraic combinatorics viewpoint on complexity theory.

This paper focuses on the case of *Schubert polynomials*. These form a basis of all polynomials and appear in the study of cohomology rings of flag manifolds. We give a tableau criterion for nonvanishing, from which we deduce the first polynomial time algorithm. These results are obtained from new characterizations of the *Schubitope*, a generalization of the permutahedron defined for any subset of the $n \times n$ grid.

CONTENTS

Date: October 24, 2018.

1. INTRODUCTION

1.1. **Nonvanishing decision problems and SNP.** Algebraic combinatorics studies families of polynomials

$$F_{\diamondsuit} = \sum_{\alpha} c_{\alpha,\diamondsuit} x^{\alpha} = \sum_{s\in\mathcal{S}} \mathsf{wt}(s) \in \mathbb{Z}[x_1, x_2, \ldots, x_n],$$

each viewed as the multivariate generating series for some discrete set $\mathcal{S}$.

Example 1.1 (Schur polynomials). $F_{\diamondsuit} = s_{\lambda}$ is a *Schur polynomial*, where $\diamondsuit = \lambda$ is an integer partition. The classical definition is $s_{\lambda} = a_{\lambda+\delta}/a_{\delta}$ where $a_{\gamma} := \det(x_i^{\gamma_j})_{i,j=1}^{n}$ and $\delta = (n-1, n-2, \ldots, 2, 1, 0)$. This establishes symmetry of s_{λ}, but not that $c_{\alpha,\lambda} \in \mathbb{Z}_{\geq 0}$. $\mathcal{S}$ is the set of *semistandard Young tableaux* of shape λ. Here, $\mathsf{wt}(s) = \prod_i x_i^{\# i \in s}$. Schur polynomials are an important basis of the vector space of all symmetric polynomials.

Example 1.2 (Stanley's chromatic symmetric polynomial). Another symmetric polynomial is Stanley's *chromatic polynomial* $F_{\diamondsuit} = \chi_G$ [33]. This time $\diamondsuit = G = (V, E)$ is a simple graph, $\mathcal{S}$ is the set of proper n-colorings of G, i.e., functions $s : V \to \{1, 2, \ldots, n\}$ such that $s(i) \neq s(j)$ if $\{i, j\} \in E$, and $\mathsf{wt}(s) = \prod_i x_i^{\# s^{-1}(i)}$.

Example 1.3 (Schubert polynomials). The central example of this paper is non-symmetric. It is the family of *Schubert polynomials* $F_{\diamondsuit} = \mathfrak{S}_w$, a basis of all polynomials. Now, $\diamondsuit = w$ is a permutation. There are many choices for $\mathcal{S}$, such as the *reduced compatible sequences* of [3]. Definitions are given in Section 1.4.

Problem 1.4 (nonvanishing)**.** *What is the complexity of deciding $c_{\alpha,\diamondsuit} \neq 0$, as measured in the input size of α and $\diamondsuit$ (under the assumption that arithmetic operations take constant time)?*

In general, nonvanishing may be undecidable: fix $S \subseteq \mathbb{N}$ that is not recursively enumerable, and let $F_m = \sum_{i=1}^{m} c_{i,m} x^m$ with $c_{i,m} = 1$ if $i \in S$ and 0 otherwise. Such sets S exist because there are uncountably many subsets of $\mathbb{N}$, but only countably many algorithms. One can explicitly take S to be the set of halting Turing machines under some numerical encoding [36], or the set of Gödel encodings [12] of statements about $(\mathbb{N}, +, \times)$ provable in first-order Peano arithmetic. All this said, in our cases of interest, $c_{\alpha,\diamondsuit} \in \mathbb{Z}_{\geq 0}$ has *combinatorial positivity*: it is given by a counting rule that implies nonvanishing is in the class NP of problems with a polynomial time checkable certificate of a YES decision.

Evidently, nonvanishing concerns the *Newton polytope*,

$$\mathsf{Newton}(F_{\diamondsuit}) = \mathrm{conv}\{\alpha : c_{\alpha,\diamondsuit} \neq 0\} \subseteq \mathbb{R}^n.$$

C. Monical, N. Tokcan and the third author [27] showed that for many examples, $F_{\diamondsuit}$ has *saturated Newton polytope* (SNP), i.e., $\gamma \in \mathsf{Newton}(F_{\diamondsuit}) \cap \mathbb{Z}^n \iff c_{\gamma,\diamondsuit} \neq 0$. The relevance of SNP to Problem 1.4 is:

> SNP $\Rightarrow$ nonvanishing$(F_{\diamondsuit})$ is equivalent to checking membership of a lattice point in $\mathsf{Newton}(F_{\diamondsuit})$.

Example 1.5 (nonvanishing(s_{λ}) is in P). $\mathsf{Newton}(s_{\lambda})$ is the *λ-permutahedron* $\mathcal{P}_{\lambda}$, the convex hull of the S_n-orbit of $\lambda \in \mathbb{R}^n$. By symmetry one may assume α is a partition. Thus $c_{\alpha,\lambda}$ is the *Kostka coefficient*, and $c_{\alpha,\lambda} = 0$ if and only if $\alpha \leq \lambda$ in *dominance order*. So nonvanishing(s_{λ}) is in the class P of polynomial time problems.

Does the "niceness" of having combinatorial positivity and SNP transfer to complexity?

Question 1.6. *Under what conditions does combinatorial positivity and SNP of* $\{F_\diamondsuit\}$ *imply* $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{P}$*, or at least that* $\mathsf{nonvanishing}(F_\diamondsuit) \notin \mathsf{NP}$*-complete?*

On the other hand, χ_G is not generally SNP [27] and $\mathsf{nonvanishing}(\chi_G)$ is hard:

Example 1.7 (χ_G-nonvanishing is NP-complete). For χ_G, nonvanishing is clearly in NP. In fact, for $n \geq 3$ it is NP-complete. The n-coloring problem of deciding if a graph has a n-proper coloring is NP-complete for $n \geq 3$. Given an efficient oracle to solve $\mathsf{nonvanishing}(\chi_G)$, call it on each partition of $|V|$ with n parts to decide if there exists a proper n-coloring. This requires only $O(|V|^n)$ calls, so it is a polynomial reduction of n-coloring to $\mathsf{nonvanishing}(\chi_G)$.

1.2. **Context from theoretical computer science.** Examples 1.5 and 1.7 show that nonvanishing can achieve the extremes of NP. What about the non-extremes?

The class NP-intermediate consists of NP problems that are neither in P nor NP-complete. *Ladner's theorem* states that if P $\neq$ NP there exists an (artificial) NP-intermediate problem. Many natural problems from, e.g., algebra, number theory, game theory and combinatorics are *suspected* to be NP-intermediate. An example is the *Graph Isomorphism problem*.

The class coNP consists of problems whose complements are in NP, i.e., those with a polynomial time checkable certificate of a `NO` decision.

> SNP $\Rightarrow$ given a halfspace description of the Newton polytope, an inequality violation checkable in polynomial time gives a coNP certificate.

The above implication of SNP says that any solution $\{F_\diamondsuit\}$ to the following problem gives $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{NP} \cap \mathsf{coNP}$.

Problem 1.8. *For a combinatorially positive family of SNP polynomials* $\{F_\diamondsuit\}$*, determine efficient half space descriptions of* $\mathsf{Newton}(F_\diamondsuit)$*.*

The class NP $\cap$ coNP is intriguing. Membership of a problem in NP $\cap$ coNP sometimes foreshadows the harder proof that it is in P. For example, consider

$$\mathsf{primes} = \text{"is a positive integer } n \text{ prime?"}$$

Clearly, primes $\in$ coNP. In 1975, V. Pratt [30] showed primes $\in$ NP. It was about thirty years before the celebrated discovery of the *AKS primality test* of M. Agrawal, N. Kayal, and N. Saxena [2], establishing primes $\in$ P.

In retrospect, another example is the *linear programming* problem

$$\mathsf{LPfeasibility} = \text{"is } A\mathbf{x} = b, \mathbf{x} \geq 0 \text{ feasible?"}$$

Clearly LPfeasibility $\in$ NP. Membership in coNP is a consequence of *Farkas' Lemma* (1902), which is a foundation for LP duality, conjectured by J. von Neumann and proved by G. Dantzig in 1948 (cf. [5]). Yet, it was not until 1979, with L. Khachiyan's work on the *ellipsoid method* that LPfeasibility $\in$ P was proved; see, e.g., the textbook [31].

These examples suggest P $=$ NP $\cap$ coNP. However, one has *integer factorization*

$$\mathsf{factorization} = \text{"given } 1 < a < b \text{ does there exist a divisor } d \text{ of } b \text{ where } 1 \leq d \leq a\text{?"}$$

An NP certificate is a divisor. A coNP certificate is a prime factorization (verified using the AKS test). Most public key cryptography in use (such as RSA) relies on P $\neq$ NP $\cap$ coNP.

The debate $\mathsf{P} \stackrel{?}{=} \mathsf{NP} \cap \mathsf{coNP}$ may be rephrased as "are problems with good characterizations in P?". One wants new examples of members of $\mathsf{NP} \cap \mathsf{coNP}$ that are not known to be in P. If such examples are proved to be in P, this adds evidence for "=". Yet, relatively few examples are known. In addition to integer factorization, one has (decision) *Discrete Log*, *Stochastic Games* [4], *Parity Games* [16] and *Lattice Problems* [1]. (It is open whether *Graph Isomorphism* is in coNP.)

We now connect this discussion with Example 1.7.

Problem 1.9. *Does restricting to a subclass of graphs G where χ_G is SNP (or Schur positive) change the complexity of n-coloring?*

Conjecture 1.10 (R. P. Stanley [33])**.** *If G is claw-free (i.e., it contains no induced $K_{1,3}$ subgraph), then χ_G is Schur positive.*

Conjecture 1.11 (C. Monical [26])**.** *If χ_G is Schur positive, then it is SNP.*

Combining these two conjectures gives

Conjecture 1.12. *If G is claw-free then χ_G is SNP.*

If coNP contains a NP-complete problem then $\mathsf{NP} = \mathsf{coNP}$ [13], solving an open problem with "=".[1] Now, by [15], n-coloring claw-free graphs is NP-complete. Therefore:

> If Conjecture 1.12 holds, Problem 1.9 and Question 1.6 are answered negatively. Moreover, a solution to Problem 1.8 proves $\mathsf{nonvanishing}(\chi_{\text{claw-free } G})$ is coNP, and hence $\mathsf{NP} = \mathsf{coNP}$.

This suggests a new complexity-theoretic rationale for the study of χ_G.

1.3. **An algebraic combinatorics paradigm for complexity.** Summarizing, we are motivated by complexity to study nonvanishing in algebraic combinatorics. Many polynomial families $\{F_\diamondsuit\}$ (conjecturally) have combinatorial positivity and SNP [27]. Together, with a solution to Problem 1.8, $\mathsf{nonvanishing} \in \mathsf{NP} \cap \mathsf{coNP}$.

In each such case $\{F_\diamondsuit\}$ one arrives at one of four logical outcomes, depending on the complexity of $\mathsf{nonvanishing}(F_\diamondsuit)$ within $\mathsf{NP} \cap \mathsf{coNP}$:

- (I) Unknown: it is a problem, in and of itself, to find additional problems that are in $\mathsf{NP} \cap \mathsf{coNP}$ that are not *known* to be in P.
- (II) P: Give an algorithm. It will likely illuminate some special structure, of independent combinatorial interest.
- (III) NP-complete: proof solves $\mathsf{NP} \stackrel{?}{=} \mathsf{coNP}$ with (a suprising) "=".
- (IV) NP-intermediate: proof solves NP-intermediate$\stackrel{?}{=} \emptyset$ with "$\neq$" (hence $\mathsf{P} \neq \mathsf{NP}$).

We illustrate (II) for Schubert polynomials.

1.4. **Schubert polynomials.** *Schubert polynomials* form a linear basis of all polynomials $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. They were introduced by A. Lascoux–M.-P. Schützenberger [20] to study the cohomology ring of the flag manifold. These polynomials represent the Schubert classes under the Borel isomorphism. A reference is the textbook [11].

[1] In this circumstance, the (complexity) polynomial hierarchy unexpectedly collapses to the first level.

If $w_0 = n\ n-1\ \cdots 2\ 1$ is the longest length permutation in S_n, then

$$\mathfrak{S}_{w_0}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

Otherwise, $w \neq w_0$ and there exists i such that $w(i) < w(i+1)$. Then one sets

$$\mathfrak{S}_w(x_1, \ldots, x_n) = \partial_i \mathfrak{S}_{ws_i}(x_1, \ldots, x_n),$$

where $\partial_i f := \frac{f - s_i f}{x_i - x_{i+1}}$, and s_i is the transposition swapping i and $i+1$. Since ∂_i satisfies

$$\partial_i \partial_j = \partial_j \partial_i \text{ for } |i-j| > 1, \text{ and } \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1},$$

the above description of $\mathfrak{S}_w$ is well-defined. In addition, under the inclusion $\iota : S_n \hookrightarrow S_{n+1}$ defined by $w(1) \cdots w(n) \mapsto w(1) \cdots w(n)\, n+1$, $\mathfrak{S}_w = \mathfrak{S}_{\iota(w)}$. Thus one unambiguously refers to $\mathfrak{S}_w$ for each $w \in S_\infty = \bigcup_{n \geq 1} S_n$.

To each $w \in S_\infty$ there is a unique associated *code*, $\mathsf{code}(w) = (c_1, c_2, \ldots, c_L) \in \mathbb{Z}_{\geq 0}^L$, where c_i counts the number of boxes in the i-th row of the Rothe diagram $D(w)$ of w. If w is the identity then $\mathsf{code}(w) = \emptyset$; otherwise, $c_L > 0$ (i.e., we truncate any trailing zeroes).

Now, $c_{\alpha,w} = 0$ unless $\alpha_i = 0$ for $i > L$, and moreover, $c_{\alpha,w} \in \mathbb{Z}_{\geq 0}$. Let Schubert be the nonvanishing problem for Schubert polynomials. The `INPUT` is $\mathsf{code} = (c_1, \ldots c_L) \in \mathbb{Z}_{\geq 0}^L$ with $c_L > 0$ and $\alpha \in \mathbb{Z}_{\geq 0}^L$. Schubert returns `YES` if $c_{\alpha,w} > 0$ and `NO` otherwise.

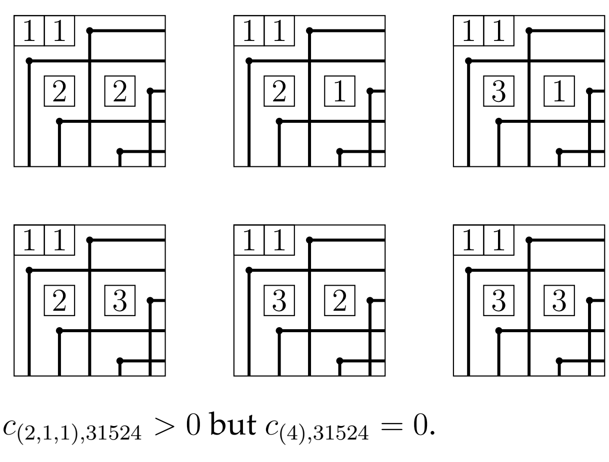
Theorem 1.13. Schubert $\in$ P.

We prove Theorem 1.13 using another result. For $w \in S_n$, let $\mathsf{Tab}_{\neq}(D(w), \alpha)$ be the fillings of $D(w)$ with α_k many k's, where entries in each column are distinct, and any entry in row i is $\leq i$. Let $\mathsf{Tab}_{<}(D(w), \alpha) \subseteq \mathsf{Tab}_{\neq}(D(w), \alpha)$ be fillings where entries in each column increase from top to bottom.

Theorem 1.14. $c_{\alpha,w} > 0 \iff \mathsf{Tab}_{\neq}(D(w), \alpha) \neq \emptyset \iff \mathsf{Tab}_{<}(D(w), \alpha) \neq \emptyset$

In general $\#\mathsf{Tab}_{\neq}(D(w), \alpha) \neq c_{\alpha,w}$ but rather $\#\mathsf{Tab}_{\neq}(D(w), \alpha) \geq c_{\alpha,w}$ (cf. [9]).

Example 1.15. Here are the tableaux in $\bigcup_\alpha \mathsf{Tab}_{<}(D(31524), \alpha)$:



Hence, for instance, $c_{(2,1,1),31524} > 0$ but $c_{(4),31524} = 0$.

To prove Theorems 1.13 and 1.14 we establish results about the *Schubitope* introduced in [27]. This polytope $\mathcal{S}_D$ generalizes the λ-permutahedron of Example 1.5. It is defined with a halfspace description for any diagram of boxes $D \subseteq [n]^2$. We prove (Theorem 2.13) that a lattice point α is in $\mathcal{S}_D$ if and only if $\mathsf{Tab}_{\neq}(D, \alpha) \neq 0$ where D is any diagram.

We then introduce a polytope $\mathcal{P}(D,\alpha)$ whose lattice points $\mathcal{P}(D,\alpha)_{\mathbb{Z}}$ are in bijection with $\mathsf{Tab}_{\neq}(D,\alpha)$. We prove that $\mathcal{P}(D,\alpha) \neq \emptyset \iff \mathcal{P}(D,\alpha)_{\mathbb{Z}} \neq \emptyset$ (Theorem 2.27). Since $\mathsf{LPfeasibility} \in \mathsf{P}$, determining if $\mathcal{P}(D,\alpha)_{\mathbb{Z}} \neq \emptyset$ (and thus if $\alpha \in \mathcal{S}_D$) is in P. We give two proofs of Theorem 2.27. The first shows $\mathcal{P}(D,\alpha)$ is totally unimodular. Hence $\mathcal{P}(D,\alpha) \neq \emptyset$ implies $\mathcal{P}(D,\alpha)$ has integral vertices. Our second proof obviates total unimodularity, and is potentially adaptable to problems lacking that property. However, only the high-level structure of the second proof is easily generalizable — the rest is necessarily *ad hoc*.

For the special case of Rothe diagrams $D = D(w)$, it was conjectured in [27] that $\mathcal{S}_{D(w)}$ is the Newton polytope of $\mathfrak{S}_w$ and moreover that $\mathfrak{S}_w$ has the SNP property. These conjectures were proved by A. Fink-K. Mészáros-A. St. Dizier [7]. This, combined with our results on the Schubitope proves Theorems 1.13 and 1.14.

The class $\#\mathsf{P}$ in L. Valiant's complexity theory of counting problems are those that count the number of accepting paths of a nondeterministic Turing machine running in polynomial time. A problem $\mathcal{P} \in \#\mathsf{P}$ is *complete* if for any problem $\mathcal{Q} \in \#\mathsf{P}$ there exists a polynomial-time counting reduction from $\mathcal{Q}$ to $\mathcal{P}$. These are the hardest of the problems in $\#\mathsf{P}$. There does not exist a polynomial time algorithm for such problems unless $\mathsf{P} = \mathsf{NP}$.

In contrast with Theorem 1.13, we prove:

Theorem 1.16. *Counting $c_{\alpha,w}$ is $\#\mathsf{P}$-complete.*[2]

Given $\{c_{\alpha,\diamondsuit} \in \mathbb{Z}_{\geq 0}\}$ it is standard to ask for a counting rule for $c_{\alpha,\diamondsuit}$. A complexity motivation is an *appropriate* rule that establishes a counting problem is in $\#\mathsf{P}$ with respect to given input (length). The rule of [3] is clearly in $\#\mathsf{P}$ if the input is (w,α) but not if the input is $(\mathsf{code}(w),\alpha)$. For the latter input assumption, we use the transition algorithm of [19] and its *graphical* reformulation from [17]. This allows us to give a polynomial time counting reduction to the $\#\mathsf{P}$-complete problem of counting Kostka coefficients [28].[3]

Our proof of Theorem 1.16 combined with the Schubitope inequalities proves that $\mathsf{Schubert} \in \mathsf{NP} \cap \mathsf{coNP}$. This put us in outcome (I) along the way to (II), i.e., Theorem 1.14.

1.5. **Further discussion.** This paper's complexity paradigm motivates examination of other polynomials from algebraic combinatorics. For example:

(1) The *resultant* is $F_{m,n} = \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - y_j)$. Here, $[x^\alpha y^\beta]F$ is the number of $m \times n$ $(0,1)$-matrices whose row sums are given by α and column sums are given by β; see, e.g., [34, Proposition 7.4.3]. Now, $\mathsf{nonvanishing}(F_{m,n}) \in \mathsf{P}$, by the *Gale-Ryser theorem*.

(2) Suppose $F_\diamondsuit = \sum_\mu c_{\mu,\diamondsuit} s_\mu$ with $c_{\lambda,\diamondsuit} \geq 0$ and such that there exists λ (depending on $\diamondsuit$) with $c_{\lambda,\diamondsuit} > 0$ and $c_{\mu,\diamondsuit} > 0 \Rightarrow \mu \leq \lambda$ in dominance order. Then $F_\diamondsuit$ is SNP [27, Proposition 2.5] and $\mathsf{Newton}(F_\diamondsuit) = \mathcal{P}_\lambda$. Hence, under modest assumptions, $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{P}$. This case includes, e.g., Schur P functions, Stanley's symmetric polynomials for reduced words, among others.

---

[2] The contrast of hard counting versus efficient nonvanishing has an antecedent. The original $\#\mathsf{P}$-complete problem [37] is to compute the *permanent* of an $n \times n$ matrix $A = (a_{ij})$ where $a_{ij} \in \{0,1\}$. However, determining if $\mathrm{per}(A) > 0$ is equivalent to deciding if a bipartite graph G with incidence matrix A has a matching. This can be solved in polynomial time, using the *Edmonds-Karp algorithm*.

[3] The input issue might appear pedantic, but an analogy is the knapsack problem. If the input is in binary, the problem is NP-complete. However, in unary, it is in P, via dynamic programming. The latter fact does not imply $\mathsf{P} = \mathsf{NP}$ since unary knapsack is not known to be NP-complete.

(3) The *Kronecker product* $s_\lambda * s_\mu$ is Schur positive; coefficients of the Schur expansion are the *Kronecker coefficients* of the symmetric group.[4] The famous open problem is to give a combinatorial rule for these coefficients. It was conjectured in [27] that $s_\lambda * s_\mu$ is SNP. However, examples show that SNPness does not follow from (1). Now $\mathsf{nonvanishing}(s_\lambda s_\mu) \in \mathsf{P}$ by (2). Is $\mathsf{nonvanishing}(s_\lambda * s_\mu) \in \mathsf{P}$?

(4) Examples show that the symmetric (modified) Macdonald polynomials $\widetilde{H}_\lambda(X; q, t)$ are not SNP. A combinatorial rule for the coefficients was given by J. Haglund-M. Haiman-N. Loehr [14]. Is $\mathsf{nonvanishing}(\widetilde{H}_\lambda) \notin \mathsf{P}$? A difficult problem is to give a combinatorial rule for the q, t*-Kostka coefficients* $K_{\lambda,\mu}(q,t) := [s_\mu]\widetilde{H}_\lambda$. When $q = t = 0$, these coefficients are the Kostka coefficients. These are not SNP in general [27]. Is $\mathsf{nonvanishing}(K_{\lambda,\mu}(q,t)) \notin \mathsf{P}$?

(5) *Key polynomials* are a specialization of the non-symmetric Macdonald polynomials. These were conjectured in [27] to be SNP; this is proved in [7]. Nonvanishing is also in P, provable using results of Section 2 in a manner analogous to Section 4.1.

(6) An inhomgeneous deformation of the Schubert polynomials are the *Grothendieck polynomials*. It was conjectured in [27] that these are SNP. A. Fink-K. Mészáros-A. St. Dizier conjectured in [21] that the Newton polytope is a generalized permutahedron. In [6], it was shown that the A. Fink-K. Mészáros-A. St. Dizier conjecture is true for all symmetric Grothendieck polynomials G_λ. Using this, the second author has found an extension of the results in Example 1.5; this gives a polynomial time algorithm to decide if a K-theoretic Kostka coefficient is nonzero. Finally, what is an analogue of Theorem 1.14?

2. THE SCHUBITOPE

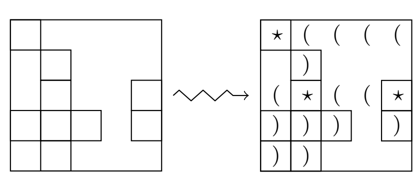
Fix $n \in \mathbb{Z}_{>0}$ and let $D \subseteq [n]^2$. We call D a *diagram* and visualize D as a subset of an $n \times n$ grid of boxes, oriented so that $(r, c) \in [n]^2$ represents the box in the rth row from the top and the cth column from the left. Given $S \subseteq [n]$ and a column $c \in [n]$, construct a string denoted $\mathsf{word}_{c,S}(D)$ by reading column c from top to bottom and recording

- (if $(r, c) \notin D$ and $r \in S$,
-) if $(r, c) \in D$ and $r \notin S$, and
- $\star$ if $(r, c) \in D$ and $r \in S$.

Let $\theta_D^c(S) = \#\{\star\text{'s in } \mathsf{word}_{c,S}(D)\} + \#\{\text{paired ()'s in } \mathsf{word}_{c,S}(D)\}$ and

$$\theta_D(S) = \sum_{c=1}^{n} \theta_D^c(S).$$

Example 2.1. In the diagram D below, we labelled the corresponding strings for $\mathsf{word}_{c,S}(D)$ for $S = \{1, 3\}$. For instance, we see $\mathsf{word}_{5,\{1,3\}}(D) = (\star)$.



[4] Although this paper is not directly related to the *Geometric Complexity Theory* attack [Mul01, MulSoh01] on $\mathsf{P} \neq \mathsf{NP}$, the Kronecker coefficients do play a prominent role in that approach.

The *Schubitope* $\mathcal{S}_D$, as defined in [27], is the polytope

$$(1) \quad \left\{ (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0} : \alpha_1 + \cdots + \alpha_n = \#D \text{ and } \sum_{i \in S} \alpha_i \leq \theta_D(S) \text{ for all } S \subseteq [n] \right\}.$$

2.1. **Characterizations via tableaux.** A *tableau* of *shape* D is a map

$$\tau : D \to [n] \cup \{\circ\},$$

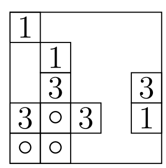
where $\tau(r,c) = \circ$ indicates that the box (r,c) is unlabelled. Let $\mathsf{Tab}(D)$ denote the set of such tableaux.

It will be useful to reformulate the original definition of $\theta_D(S)$ into the language of tableaux. Given $S \subseteq [n]$, define $\pi_{D,S} \in \mathsf{Tab}(D)$ by

$$(2) \qquad \pi_{D,S}(r,c) = \begin{cases} r & \text{if } (r,c) \text{ contributes a ``}\star\text{'' to } \mathsf{word}_{c,S}(D), \\ s & \text{if } (r,c) \text{ contributes a ``)'' to } \mathsf{word}_{c,S}(D) \text{ which is} \\ & \text{paired with an ``('' from } (s,c), \\ \circ & \text{otherwise.} \end{cases}$$

In (2) and throughout, we pair by the standard "inside-out" convention.

Example 2.2. Continuing Example 2.1, below is $\pi_{D,\{1,3\}}(D)$

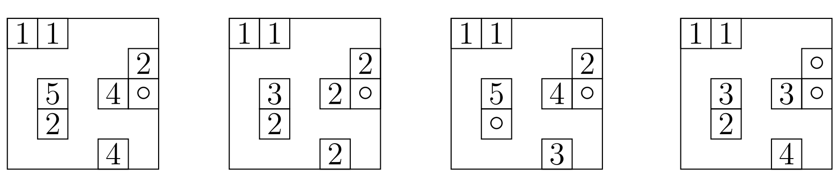


Proposition 2.3. *For all* $D \subseteq [n]^2$ *and* $S \subseteq [n]$, *we have* $\theta_D(S) = \#\pi^{-1}_{D,S}(S)$.

Proof. $\pi_{D,S}(r,c) \in S$ if and only if (r,c) falls into one of the first two cases in (2). ☐

Say $\tau \in \mathsf{Tab}(D)$ is *flagged* if $\tau(r,c) \leq r$ whenever $\tau(r,c) \neq \circ$. It is *column-injective* if $\tau(r,c) \neq \tau(r',c)$ whenever $r \neq r'$ and $\tau(r,c) \neq \circ$. Let $\mathsf{FCITab}(D) \subseteq \mathsf{Tab}(D)$ be the set of tableaux of shape D which are flagged and column-injective.

Example 2.4. Of the tableaux of shape D below, only the second and fourth are flagged, and only the third and fourth are column-injective.



Proposition 2.5. $\pi_{D,S} \in \mathsf{FCITab}(D)$ *for all* $D \subseteq [n]^2$ *and* $S \subseteq [n]$.

Proof. This is immediate from (2). ☐

A simple consequence of being flagged and column-injective is the following.

Proposition 2.6. *Let $\tau \in \mathsf{FCITab}(D)$. Then for all $(r,c) \in [n]^2$ and $S \subseteq [n]$, we have*

$$\#\{(i,c) \in \tau^{-1}(S) : i < r\} \leq \#\{i \in S : i \leq r\}, \tag{3}$$

with strict inequality whenever $(r,c) \in \tau^{-1}(S)$.

Proof. The map $(i,c) \mapsto \tau(i,c)$ from $\{(i,c) \in \tau^{-1}(S) : i \leq r\}$ to $\{i \in S : i \leq r\}$ is well-defined since τ is flagged. It is injective since τ is column-injective. Thus (3) holds, and

$$\#\{(i,c) \in \tau^{-1}(S) : i < r\} < \#\{(i,c) \in \tau^{-1}(S) : i \leq r\} \leq \#\{i \in S : i \leq r\}$$

whenever $(r,c) \in \tau^{-1}(S)$, establishing the strict inequality assertion. □

In fact, a stronger assertion holds when $\tau = \pi_{D,S}$.

Proposition 2.7. *If $(r,c) \in D \subseteq [n]^2$ and $S \subseteq [n]$, then*

$$(r,c) \in \pi_{D,S}^{-1}(S) \iff \#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\}.$$

Proof. ($\Rightarrow$) This direction follows from Propositions 2.5 and 2.6.
($\Leftarrow$) If $r \in S$, then (r,c) contributes a "$\star$" to $\mathsf{word}_{c,S}(D)$, so $\pi_{D,S}(r,c) = r \in S$, as desired. Thus we assume $r \notin S$. The hypothesis combined with this assumption says

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\} = \#\{i \in S : i < r\}.$$

Thus, there is a maximal $s \in S$ with $s < r$ such that $\pi_{D,S}(r',c) \neq s$ whenever $r' < r$. If $(s,c) \in D$, then (s,c) contributes a "$\star$" to $\mathsf{word}_{c,S}(D)$, so $\pi_{D,S}(s,c) = s$, contradicting our choice of s. Therefore, (s,c) contributes an "(" to $\mathsf{word}_{c,S}(D)$. If this "(" is paired by a ")" contributed by $(r',c) \in D$ with $r' < r$, then $\pi_{D,S}(r',c) = s$, again a contradiction. Thus, this "(" pairs the ")" from (r,c), so $\pi_{D,S}(r,c) = s \in S$. Hence, $(r,c) \in \pi_{D,S}^{-1}(S)$ as desired. □

The previous two propositions combined assert that $\{(r,c) \in \pi_{D,S}^{-1}(S)\}$ is characterized by greedy selection as one moves down each column c. The next proposition shows that this greedy algorithm maximizes $\#\tau^{-1}(S)$ among all $\tau \in \mathsf{FCITab}(D)$.

Proposition 2.8. *Let $D \subseteq [n]^2$ and $S \subseteq [n]$. Then $\#\pi_{D,S}^{-1}(S) \geq \#\tau^{-1}(S)$ for all $\tau \in \mathsf{FCITab}(D)$.*

Proof. If not, then there exist $\tau \in \mathsf{FCITab}(D)$ and $(r,c) \in [n]^2$ satisfying

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i \leq r\} < \#\{(i,c) \in \tau^{-1}(S) : i \leq r\}$$

and we can choose these such that r is minimized. Then because r is minimal,

$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} = \#\{(i,c) \in \tau^{-1}(S) : i < r\}$$

and $(r,c) \in \tau^{-1}(S) \setminus \pi_{D,S}^{-1}(S)$, so in particular $(r,c) \in D$. Thus Proposition 2.6 implies

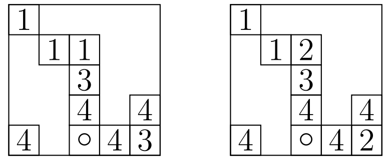
$$\#\{(i,c) \in \pi_{D,S}^{-1}(S) : i < r\} = \#\{(i,c) \in \tau^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\}.$$

But then we must have $(r,c) \in \pi_{D,S}^{-1}(S)$ by Proposition 2.7, a contradiction. □

If τ has shape a subset of $[n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0}$, say τ *exhausts* α *over* S if

$$\sum_{i \in S} \alpha_i \leq \#\tau^{-1}(S).$$

Example 2.9. Only the left tableau below exhausts $\alpha = (3,2,2,4)$ over $S = \{1,3\}$.



Theorem 2.10. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then $\alpha \in \mathcal{S}_D$ if and only if for each $S \subseteq [n]$, there exists $\tau_{D,S} \in \mathsf{FCITab}(D)$ which exhausts α over S.*

Proof of Theorem 2.10. ($\Rightarrow$) The inequalities in (1) combined with Proposition 2.3 imply

$$\sum_{i \in S} \alpha_i \leq \theta_D(S) = \#\pi^{-1}_{D,S}(S).$$

Thus, $\tau_{D,S} := \pi_{D,S}$ exhausts α over S.

($\Leftarrow$) By Propositions 2.8 and 2.3,

$$\sum_{i \in S} \alpha_i \leq \#\tau^{-1}_{D,S}(S) \leq \#\pi^{-1}_{D,S}(S) = \theta_D(S),$$

so the inequalities in (1) hold. □

Remark 2.11. The proof of ($\Rightarrow$) shows that we can take $\tau_{D,S} = \pi_{D,S}$ in Theorem 2.10.

It would be nice if $\tau_{D,S}$ did not depend on S, i.e., if some τ_D exhausted α over all $S \subseteq [n]$, so we could take $\tau_{D,S} = \tau_D$ in Theorem 2.10. Indeed, this is shown in Theorem 2.13.

Say $\tau \in \mathsf{Tab}(D)$ has *content* α if $\#\tau^{-1}(\{i\}) = \alpha_i$ for each $i \in [n]$. Let $\mathsf{Tab}(D, \alpha)$ and $\mathsf{FCITab}(D, \alpha)$ be the subsets of $\mathsf{Tab}(D)$ and $\mathsf{FCITab}(D)$, respectively, of those tableaux which have content α. In addition, call a tableau $\tau \in \mathsf{Tab}(D)$ *perfect* if $\tau \in \mathsf{FCITab}(D)$, and if no boxes are left unlabelled, i.e., $\tau^{-1}(\{\circ\}) = \emptyset$. Let $\mathsf{PerfectTab}(D, \alpha) \subseteq \mathsf{FCITab}(D, \alpha)$ denote the set of perfect tableaux of content α.

Proposition 2.12. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathsf{FCITab}(D, \alpha) \neq \emptyset$.*

Proof. ($\Rightarrow$) Let $\tau \in \mathsf{PerfectTab}(D, \alpha)$. Then $\tau \in \mathsf{FCITab}(D, \alpha)$, and since τ has content α and satisfies $\tau^{-1}(\{\circ\}) = \emptyset$,

$$\alpha_1 + \cdots + \alpha_n = \#\tau^{-1}(\{1\}) + \cdots + \#\tau^{-1}(\{n\}) = \#D.$$

($\Leftarrow$) Let $\tau \in \mathsf{FCITab}(D, \alpha)$. Then since τ has content α,

$$\#\tau^{-1}(\{\circ\}) = \#D - \#\tau^{-1}(\{1\}) - \cdots - \#\tau^{-1}(\{n\}) = \#D - \alpha_1 - \cdots - \alpha_n = 0.$$

Thus, $\tau \in \mathsf{PerfectTab}(D, \alpha)$. □

Theorem 2.13. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\alpha \in \mathcal{S}_D$ if and only if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$.*

The proof will require a lemma regarding tableaux of the form $\tau = \pi_{D,S}$.

Lemma 2.14. *Let $D \subseteq [n]^2$, and $S, T \subseteq [n]$ be disjoint. Set*

$$\tilde{D} = D \smallsetminus \pi^{-1}_{D,S}(S) \text{ and } U = S \cup T.$$

Then

$$\pi^{-1}_{D,U}(U) = \pi^{-1}_{D,S}(S) \cup \pi^{-1}_{\tilde{D},T}(T).$$

Proof. Let $(r, c) \in D$, and assume by induction on r that

$$(i, c) \in \pi_{D,U}^{-1}(U) \iff (i, c) \in \pi_{D,S}^{-1}(S) \cup \pi_{\tilde{D},T}^{-1}(T) \tag{4}$$

whenever $i < r$. This clearly holds in the base case $r = 1$. By Proposition 2.7, $(r, c) \in \pi_{D,U}^{-1}(U)$ if and only if

$$\#\{(i, c) \in \pi_{D,U}^{-1}(U) : i < r\} < \#\{i \in U : i \leq r\}. \tag{5}$$

By (4) and the fact that

$$\pi_{D,S}^{-1}(S) \cap \tilde{D} = \emptyset = S \cap T,$$

(5) is equivalent to

$$\#\{(i, c) \in \pi_{D,S}^{-1}(S) : i < r\} + \#\{(i, c) \in \pi_{\tilde{D},T}^{-1}(T) : i < r\} < \#\{i \in S : i \leq r\} + \#\{i \in T : i \leq r\}.$$

By applying Proposition 2.6 twice, we see that this holds if and only if at least one of (i) and (ii) below hold.

(i) $\#\{(i, c) \in \pi_{D,S}^{-1}(S) : i < r\} < \#\{i \in S : i \leq r\}$
(ii) $\#\{(i, c) \in \pi_{\tilde{D},T}^{-1}(T) : i < r\} < \#\{i \in T : i \leq r\}$

By Proposition 2.7, (i) is equivalent to $(r, c) \in \pi_{D,S}^{-1}(S)$. If indeed $(r, c) \in \pi_{D,S}^{-1}(S)$ holds, then our induction step is complete. Otherwise, $(r, c) \notin \pi_{D,S}^{-1}(S)$, so by definition, $(r, c) \in \tilde{D}$. Thus, applying Proposition 2.7 to $\tilde{D}$, $T \subseteq [n]$ and $(r, c) \in \tilde{D}$, (ii) is equivalent to $(r, c) \in \pi_{\tilde{D},T}^{-1}(T)$. Hence, (4) holds for all $i \leq r$. ☐

Corollary 2.15. *Let $D \subseteq [n]^2$ and $S \subseteq U \subseteq [n]$. Then $\pi_{D,S}^{-1}(S) \subseteq \pi_{D,U}^{-1}(U)$.*

Proof. Take $T = U \setminus S$ in Lemma 2.14. ☐

Finally, we are ready to prove Theorem 2.13.

Proof of Theorem 2.13. ($\Leftarrow$) Let $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$. Then $\alpha_1 + \cdots + \alpha_n = \#D$ by Proposition 2.12. Also, for each $S \subseteq [n]$,

$$\sum_{i \in S} \alpha_i = \sum_{i \in S} \#\tau_D^{-1}(\{i\}) = \#\tau_D^{-1}(S),$$

so τ_D exhausts α over S. Thus, $\alpha \in \mathcal{S}_D$ by Theorem 2.10.

($\Rightarrow$) We induct on the sum of the row indices of each box in D, i.e., $\sum_{(i,j) \in D} i$. The base case of an empty diagram is trivial, so we may assume $D \neq \emptyset$. Then since $\alpha \in \mathcal{S}_D$, (1) implies $\alpha_1 + \cdots + \alpha_n = \#D > 0$, so we can choose m maximal such that $\alpha_m > 0$.

Case 1: (D contains boxes below row m). Pick $(r, c) \in D$ below row m (so $r > m$).

Claim 2.16. *There exists $r_1 < r$ such that $(r_1, c) \notin D$.*

Proof of Claim 2.16. By Theorem 2.10, there exists $\tau_{D,[m]} \in \mathsf{FCITab}(D)$ such that

$$\#\tau_{D,[m]}^{-1}([m]) \geq \alpha_1 + \cdots + \alpha_m = \alpha_1 + \cdots + \alpha_n = \#D. \tag{6}$$

Thus, $\tau_{D,[m]}(D) \subseteq [m]$. Consequently, by column-injectivity of $\tau_{D,[m]}$, there can be at most m boxes in each column of D. Since $(r, c) \in D$ with $r > m$, there are more than m boxes in column c if $(r_1, c) \in D$ for all $r_1 < r$. Hence there must be some $r_1 < r$ for which $(r_1, c) \notin D$, as asserted. ☐

By Claim 2.16, we can choose $r_1 < r$ maximal such that $(r_1, c) \notin D$. Let

$$\tilde{D} = (D \setminus \{(r,c)\}) \cup \{(r_1, c)\}.$$

Claim 2.17. $\alpha \in \mathcal{S}_{\tilde{D}}$.

Proof of Claim 2.17. Since $\alpha \in \mathcal{S}_D$, $(r,c) \in D$, and $(r_1, c) \notin D$, we have

$$\alpha_1 + \cdots + \alpha_n = \#D = \#\tilde{D}.$$

Let $S \subseteq [n]$ and $T = S \cap [m]$. Then define $\tau_{\tilde{D},S} \in \mathsf{Tab}(\tilde{D})$ by

$$\tau_{\tilde{D},S}(i,j) = \begin{cases} \pi_{D,T}(r,c) & \text{if } (i,j) = (r_1,c), \\ \pi_{D,T}(i,j) & \text{otherwise.} \end{cases}$$

If $\pi_{D,T}(r,c) = \circ$, then certainly $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$. Otherwise, let $s = \pi_{D,T}(r,c)$. Since $(r,c) \in D$ but $r \notin T$, (r,c) contributes a ")" to $\mathsf{word}_{c,S}(D)$. Thus, by (2), (s,c) contributes an "(", so in particular $(s,c) \notin D$. From our choice of r_1, we must therefore have $s \leq r_1$, so $\tau_{\tilde{D},S}$ is flagged. Hence, $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$.

By construction,

$$\#\tau_{\tilde{D},S}^{-1}(\{i\}) = \#\pi_{D,T}^{-1}(\{i\})$$

for each $i \in [n]$, so $\tau_{\tilde{D},S}$ exhausts α over T by Theorem 2.10 and in particular Remark 2.11. Since $\alpha_i = 0$ for all $i > m$, we can write

$$\sum_{i \in S} \alpha_i = \sum_{i \in T} \alpha_i \leq \#\tau_{\tilde{D},S}^{-1}(T) \leq \#\tau_{\tilde{D},S}^{-1}(S).$$

Therefore, $\tau_{\tilde{D},S} \in \mathsf{FCITab}(\tilde{D})$ exhausts α over S, so $\alpha \in \mathcal{S}_{\tilde{D}}$ by Theorem 2.10. $\square$

Since $r_1 < r$,

$$\sum_{(i,j) \in \tilde{D}} i < \sum_{(i,j) \in D} i.$$

Thus, Claim 2.17 and induction yields $\tau_{\tilde{D}} \in \mathsf{PerfectTab}(\tilde{D}, \alpha)$. Define $\tau_D \in \mathsf{Tab}(D)$ by

$$\tau_D(i,j) = \begin{cases} \tau_{\tilde{D}}(r_1,c) & \text{if } (i,j) = (r,c), \\ \tau_{\tilde{D}}(i,j) & \text{otherwise.} \end{cases}$$

Then it is easy to check that $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$, so Case 1 is complete.

Case 2: (D does not contain boxes below row m). We say an inequality $\sum_{i \in S} \alpha_i \leq \theta_D(S)$ from (1) is *nontrivial* if

(7) $$\sum_{i \in S} \alpha_i > 0 \qquad \text{and} \qquad \theta_D(S) < \#D.$$

Case 2a: (All nontrivial inequalities from (1) are strict). Thus if (7) holds, then

(8) $$\sum_{i \in S} \alpha_i < \theta_D(S).$$

Claim 2.18. *There exists $c \in [n]$ such that $(m,c) \in D$.*

Proof of Claim 2.18. By Theorem 2.10, there exists some $\tau_{D,\{m\}} \in \mathsf{FCITab}(D)$ which exhausts α over $\{m\}$. Then

$$\#\tau^{-1}_{D,\{m\}}(\{m\}) \geq \alpha_m > 0,$$

so $\tau_{D,\{m\}}(r,c) = m$ for some $(r,c) \in D$. Since $\tau_{D,\{m\}}$ is flagged, we must have $r \geq m$. But by the assumption of Case 2, there are no boxes below row m, so $r = m$. □

Pick $c \in [n]$ as in Claim 2.18. Then let $\tilde{D} = D \setminus \{(m,c)\}$ and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) := (\alpha_1, \ldots, \alpha_{m-1}, \alpha_m - 1, 0, \ldots, 0)$.

Claim 2.19. $\tilde{\alpha} \in \mathcal{S}_{\tilde{D}}$.

Proof of Claim 2.19. Since $\alpha_i = 0$ for all $i > m$, and $(m,c) \in D$, we have

$$\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_n = \alpha_1 + \cdots + \alpha_n - 1 = \#D - 1 = \#\tilde{D}. \tag{9}$$

For each $S \subseteq [n]$, let

$$\tau_{\tilde{D},S} = \pi_{D,S}|_{\tilde{D}} \in \mathsf{FCITab}(\tilde{D})$$

be the restriction of $\pi_{D,S}$ to $\tilde{D}$. Then by Proposition 2.3,

$$\#\tau^{-1}_{\tilde{D},S}(S) \geq \#\pi^{-1}_{D,S}(S) - 1 = \theta_D(S) - 1. \tag{10}$$

If $\sum_{i\in S} \alpha_i = 0$, then

$$\sum_{i\in S} \tilde{\alpha}_i = 0 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

If $\theta_D(S) = \#D$, then by (9) and (10),

$$\sum_{i\in S} \tilde{\alpha}_i \leq \tilde{\alpha}_1 + \cdots + \tilde{\alpha}_n = \#D - 1 = \theta_D(S) - 1 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

Finally, if $\sum_{i\in S} \alpha_i > 0$ and $\theta_D(S) < \#D$, then (8) must hold, so by (8) and (10),

$$\sum_{i\in S} \tilde{\alpha}_i \leq \sum_{i\in S} \alpha_i \leq \theta_D(S) - 1 \leq \#\tau^{-1}_{\tilde{D},S}(S).$$

In all three cases, $\tau_{\tilde{D},S}$ exhausts $\tilde{\alpha}$ over S, so $\tilde{\alpha} \in \mathcal{S}_{\tilde{D}}$ by Theorem 2.10. □

By construction,

$$\sum_{(i,j)\in\tilde{D}} i < \sum_{(i,j)\in D} i.$$

Thus, Claim 2.19 and induction yield $\tau_{\tilde{D}} \in \mathsf{PerfectTab}(\tilde{D}, \tilde{\alpha})$. Define $\tau_D \in \mathsf{Tab}(D)$ by

$$\tau_D(i,j) = \begin{cases} m & \text{if } (i,j) = (m,c), \\ \tilde{\tau}(i,j) & \text{otherwise.} \end{cases}$$

Clearly, τ_D is flagged, has content α, and satisfies $\tau_D^{-1}(\{\circ\}) = \emptyset$. The only potential obstruction to column-injectivity is that there could be some $r \neq m$ for which $\tau_D(r,c) = m$. This is impossible, since τ_D is flagged, so such an r must be greater than m, but by the assumption of Case 2 there are no boxes below row m. Thus, $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$, so Case 2a is complete.

Case 2b: (There exists a tight, nontrivial inequality in (1)). Thus, there exists $A \subseteq [n]$ satisfying

$$0 < \sum_{i \in A} \alpha_i = \theta_D(A) < \#D. \tag{11}$$

Let $D^{(1)} = \pi^{-1}_{D,A}(A)$ and $D^{(2)} = D \smallsetminus D^{(1)}$. Then for each $i \in [n]$, set

$$\alpha_i^{(1)} = \begin{cases} \alpha_i & \text{if } i \in A, \\ 0 & \text{if } i \notin A \end{cases} \qquad \text{and} \qquad \alpha_i^{(2)} = \begin{cases} \alpha_i & \text{if } i \notin A, \\ 0 & \text{if } i \in A. \end{cases}$$

Claim 2.20. $\alpha^{(1)} := (\alpha_1^{(1)}, \dots, \alpha_n^{(1)}) \in \mathcal{S}_{D^{(1)}}$.

Proof of Claim 2.20. By (11) and Proposition 2.3, we have

$$\alpha_1^{(1)} + \cdots + \alpha_n^{(1)} = \sum_{i \in A} \alpha_i = \theta_D(A) = \#\pi^{-1}_{D,A}(A) = \#D^{(1)}.$$

Let $S \subseteq [n]$ and $T = S \cap A$. Then set

$$\tau_{D^{(1)},S} = \pi_{D,T}|_{D^{(1)}} \in \mathsf{FCITab}(D^{(1)}).$$

By Corollary 2.15, $\pi^{-1}_{D,T}(T) \subseteq D^{(1)}$, so $\tau^{-1}_{D^{(1)},S}(T) = \pi^{-1}_{D,T}(T)$. Thus, by Remark 2.11, $\tau_{D^{(1)},S}$ exhausts α over T. Hence,

$$\sum_{i \in S} \alpha_i^{(1)} = \sum_{i \in T} \alpha_i \leq \#\tau^{-1}_{D^{(1)},S}(T) \leq \#\tau^{-1}_{D^{(1)},S}(S),$$

so $\tau_{D^{(1)},S}$ exhausts $\alpha^{(1)}$ over S, and consequently $\alpha^{(1)} \in \mathcal{S}_{D^{(1)}}$ by Theorem 2.10. ☐

Claim 2.21. $\alpha^{(2)} := (\alpha_1^{(2)}, \dots, \alpha_n^{(2)}) \in \mathcal{S}_{D^{(2)}}$.

Proof of Claim 2.21. By (11) and Proposition 2.3,

$$\alpha_1^{(2)} + \cdots + \alpha_n^{(2)} = \alpha_1 + \cdots + \alpha_n - \sum_{i \in A} \alpha_i = \#D - \theta_D(A) = \#D - \#\pi^{-1}_{D,A}(A) = \#D^{(2)}.$$

Let $S \subseteq [n]$, $T = S \smallsetminus A$, and $U = A \cup T$. Then by Theorem 2.10, Remark 2.11, (11), Proposition 2.3, and Lemma 2.14, we can write

$$\begin{aligned} \sum_{i \in S} \alpha_i^{(2)} = \sum_{i \in U} \alpha_i - \sum_{i \in A} \alpha_i &\leq \#\pi^{-1}_{D,U}(U) - \theta_D(A) \\ &= \#\pi^{-1}_{D,U}(U) - \#\pi^{-1}_{D,A}(A) = \#\pi^{-1}_{D^{(2)},T}(T) \leq \#\pi^{-1}_{D^{(2)},T}(S). \end{aligned}$$

Thus, $\tau_{D^{(2)},S} := \pi_{D^{(2)},T}$ exhausts $\alpha^{(2)}$ over S, so $\alpha^{(2)} \in \mathcal{S}_{D^{(2)}}$ by Theorem 2.10. ☐

By (11) and Proposition 2.3, we have

$$0 < \#\pi^{-1}_{D,A}(A) < \#D,$$

so $D^{(1)}, D^{(2)} \subsetneq D$. Thus, by Claims 2.20 and 2.21 and induction, there exist

$$\tau_{D^{(1)}} \in \mathsf{PerfectTab}(D^{(1)}, \alpha^{(1)}) \text{ and } \tau_{D^{(2)}} \in \mathsf{PerfectTab}(D^{(2)}, \alpha^{(2)}).$$

Define $\tau_D = \tau_{D^{(1)}} \cup \tau_{D^{(2)}} \in \mathsf{Tab}(D)$ by

$$\tau_D(i,j) = \begin{cases} \tau_{D^{(1)}}(i,j) & \text{if } (i,j) \in D^{(1)}, \\ \tau_{D^{(2)}}(i,j) & \text{if } (i,j) \in D^{(2)}. \end{cases}$$

Clearly τ_D is flagged and satisfies $\tau_D^{-1}(\{\circ\}) = \emptyset$. It has content α because $\alpha = \alpha^{(1)} + \alpha^{(2)}$, and it is column-injective because the images of $\tau_{D^{(1)}}$ and $\tau_{D^{(2)}}$ are disjoint. Therefore, $\tau_D \in \mathsf{PerfectTab}(D, \alpha)$ and $\mathsf{Case\ 2b}$ is complete.

This completes the proof of Theorem 2.13. $\square$

2.2. **Polytopal descriptions of perfect tableaux.** Given $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, define

$$\mathcal{P}(D, \alpha) \subseteq \mathbb{R}^{n^2}$$

to be the polytope with points of the form $(\alpha_{ij})_{i,j\in[n]} = (\alpha_{11}, \ldots, \alpha_{n1}, \ldots, \alpha_{1n}, \ldots, \alpha_{nn})$ governed by the inequalities (A)-(C) below.

(A) Column-Injectivity Conditions: For all $i, j \in [n]$,

$$0 \leq \alpha_{ij} \leq 1.$$

(B) Content Conditions: For all $i \in [n]$,

$$\sum_{j=1}^{n} \alpha_{ij} = \alpha_i.$$

(C) Flag Conditions: For all $s, j \in [n]$,

$$\sum_{i=1}^{s} \alpha_{ij} \geq \#\{(i,j) \in D : i \leq s\}.$$

Proposition 2.22. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$ with $\alpha_1 + \cdots + \alpha_n = \#D$. If $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$, then for each $j \in [n]$, we have*

$$\sum_{i=1}^{n} \alpha_{ij} = \#\{(i,j) \in D : i \in [n]\}.$$

Proof. From the flag conditions (C) where $s = n$, we have that

$$\sum_{i=1}^{n} \alpha_{ij} \geq \#\{(i,j) \in D : i \in [n]\}.$$

If this inequality is strict for any j, then using the content conditions (B), we can write

$$\#D = \alpha_1 + \cdots + \alpha_n = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} \alpha_{ij} > \sum_{j=1}^{n} \#\{(i,j) \in D : i \in [n]\} = \#D,$$

a contradiction. $\square$

Theorem 2.23. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$. Then $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$.*

Proof. (⇒) By Proposition 2.12, we have $\alpha_1 + \cdots + \alpha_n = \#D$. Let $\tau \in \mathsf{PerfectTab}(D, \alpha)$. Then for each $i, j \in [n]$, set

$$\alpha_{ij} = \#\{r \in [n] : \tau(r, j) = i\} = \begin{cases} 1 & \text{if } \tau(r, j) = i \text{ for some } r \in [n], \\ 0 & \text{otherwise,} \end{cases}$$

where the second equality follows from the fact that τ is column-injective.

Claim 2.24. $(\alpha_{ij}) \in \mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2}$.

Proof of Claim 2.24. Clearly $(\alpha_{ij}) \in \mathbb{Z}^{n^2}$ and the column-injectivity conditions (A) hold. Since τ has content α,

$$\sum_{j=1}^{n} \alpha_{ij} = \sum_{j=1}^{n} \#\{r \in [n] : \tau(r, j) = i\} = \#\tau^{-1}(\{i\}) = \alpha_i$$

for each $i \in [n]$, so the content conditions (B) hold. Finally, for each $s, j \in [n]$, we have

$$\sum_{i=1}^{s} \alpha_{ij} = \#\{r \in [n] : \tau(r, j) \leq s\} \geq \#\{(r, j) \in D : r \leq s\}$$

since τ is flagged. Thus, the flag conditions (C) also hold. □

(⇐) Let $(\alpha_{ij}) \in \mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2}$. By the column-injectivity conditions (A), $\alpha_{ij} \in \{0, 1\}$. Thus, by Proposition 2.22, there exists for each $j \in [n]$ a bijection

$$\varphi_j : \{i \in [n] : (i, j) \in D\} \to \{i \in [n] : \alpha_{ij} = 1\}$$

that is order-preserving, i.e., φ_j satisfies $\varphi_j(i) < \varphi_j(i')$ whenever $i < i'$. Define $\tau \in \mathsf{Tab}(D)$ by $\tau(i, j) = \varphi_j(i)$.

Claim 2.25. $\tau \in \mathsf{PerfectTab}(D, \alpha)$.

Proof of Claim 2.25. By construction, $\tau^{-1}(\{\circ\}) = \emptyset$. Since φ_j is injective and order-preserving, τ is strictly increasing along columns, hence column-injective. For each $i \in [n]$, the content conditions (B) imply

$$\tau^{-1}(\{i\}) = \sum_{j=1}^{n} \#\varphi_j^{-1}(\{i\}) = \sum_{j=1}^{n} \alpha_{ij} = \alpha_i,$$

so τ has content α. Finally, the flag conditions (C) show that for each $s, j \in [n]$,

$$\#\{i \leq s : (i, j) \in D\} \leq \sum_{i=1}^{s} \alpha_{ij} = \#\{i \leq s : \alpha_{ij} = 1\},$$

so $\varphi_j(i) \leq i$ for each $(i, j) \in D$ since φ_j is order-preserving. Thus, $\tau(i, j) = \varphi_j(i) \leq i$ and τ is flagged. Hence, $\tau \in \mathsf{PerfectTab}(D, \alpha)$. □

This shows that $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$ and completes the proof of the theorem. □

Remark 2.26. The proof of Claim 2.25 shows that if $\mathsf{PerfectTab}(D, \alpha) \neq \emptyset$, then we can find $\tau \in \mathsf{PerfectTab}(D, \alpha)$ which is not only column-injective, but also strictly increasing along columns, so $\tau(i, j) < \tau(i', j)$ whenever $i < i'$.

Theorem 2.23 formulates the problem of determining if $\mathsf{PerfectTab}(D,\alpha) \neq \emptyset$ in terms of feasibility of an integer linear programming problem. In general, integral feasibility is NP-complete. We now show that in our case, feasibility of the problem is equivalent to feasibility of its LP-relaxation:

Theorem 2.27. *Let $D \subseteq [n]^2$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha_1 + \cdots + \alpha_n = \#D$. Then $\mathcal{P}(D,\alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$ if and only if $\mathcal{P}(D,\alpha) \neq \emptyset$.*

For reasons given in the Introduction, we provide two proofs of this fact.

Proof 1 of Theorem 2.27. We write the constraints (A)-(C) in the form $M\vec{x} \leq \vec{b}$ where M is a $(3n^2+n) \times n^2$ block matrix and $\vec{b}$ is a vector of length $3n^2+n$ of the form

$$M = \begin{pmatrix} M_{A_1} \\ M_{A_2} \\ M_B \\ M_C \end{pmatrix} \text{ and } \vec{b} = (b_i)_{i=1}^{3n^2+n}.$$

Let $\vec{b}_I$ denote the subvector of $\vec{b}$ containing those b_i with $i \in I \subseteq [3n^2+n]$. Also, we use the following coordinatization:

$$\vec{x} = (\alpha_{11}, \ldots, \alpha_{n1}, \alpha_{12}, \ldots, \alpha_{n2}, \ldots, \alpha_{nn})^T.$$

- M_{A_1} is the $n^2 \times n^2$ block corresponding to the condition $0 \leq \alpha_{ij}$ from (A). Thus, $M_{A_1} = -I_{n^2}$ and $b_r = 0$ for $r \in [1, n^2]$.
- M_{A_2} is the $n^2 \times n^2$ block corresponding to $\alpha_{ij} \leq 1$ from (A). Hence, $M_{A_2} = I_{n^2}$ and $b_r = 1$ for $r \in [n^2+1, 2n^2]$.
- M_C is the $n^2 \times n^2$ matrix for (C). Thus,

$$M_C = \begin{pmatrix} M_{C_T} & 0 & \ldots & 0 \\ 0 & M_{C_T} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_{C_T} \end{pmatrix}$$

  where $M_{C_T} = (c_{ij})_{1 \leq i,j \leq n}$ is lower triangular such that $c_{ij} = -1$ for $i \geq j$. Also,

$$b_{(2n^2+n)+n(j-1)+s} = -\#\{(i,j) \in D : i \leq s\}, \text{ for } s, j \in [n].$$

- M_B is the $n \times n^2$ block encoding (B). Take $M_B = \begin{pmatrix} I_n & I_n & \ldots & I_n \end{pmatrix}$ and $\vec{b}_{[2n^2+1, 2n^2+n]} = (\alpha_i)_{i \in [n]}$. Clearly $M_B\vec{x} \leq (\alpha_i)_{i \in [n]}$ encodes the inequalities $\sum_{j=1}^n \alpha_{ij} \leq \alpha_i$. Now, (B) requires $\sum_{j=1}^n \alpha_{ij} = \alpha_i$. However, $\alpha_1 + \cdots + \alpha_n = \#D$ ensures that

$$\begin{pmatrix} M_B \\ M_C \end{pmatrix} \vec{x} \leq \vec{b}_{[2n^2+1, 3n^2+n]} \text{ only if } M_B\vec{x} = (\alpha_i)_{i \in [n]}.$$

Summarizing, $M\vec{x} \leq \vec{b}$ indeed encodes (A)-(C).

Example 2.28. For $n = 2$ consider $\vec{x} = (\alpha_{11}, \alpha_{21}, \alpha_{12}, \alpha_{22})^T$ with $D = \{(1,1), (1,2), (2,2)\} \subset [2] \times [2]$ and $\alpha = (2,1)$.

We have

$$M_{A_1}\vec{x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$M_{A_2}\vec{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \le \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$M_B\vec{x} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \le \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$M_C\vec{x} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{12} \\ \alpha_{22} \end{pmatrix} \le \begin{pmatrix} -\#\{(i,1) \in D : i \le 1\} \\ -\#\{(i,1) \in D : i \le 2\} \\ -\#\{(i,2) \in D : i \le 1\} \\ -\#\{(i,2) \in D : i \le 2\} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -2 \end{pmatrix}$$

Theorem 2.29. *M is a totally unimodular matrix; that is, every minor of M equals $0, 1$, or -1.*

Proof. Suppose M' is a square submatrix of M with k rows from M_{A_1} or M_{A_2}. We show by induction on k that $\det(M') \in \{0, \pm 1\}$.

For the base case $k = 0$, consider M' an $\ell \times \ell$ submatrix of M with only rows from M_B and M_C. Let M'_B, M'_C be the corresponding blocks of M', i.e. $M' = \begin{pmatrix} M'_B \\ M'_C \end{pmatrix}$ where M'_B, or M'_C, is the submatrix of M_B, or M_C respectively, using the rows and columns of M'. Since M_B has one 1 per column, M'_B has at most one 1 per column. By the form of M_C, it is straightforward to row reduce M'_C to obtain a $(0, -1)$-matrix M''_C with at most one -1 in each column. Let $M'' = \begin{pmatrix} M'_B \\ M''_C \end{pmatrix}$, an $\ell \times \ell$ matrix. It is textbook (see [29, Theorem 13.3]) that if a $(0, \pm 1)$-matrix N has at most one 1 and at most one -1 in each column, N is totally unimodular; hence $\det(M') = \pm \det(M'') \in \{0, -1, 1\}$ as desired. Thus the base case holds.

Now suppose M' is a square submatrix of M that contains $k \ge 1$ rows from M_{A_1} or M_{A_2}. Let R be such a row from M_{A_1} or M_{A_2}. If R contains all 0's, $\det(M') = 0$, and we are done. Otherwise R contains a single ± 1. Hence the cofactor expansion for $\det(M')$ along R gives $\det(M') = \pm \det(M'')$ where M'' is a submatrix of M with $k - 1$ rows from M_{A_1} or M_{A_2}. So by induction, $\det(M') \in \{0, \pm 1\}$, as required. □

Since M is totally unimodular then any vertices of $M\vec{x} \le \vec{b}$ are integral [29, Theorem 13.2]. Thus, if $\mathcal{P}(D, \alpha) \neq \emptyset$ then its vertices are integral, i.e., $\mathcal{P}(D, \alpha) \cap \mathbb{Z}^{n^2} \neq \emptyset$. □

Proof 2 of Theorem 2.27. Given a point $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$, we say a pair of sequences

$$(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m) \in [n]^{m+1} \times [n]^m,$$

for some $m \in \mathbb{Z}_{>0}$, is *stable* at (α_{ij}) if the properties (i)-(iv) below hold. The purpose of each property will become clear later.

(i) $r_{m+1} = r_1$.
(ii) For all $k \in [m]$, $\alpha_{r_k c_k}, \alpha_{r_{k+1} c_k} \notin \mathbb{Z}$.
(iii) For all $k \in [m]$, if $i > r_{k+1}$ and $\alpha_{i c_k} \notin \mathbb{Z}$, then $i = r_k$.

(iv) There exists $(r, c) \in [n]^2$ such that

$$\#\{k \in [m] : (r, c) = (r_k, c_k)\} \neq \#\{k \in [m] : (r, c) = (r_{k+1}, c_k)\}.$$

Claim 2.30. *For any* $(\alpha_{ij}) \in \mathcal{P}(D, \alpha) \setminus \mathbb{Z}^{n^2}$*, there exists* $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ *stable at* (α_{ij})*.*

Proof of Claim 2.30. Choose r_1, c_1 such that $\alpha_{r_1c_1} \notin \mathbb{Z}$, and assume that we have fixed r_k, c_k such that $\alpha_{r_kc_k} \notin \mathbb{Z}$. By Proposition 2.22, we have

$$\sum_{i=1}^{n} \alpha_{ic_k} = \#\{(i, c_k) \in D : i \in [n]\} \in \mathbb{Z}.$$

Thus, as $\alpha_{r_kc_k} \notin \mathbb{Z}$, it makes sense to set

$$r_{k+1} = \max\{i \neq r_k : \alpha_{ic_k} \notin \mathbb{Z}\}. \tag{12}$$

If $r_{k+1} = r_\ell$ for some $\ell \in [k]$, then end the construction of these sequences. Otherwise, the content conditions (B) say that

$$\sum_{j=1}^{n} \alpha_{r_{k+1}j} = \alpha_{r_{k+1}} \in \mathbb{Z},$$

and since $\alpha_{r_{k+1}c_k} \notin \mathbb{Z}$, we can choose $c_{k+1} \neq c_k$ such that $\alpha_{r_{k+1}c_{k+1}} \notin \mathbb{Z}$, completing the recursive definition. By the pigeonhole principle, this process must halt, yielding sequences $r_1, \ldots, r_\ell, \ldots, r_{m+1}$ and $c_1, \ldots, c_\ell, \ldots, c_m$ with $r_{m+1} = r_\ell$.

By disregarding the first $\ell - 1$ terms of each sequence, we may assume $\ell = 1$ without loss of generality. Then we assert that $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ is stable at (α_{ij}). Indeed, (i) and (ii) are immediate from the construction, (iii) follows from (12), and (iv) holds because $(r, c) := (r_2, c_2)$ exists and satisfies

$$\#\{k \in [m] : (r, c) = (r_k, c_k)\} = 1 \qquad \text{and} \qquad \#\{k \in [m] : (r, c) = (r_{k+1}, c_k)\} = 0,$$

since $c_2 \neq c_1$ and $r_2 \neq r_k$ for all $k \neq 2$. □

We now fix a pair of sequences $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$. Given (α_{ij}) and $\delta > 0$, set

$$\alpha_{ij}^\delta = \alpha_{ij} + \delta[\#\{k \in [m] : (i, j) = (r_k, c_k)\} - \#\{k \in [m] : (i, j) = (r_{k+1}, c_k)\}]. \tag{13}$$

Claim 2.31. *If* $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ *is stable at* $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$*, then* $(\alpha_{ij}^\delta) \in \mathcal{P}(D, \alpha)$ *for some* $\delta > 0$*.*

Proof of Claim 2.31. First, note that the content conditions (B) are preserved regardless of our choice of δ. Indeed, for each $i \in [n]$,

$$\begin{aligned}\sum_{j=1}^{n} \alpha_{ij}^\delta &= \sum_{j=1}^{n} [\alpha_{ij} + \delta[\#\{k \in [m] : (i, j) = (r_k, c_k)\} - \#\{k \in [m] : (i, j) = (r_{k+1}, c_k)\}]] \\ &= \alpha_i + \delta[\#\{k \in [m] : i = r_k\} - \#\{k \in [m] : i = r_{k+1}\}],\end{aligned}$$

and the term in brackets vanishes by (i).

We next check the flag conditions (C). For each $s, j \in [n]$, we can write

$$\begin{aligned}\sum_{i=1}^{s} \alpha_{ij}^{\delta} &= \sum_{i=1}^{s} [\alpha_{ij} + \delta[\#\{k \in [m] : (i,j) = (r_k, c_k)\} - \#\{k \in [m] : (i,j) = (r_{k+1}, c_k)\}]] \\ &= \sum_{i=1}^{s} \alpha_{ij} + \delta[\#\{k \in [m] : s \geq r_k \text{ and } j = c_k\} - \#\{k \in [m] : s \geq r_{k+1} \text{ and } j = c_k\}] \\ &\geq \sum_{i=1}^{s} \alpha_{ij} - \delta[\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\}]. \quad (14)\end{aligned}$$

Thus, if $\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\} = 0$, then the flag condition (C) for these s, j is preserved.

Otherwise, $r_{k+1} \leq s < r_k$ and $j = c_k$ for some $k \in [m]$, so (ii) and (iii) tell us that there is exactly one $i > s$ for which $\alpha_{ij} \notin \mathbb{Z}$, namely $i = r_k$. This, combined with Proposition 2.22, shows that

$$\sum_{i=1}^{s} \alpha_{ij} = \sum_{i=1}^{n} \alpha_{ij} - \sum_{i=s+1}^{n} \alpha_{ij} = \#\{(i,j) \in D : i \in [n]\} - \sum_{i=s+1}^{n} \alpha_{ij} \notin \mathbb{Z}. \quad (15)$$

By the nonintegrality from (15), the flag inequalities (C) for $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$ are strict:

$$\sum_{i=1}^{s} \alpha_{ij} > \#\{(i,j) \in D : i \leq s\}. \quad (16)$$

Hence, by taking δ sufficiently small and applying (14) and (16), we can ensure

$$\sum_{i=1}^{s} \alpha_{ij}^{\delta} \geq \sum_{i=1}^{s} \alpha_{ij} - \delta[\#\{k \in [m] : r_{k+1} \leq s < r_k \text{ and } j = c_k\}] \geq \#\{(i,j) \in D : i \leq s\}$$

for all $s, j \in [n]$, so the flag conditions (C) will be preserved. If $\alpha_{ij} \neq \alpha_{ij}^{\delta}$ then by (13) we must have $(i,j) = (r_k, c_k)$ or $(i,j) = (r_{k+1}, c_k)$ for some k, which by (ii) implies $0 < \alpha_{ij} < 1$. So we can require in addition that δ be small enough that $0 \leq \alpha_{ij}^{\delta} \leq 1$ for all $i, j \in [n]$. For such δ, the conditions (A)-(C) all hold, so $(\alpha_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$. □

Finally, choose a point $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$ with the maximum number of integer coordinates. If $(\alpha_{ij}) \in \mathbb{Z}^{n^2}$, then we are done. Otherwise, there exists $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ that is stable at (α_{ij}) by Claim 2.30. By (iv), there exists $(r, c) \in [n]^2$ such that $|\alpha_{rc}^{\delta}| \to \infty$ as $\delta \to \infty$, so α_{rc}^{δ} violates the column-injectivity conditions (A) for large δ. This, combined with Claim 2.31, shows that the set $S = \{\delta > 0 : (\alpha_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)\}$ is nonempty and bounded above. Thus, we can define $\eta = \sup S$ and set $(\tilde{\alpha}_{ij}) = (\alpha_{ij}^{\eta})$. Since $\mathcal{P}(D, \alpha)$ is closed and the map $\delta \mapsto (\alpha_{ij}^{\delta})$ from S to $\mathcal{P}(D, \alpha)$ is continuous, this supremum is in fact a maximum, and $(\tilde{\alpha}_{ij}) \in \mathcal{P}(D, \alpha)$. By our choice of (α_{ij}), we cannot have $\tilde{\alpha}_{r_k c_k} \in \mathbb{Z}$ or $\tilde{\alpha}_{r_{k+1} c_k} \in \mathbb{Z}$ for any $k \in [m]$, since then $(\tilde{\alpha}_{ij})$ has more integer coordinates than (α_{ij}). Thus, $(r_1, \ldots, r_{m+1}; c_1, \ldots, c_m)$ is stable at $(\tilde{\alpha}_{ij})$, so by Claim 2.31, there exists $\delta > 0$ for which $(\tilde{\alpha}_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$. But then $(\alpha_{ij}^{\eta+\delta}) = (\tilde{\alpha}_{ij}^{\delta}) \in \mathcal{P}(D, \alpha)$, contradicting the maximality of η. □

If $D \subseteq [n]^2$ has many identical columns, then many of the flag conditions (C) will look essentially the same. Therefore, our final goal will be to construct a "compressed" version of $\mathcal{P}(D, \alpha)$ that removes some of the repetitive inequalities.

A tuple $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\lambda_k\}_{k=1}^{\ell})$ is a *compression* of $D \subseteq [n]^2$ if:

- $m \leq n$ is a nonnegative integer such that $(r,p) \notin D$ whenever $r > m$ and $p \in [n]$,
- $P = P_1 \dot\cup \cdots \dot\cup P_\ell \subseteq [n]$ such that if $p, p' \in P_k$ then
$$\{r \in [n] : (r,p) \in D\} = \{r \in [n] : (r,p') \in D\},$$
and moreover if D is nonempty in column p then $p \in P_k$ for some $k \in [\ell]$.
- $p_k \in P_k$ a representative for each $k \in [\ell]$, and
- $\lambda_k = \#P_k$ for each $k \in \ell$.

For $D \subseteq [n]^2$, a compression $\mathcal{C}$ of D, and $\tilde\alpha = (\tilde\alpha_1, \ldots, \tilde\alpha_m) \in \mathbb{Z}_{\geq 0}^m$ define

$$\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) \subseteq \mathbb{R}^{m\ell} \tag{17}$$

to be the polytope with points of the form $(\tilde\alpha_{ik})_{i\in[m],k\in[\ell]}$ satisfying (A′)-(C′) below.

(A′) Column-Injectivity Conditions: For all $i \in [m], k \in [\ell]$,
$$0 \leq \tilde\alpha_{ik} \leq 1.$$

(B′) Content Conditions: For all $i \in [m]$,
$$\sum_{k=1}^{\ell} \lambda_k \tilde\alpha_{ik} = \alpha_i.$$

(C′) Flag Conditions: For all $s \in [m], k \in [\ell]$,
$$\sum_{i=1}^{s} \tilde\alpha_{ik} \geq \#\{(i, p_k) \in D : i \leq s\}.$$

Remark 2.32. We can always take $m = \ell = n$ and $P_k = \{k\}$ for each $k \in [\ell]$, in which case $\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) = \mathcal{P}(D, \alpha) \subseteq \mathbb{R}^{n^2}$.

Theorem 2.33. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$ and $\tilde\alpha = (\tilde\alpha_1, \ldots, \tilde\alpha_m) := (\alpha_1, \ldots, \alpha_m)$. Then $\alpha_1 + \cdots + \alpha_n = \#D$ and $\mathcal{P}(D, \alpha) \neq \emptyset$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde\alpha) \neq \emptyset$.*

Proof. ($\Rightarrow$) Let $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$. Then by the content and flag conditions (B) and (C),

$$\begin{aligned}\#D = \alpha_1 + \cdots + \alpha_n \geq \alpha_1 + \cdots + \alpha_m &= \sum_{i=1}^{m}\sum_{j=1}^{n} \alpha_{ij} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} \alpha_{ij} \geq \sum_{j=1}^{n} \#\{(i,j) \in D : i \leq m\} = \#D.\end{aligned}$$

Thus, $\alpha_1 + \cdots + \alpha_m = \#D$ and $\alpha_{m+1} = \cdots = \alpha_n = 0$. Now, for each $i \in [m]$ and $k \in [\ell]$, set
$$\tilde\alpha_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \alpha_{ij}.$$

We claim that $(\tilde\alpha_{ik}) \in \mathcal{Q}(D, \mathcal{C}, \alpha)$. First, for each $i \in [m]$ and $k \in [\ell]$, we have
$$0 \leq \tilde\alpha_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \alpha_{ij} \leq \frac{1}{\lambda_k} \sum_{j \in P_k} 1 = 1,$$

so the column-injectivity conditions (A′) are satisfied. Next, for each $i \in [m]$, (B) implies

$$\sum_{k=1}^{\ell} \lambda_k \tilde{\alpha}_{ik} = \sum_{k=1}^{\ell} \sum_{j \in P_k} \alpha_{ij} = \sum_{j=1}^{n} \alpha_{ij} = \alpha_i,$$

so the content conditions (B′) are satisfied. Finally, for each $s \in [m]$ and $k \in [\ell]$, (C) implies

$$\sum_{i=1}^{s} \tilde{\alpha}_{ik} = \frac{1}{\lambda_k} \sum_{j \in P_k} \sum_{i=1}^{s} \alpha_{ij} \geq \frac{1}{\lambda_k} \sum_{j \in P_k} \#\{(i,j) \in D : i \leq s\} = \#\{(i,p_k) \in D : i \leq s\},$$

so the flag conditions (C′) are satisfied.

(⇐) Clearly $\alpha_1 + \cdots + \alpha_n = \#D$. Let $(\tilde{\alpha}_{ik}) \in \mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$. For each $i, j \in [n]$, set

$$\alpha_{ij} = \begin{cases} 0 & \text{if } i > m, \\ \tilde{\alpha}_{ik} & \text{if } i \leq m \text{ and } j \in P_k. \end{cases}$$

We claim that $(\alpha_{ij}) \in \mathcal{P}(D, \alpha)$. The column-injectivity conditions (A) are clear. If $i > m$,

$$\sum_{j=1}^{n} \alpha_{ij} = 0 = \alpha_i.$$

Otherwise $i \leq m$, and (B′) implies

$$\sum_{j=1}^{n} \alpha_{ij} = \sum_{k=1}^{\ell} \sum_{j \in P_k} \tilde{\alpha}_{ik} = \sum_{k=1}^{\ell} \lambda_k \tilde{\alpha}_{ik} = \alpha_i.$$

Thus, the content conditions (B) hold. Finally, if $s \in [n]$ and $j \in P_k$, then (C′) implies

$$\sum_{i=1}^{s} \alpha_{ij} = \sum_{i=1}^{\min\{s,m\}} \tilde{\alpha}_{ik} \geq \#\{(i,p_k) \in D : i \leq \min\{s,m\}\} = \#\{(i,j) \in D : i \leq s\}.$$

Hence, the flag conditions (C) hold as well. □

2.3. **Deciding membership in the Schubitope.** We use the above results of this section to give a polynomial time algorithm to check if a lattice point is in the Schubitope.

Let $D \subseteq [n]^2$, and fix a compression $\mathcal{C} = (m, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\lambda_k\}_{k=1}^{\ell})$ of D (as in Section 2.2).

Theorem 2.34. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$. Then $\alpha \in \mathcal{S}_D$ if and only if $\alpha_1 + \cdots + \alpha_m = \#D$, $\alpha_{m+1} = \cdots = \alpha_n = 0$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \neq \emptyset$, where $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) := (\alpha_1, \ldots, \alpha_m)$.*

Proof. This follows from Theorems 2.13, 2.23, 2.27, and 2.33. □

For each $k \in [\ell]$, let $R_k(\mathcal{C}) = \{r \in [n] : (r, p_k) \in D\} \subseteq [m]$.

Theorem 2.35. *Given as input $\{R_k(\mathcal{C})\}_{k=1}^{\ell}$, $\{\lambda_k\}_{k=1}^{\ell}$, and $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) \in \mathbb{Z}_{\geq 0}^m$ satisfying $\tilde{\alpha}_1 + \cdots + \tilde{\alpha}_m = \#D$, one can decide if $\alpha := (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m, 0, \ldots, 0) \in \mathbb{Z}_{\geq 0}^n$ lies in $\mathcal{S}_D$ in polynomial time in m and ℓ.*

Remark 2.36. In view of Theorem 2.34, this input is most natural, because the conditions $\alpha_1 + \cdots + \alpha_m = \#D$ and $\alpha_{m+1} = \cdots = \alpha_n = 0$ are clearly necessary, and it contains the minimum amount of information we need to compute $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$.

Remark 2.37. As in Remark 2.32, we can take $m = \ell = n$ and $P_k = \{k\}$ for each $k \in [\ell]$, so we can check if α is in $\mathcal{S}_D$ in polynomial time in n regardless of the structure of D.

Proof of Theorem 2.35. Since $R_k(\mathcal{C})$ takes m bits to encode for each $k \in [\ell]$, and $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha}) \subseteq \mathbb{R}^{m\ell}$ is governed by $O(m\ell)$ constraints, $\mathcal{Q}(D, \mathcal{C}, \tilde{\alpha})$ can be constructed in polynomial time in m and ℓ. By Theorem 2.34, we are done since $\mathsf{LPfeasibility} \in \mathsf{P}$ (see Section 1.2). $\square$

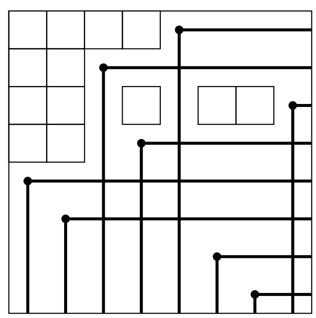
3. Rothe diagrams

The *graph* $G(w)$ of a permutation $w \in S_n$ is the $n \times n$ grid, with a $\bullet$ placed in position $(i, w(i))$ (in matrix coordinates). The *Rothe diagram* of w is given by

$$D(w) = \{(i,j) : 1 \leq i, j \leq n, j < w(i), i < w^{-1}(j)\}.$$

This is pictorially described with rays that strike out boxes south and east of each $\bullet$ in $G(w)$. $D(w)$ are the remaining boxes. If it exists, we call the connected component of $D(w)$ involving $(1,1)$ the *dominant component*, and denote it by $\mathsf{Dom}(w)$.

Example 3.1. If $w = 53841267 \in S_8$ (in one line notation) then $D(w)$ is depicted by:



In this example, $\mathsf{Dom}(w)$ has shape $(4, 2, 2, 2)$.

The *code* of w, denoted $\mathsf{code}(w)$ is the vector $(c_1, c_2, \ldots, c_L)$ where c_i is the number of boxes in the i-th row of $D(w)$ and L indexes the southmost row with a positive number of boxes. For example, $\mathsf{code}(53841267) = (4, 2, 5, 2)$. To each permutation $w \in S_\infty$ there is a unique associated code; see [25, Proposition 2.1.2].

We will repeatedly use:

Proposition 3.2. *There exists an $O(L^2)$-time algorithm to compute $(w(1), \ldots, w(L))$ from the input $\mathsf{code}(w) = (c_1, \ldots, c_L)$.*

Proof. Clearly $w(1) = c_1 + 1$. After determining $w(1), \ldots, w(i-1)$, we determine (in $O(L)$-time), $\pi := \pi^{(i)} \in S_{i-1}$ such that $(w(\pi(1)) < w(\pi(2)) < \ldots < w(\pi(i-1)))$. Next, set

$$B := (w(\pi(1)), w(\pi(2)) - w(\pi(1)), w(\pi(3)) - w(\pi(2)), \ldots, w(\pi(i-1)) - w(\pi(i-2)))).$$

Let

$$V_t := \sum_{j=1}^{t} (B_j - 1), \quad \text{for } 0 \leq t \leq i-1.$$

Set $w(i) := c_i + T + 1$ where $T := \max_{t \in [0, i-1]}\{t \ : \ c_i \geq V_t\}$. By construction, $w(1), \ldots, w(i)$ is a partial permutation with code $(c_1, \ldots, c_{i-1}, c_i)$. Each stage $1 \leq i \leq L$ takes $O(i)$-time. $\square$

The *essential set* of w consists of the maximally southeast boxes of each connected component of $D(w)$, i.e.,

$$\mathsf{Ess}(w) = \{(i,j) \in D(w) : (i+1,j), (i,j+1) \notin D(w)\}. \tag{18}$$

If it exists, the *accessible box* $\mathbf{z}_w$ is the southmost then eastmost box in $\mathsf{Ess}(w) \smallsetminus \mathsf{Dom}(w)$. In Example 3.1,

$$\mathsf{Ess}(w) = \{(1,4),(3,4),(3,7),(4,2)\} \text{ and } \mathbf{z}_w = (3,7).$$

(Although $(4,2)$ is the southmost box of $\mathsf{Ess}(w)$, it is in $\mathsf{Dom}(w)$, and hence not the accessible.)

We will need the following in Section 5:

Proposition 3.3. *Given* $\mathsf{code}(w)$, *there exists an* $O(L^2)$*-time algorithm to compute* $\mathbf{z}_w = (r,c)$ *or determine it does not exist.*

Proof. Use Proposition 3.2 to find $(w(1),\ldots,w(L))$ in $O(L^2)$-time. Next, compute

$$w_{NW}(i) := \{w(j) : w(j) \leq w(i),\ j \leq i\}.$$

Then take

$$Y(i) := \{q-1 \ : \ q \in w_{NW}(i)\} \smallsetminus w_{NW}(i), \ \text{ for } i \in [L].$$

Compute $k_i := \max Y(i)$ for $i \in [L]$ in $O(L^2)$-time (if $k_i \geq 1$, then k_i is the column index of the eastmost box of $D(w)$ in row i). In $O(L^2)$-time, calculate

$$I := \{i \in [2,\ldots,L] \ : \ k_i > \min_{j<i} w(j)\}.$$

Let $Y := \{(i,k_i) \ : \ i \in I\}$. Hence, $Y \cap \mathsf{Dom}(w) = \emptyset$. Thus, if $Y = \emptyset$, $\mathbf{z}_w$ does not exist. Otherwise, $\mathbf{z}_w \in Y$. Thus, in $O(L)$-time, determine $r := \max\{i : (i,k_i) \in Y\}$. Output $\mathbf{z}_w = (r,k_r)$. ☐

The *pivots* of $\mathbf{z}_w$ denoted $\mathsf{Piv}(\mathbf{z}_w)$ are the $\bullet$'s of $D(w)$ that are maximally southeast, among those northwest of $\mathbf{z}_w$. In our example, $\mathsf{Piv}((3,7)) = \{(2,3),(1,5)\}$.

4. Proofs of Theorems 1.13 and 1.14

4.1. **Proof of Theorem 1.13.** Fix $w \in S_\infty$ with $\mathsf{code}(w) = (c_1,\ldots,c_L)$. Let $\sigma \in S_L$ be such that $\{w(\sigma(1)) < w(\sigma(2)) < \ldots < w(\sigma(L))\}$. For convenience, set $w(\sigma(0)) := 0$.

Lemma 4.1. *For* $1 \leq h \leq L$, *and for all*

$$j_1, j_2 \in \{w(\sigma(h-1))+1, w(\sigma(h-1))+2, \ldots, w(\sigma(h))-1\},$$

we have $(i,j_1) \in D(w)$ *if and only if* $(i,j_2) \in D(w)$.

Proof. For each k, let $u_1^{(k)} < \ldots < u_k^{(k)}$ be $w(1), w(2), \ldots, w(k)$ sorted in increasing order. Set $u_0^{(k)} := 0$. The lemma follows from the inductive claim that in the first k rows of $D(w)$, the columns $u_{h-1}^{(k)}+1, u_{h-1}^{(k)}+2, \ldots, u_h^{(k)}-1$ are the same. The base case $k=1$ is clear. The inductive step is straightforward by considering how, in row $k+1$ of $D(w)$, the $\bullet$ and its ray emanating east affects the columns. ☐

Define a collection of intervals in $[n]$ by

$$P'_{2k-1} := [w(\sigma(k-1)) + 1, w(\sigma(k)) - 1] \text{ and } P'_{2k} := \{w(\sigma(k))\}, \text{ for } 1 \leq k \leq L.$$

Let $1 \leq h_1 < h_2 < \ldots < h_\ell \leq 2L$ be indices of the intervals P'_h that are nonempty. Set $P_i := P'_{h_i}$.

Lemma 4.2. *If $j_1, j_2 \in P_k$ for some k, then $(i, j_1) \in D(w) \iff (i, j_2) \in D(w)$.*

Proof. This follows by the definition of $\{P_k\}_{k=1}^{\ell}$ together with Lemma 4.1. □

Let $p_k := \min\{p \in P_k\}$ for each $k \in [\ell]$.

Proposition 4.3. *There exists an $O(L^2)$-time algorithm to compute $\{P_k\}_{k=1}^{\ell}$, $\{p_k\}_{k=1}^{\ell}$, and $\{\#P_k\}_{k=1}^{\ell}$ from the input $\mathsf{code}(w) = (c_1, \ldots, c_L)$.*

Proof. Proposition 3.2 computes $(w(1), \ldots, w(L))$ in $O(L^2)$-time. It takes $O(L\log(L))$-time to sort $(w(1), \ldots, w(L))$, i.e., to compute $\sigma \in S_L$. Computing the endpoints, and thus cardinalities, of the P'_k takes $O(L)$-time as there are at most $2L$ of them. Then we reindex $\{\#P'_k\}_{k=1}^{2L}$ to obtain $\{\#P_k\}_{k=1}^{\ell}$ in $O(L)$-time. □

For each $k \in [\ell]$, let

$$R_k := \{r \in [L] : (r, p_k) \in D(w)\}.$$

Proposition 4.4. *Computing $\{R_k\}_{k=1}^{\ell}$ from $\mathsf{code}(w)$ takes $O(L^2)$-time.*

Proof. By $D(w)$'s definition, $r \in R_k$ if and only if $w(r) > p_k$ and $p_k \notin \{w(i) : i < r\}$. Propositions 4.3 and 3.2 give $\{P_k\}_{k=1}^{\ell}$, $\{p_k\}_{k=1}^{\ell}$ and $\{w(1), \ldots, w(L)\}$ in $O(L^2)$-time. □

Conclusion of proof of Theorem 1.13: Proposition 4.3 computes $\{P_k\}_{k=1}^{\ell}$, $\{p_k\}_{k=1}^{\ell}$, and $\{\#P_k\}_{k=1}^{\ell}$ in $O(L^2)$-time. Proposition 4.4 finds $\{R_k\}_{k=1}^{\ell}$ in $O(L^2)$-time. One checks, using Lemma 4.2, that $\mathcal{C} = (L, \{P_k\}_{k=1}^{\ell}, \{p_k\}_{k=1}^{\ell}, \{\#P_k\}_{k=1}^{\ell})$ is a compression of $D(w)$. Hence we may apply Theorem 2.35 by taking $D := D(w)$, $R_k(\mathcal{C}) := R_k$, $\lambda_k := \#P_k$ for $k \in [\ell]$ and $m := L$. □

4.2. **Proof of Theorem 1.14; an application.** By [7], $\alpha \in \mathcal{S}_{D(w)} \iff c_{\alpha,w} > 0$. By Theorem 2.13, $\alpha \in \mathcal{S}_{D(w)}$ if and only if $\mathsf{PerfectTab}(D(w), \alpha) \neq \emptyset$. Notice that $\mathsf{Tab}_{\neq}(D(w), \alpha) = \mathsf{PerfectTab}(D(w), \alpha)$. This combined with Remark 2.26 proves the theorem. □

Let $n_{132}(w)$ be the number of 132-patterns in $w \in S_n$, that is,

$$n_{132}(w) = \#\{(i, j, k) : 1 \leq i < j < k \leq n, w(i) < w(k) < w(j)\}.$$

Corollary 4.5. *There are at least $n_{132}(w) + 1$ distinct vectors α such that $c_{\alpha,w} > 0$.*

Proof. Suppose $i < j < k$ index a 132 pattern in w. There is a box b of $D(w)$ in row j and column $w(k)$. There are $N := n_{132}(w)$ many such boxes, $\mathsf{b}_1, \ldots, \mathsf{b}_N$ (all distinct), listed in English language reading order. Let M_i be boxes in the same column and connected component as b_i that are weakly north of b_i and strictly south of any b_j, where $j < i$. Iteratively define fillings $F_0, F_1, F_2, \ldots, F_N$ of $D(w)$:

(F_0) Fill each box c of $D(w)$ with the row number of c.
(F_i) For $1 \leq i \leq N$, F_i is the same as F_{i-1} except that $F_i(\mathsf{c}) := F_{i-1}(\mathsf{c}) - 1$ if $\mathsf{c} \in M_i$.

Clearly, $F_0 \in \mathsf{Tab}_<(D(w)) := \bigcup_\alpha \mathsf{Tab}_<(D(w),\alpha)$. Inductively assume $F_{i-1} \in \mathsf{Tab}_<(D(w))$. Since labels only decrease, F_i satisfies the row bound condition. Next we check that each column is strictly increasing. Let m_i be the northmost box of M_i. If m_i is adjacent and directly below some b_j (for a $j < i$) then

$$F_i(\mathsf{b}_j) = F_0(\mathsf{b}_j) - 1 < F_0(\mathsf{m}_i) - 1 = F_i(\mathsf{m}_i),$$

as needed. Otherwise suppose m_i is adjacent and south of a non-diagram position. Let d_i (if it exists) be the first diagram box directly north of m_i. Then $F_0(\mathsf{d}_i) < F_0(\mathsf{m}_i) - 1$. Hence

$$F_i(\mathsf{d}_i) \leq F_0(\mathsf{d}_i) < F_0(\mathsf{m}_i) - 1 = F_i(\mathsf{m}_i),$$

verifying column increasingness here as well. That F_i is column increasing elsewhere is clear since F_{i-1} is column increasing (by induction) and only labels of M_i are changed.

It remains to check that every label of F_i is in $\mathbb{Z}_{>0}$. Since each box of $D(w)$ is decremented at most once, the only concern is there is a box x in the first row that appears in some M_i, since then $F_0(\mathsf{x}) = 1$ and $F_i(\mathsf{x}) = 0$. However, in this case b_i must be in $\mathsf{Dom}(w)$, which implies that the "1" in the 132-pattern associated to b_i could not exist, a contradiction. Thus $F_i \in \mathsf{Tab}_<(D(w))$, completing the induction.

Finally, under Theorem 1.14, each F_i corresponds to a distinct exponent vector since the sum of the labels is strictly decreasing at each step $F_{i-1} \mapsto F_i$. □

From Corollary 4.5, this result of A. Weigandt [38] is immediate:

Corollary 4.6 (A. Weigandt's 132-bound)**.** $\mathfrak{S}_w(1,1,1,\ldots,1) \geq n_{132}(w) + 1$.

As shown in [38], Corollary 4.6 in turn implies $\mathfrak{S}_w(1,1,\ldots,1) \geq 3$ if $n_{132}(w) \geq 2$, a recent conjecture of R. P. Stanley [35].

5. COUNTING $c_{\alpha,w}$ IS IN #P

5.1. **Vexillary permutations.** A permutation $w \in S_n$ is *vexillary* if there does not exist a 2143 *pattern*, i.e., indices $i < j < k < l$ such that w has the pattern $w(j) < w(i) < w(l) < w(k)$. For example, $w = \underline{5}\underline{3}\underline{8}41\underline{2}\underline{6}7$ is not vexillary; we underlined the positions of a 2143 pattern. *Fulton's criterion* states that w is vexillary if and only if there do not exist $(a,b),(c,d) \in \mathsf{Ess}(w)$ such that $a < c$ and $b < d$. In Example 3.1, w is not vexillary due to $(1,4)$ and $(3,7)$. Our main reference for this subsection is [25, Chapter 2].

We will also use this characterization of vexillary permutations:

Theorem 5.1. [19] *Given* $\mathsf{code}(w) = (c_1,\ldots,c_L) \in \mathbb{Z}^n_{\geq 0}$*,* w *vexillary if and only if*

(i) *if* i *is such that* $c_i > c_{i+1}$*, then* $c_i > c_j$ *for any* $j > i$*, and*
(ii) *if* i,h *are such that* $c_i \geq c_h$*, then* $\#\{j \,:\, i < j < h,\ c_j < c_h\} \leq c_i - c_h$.

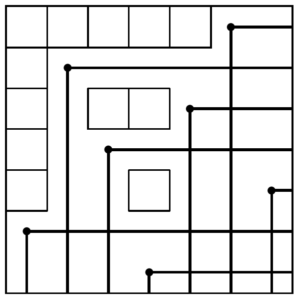
The *shape* of a vexillary permutation v is the partition $\lambda(v)$ formed by sorting $\mathsf{code}(v) = (c_1, c_2, \ldots)$ into decreasing order. Now, if $c_i \neq 0$, let e_i be the greatest integer $j \geq i$ such that $c_j \geq c_i$. The *flag*

$$\phi(v) = (\phi_1 \leq \phi_2 \leq \ldots \leq \phi_m)$$

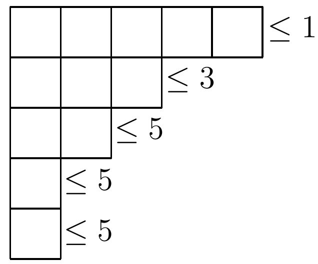
for v is the sequence of e_i's sorted into increasing order; see, e.g., [25, Definition 2.2.9].

Example 5.2. Consider $\mathsf{code}(v) = (5,1,3,1,2)$ for the vexillary $v = 6253714$. Here

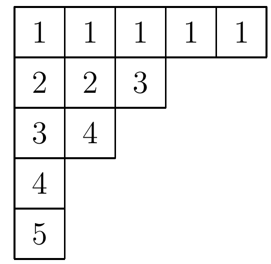
$$e = (1,5,3,5,5), \phi(v) = (1,3,5,5,5) \text{ and } \lambda(v) = (5,3,2,1,1).$$



$D(v)$



$\lambda(v)$ flagged by $\phi(v)$



$T \in \mathsf{SSYT}(\lambda(v), \phi(v))$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m > 0)$ and a flag $\phi = (\phi_1 \leq \phi_2 \leq \ldots \leq \phi_m)$ of positive integers, define the *flagged Schur function*

$$S_\lambda(\phi) = \det |h_{\lambda_i - i + j}(\phi_i)|_{i,j=1,\ldots,m},$$

where

$$h_k(n) = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} x_{i_1} \cdots x_{i_k}$$

is the complete homogeneous symmetric polynomial of degree k. Furthermore,

$$\mathfrak{S}_v = S_{\lambda(v)}(\phi(v)), \text{ for } v \text{ vexillary}. \tag{19}$$

A semistandard Young tableau of shape λ is *flagged* by ϕ if its entries in row i are $\leq \phi_i$; see Example 5.2. Denote the set of such tableaux by $\mathsf{SSYT}(\lambda, \phi)$. Then

$$S_\lambda(\phi) = \sum_{T \in \mathsf{SSYT}(\lambda,\phi)} x^{\mathsf{content}(T)}. \tag{20}$$

where $\mathsf{content}(T) = (\mu_1, \ldots, \mu_{\ell(\lambda)})$ such that μ_i is the number of i's in T.

5.2. **Graphical transition.** The transition recurrence for $\mathfrak{S}_w$ was found by A. Lascoux and M.-P. Schützenberger [19]. This is transition for the case discussed in [17]:

Theorem 5.3 ([19], cf. [17])**.** *Let* $\mathbf{z}_w = (r, c)$ *and* $w' = w \cdot (r\ k)$ *where* $k = w^{-1}(c)$*. Then*

$$\mathfrak{S}_w = x_r \mathfrak{S}_{w'} + \sum_{w'' = w' \cdot (i\ k)} \mathfrak{S}_{w''}, \tag{21}$$

where the summation is over $\{i : (i, w(i)) \in \mathsf{Piv}(\mathbf{z}_w)\}$.

We will use the *graphical* transition tree $\mathcal{T}(w)$ of [17]. This reformulates (21) in terms of Rothe diagrams and certain moves on these diagrams. By definition, $D(w)$ (equivalently w) will label the root of $\mathcal{T}(w)$. If w is vexillary, stop. Otherwise, there exists an accessible box $\mathbf{z}_w = (r, c) \in D(w)$ (if not, $D(w) = \mathsf{Dom}(w)$, contradicting w is not vexillary).

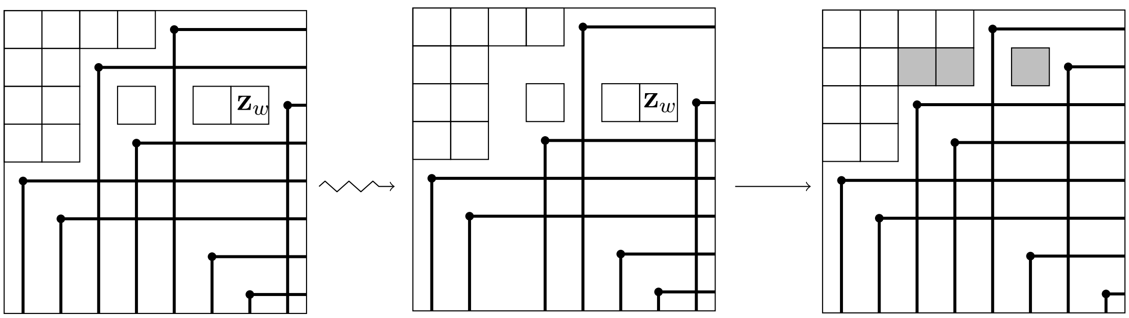
The children of $D(w)$ are Rothe diagrams resulting from two types of moves:

(T.1) *Deletion moves*: remove $\mathbf{z}_w$ from $D(w)$. The resulting diagram is $D(w')$. Add an edge $D(w) \xrightarrow{x_r} D(w')$.

(T.2) *March moves*: There is a move for each $\mathbf{x}^{(i)} = (i, w(i)) \in \mathsf{Piv}(\mathbf{z}_w)$. Let $\mathcal{R}$ be the rectangle with corners $\mathbf{z}_w$ and $\mathbf{x}^{(i)}$. Remove $\mathbf{x}^{(i)}$ and its rays from $G(w)$ to form $G^{(i)}(w)$. Order the boxes $\{\mathsf{b}_i\}_{i=1}^{r}$ in $\mathcal{R}$ in English reading order. Move b_1 strictly north and strictly west to the closest position not occupied by other boxes of $D(w)$ or rays from $G^{(i)}(w)$. Repeat with $\mathsf{b}_2, \mathsf{b}_3, \ldots$ where b_j may move into a square left unoccupied by earlier moves. The resulting diagram will be $D(w'')$ where $w'' = w' \cdot (i\ k)$. Add an edge $D(w) \xrightarrow{i} D(w'')$.

Repeat for each child $D(u)$. Stop when u vexillary; these permutations are the leaves $\mathcal{L}(w)$ of $\mathcal{T}(w)$. (Multiple leaves may be labelled by the same permutation.)

Example 5.4. Let $w = 53841267$. We compute the march move 2 for the pivot $(2, 3)$:



$w = 53841267$ remove hook at $(2, 3)$ $w'' = 57341268$

The moved boxes during $D(w) \mapsto D(w'')$ are shaded gray.

Example 5.5. Let $w = 53861247$. Using $\mathcal{T}(w)$ from Figure 1, we compute

$$\mathfrak{S}_w = x_4 \cdot \mathfrak{S}_{73541268} + x_4 \cdot \mathfrak{S}_{57341268} + x_3^2 x_4 \cdot \mathfrak{S}_{53641278} + x_3 x_4 \cdot \mathfrak{S}_{63541278} + x_3 x_4 \cdot \mathfrak{S}_{56341278}$$
$$+ \mathfrak{S}_{74531268} + \mathfrak{S}_{57431268} + x_3^2 \cdot \mathfrak{S}_{54631278} + x_3 \cdot \mathfrak{S}_{64531278} + x_3 \cdot \mathfrak{S}_{56431278}.$$

For instance, $c_{(4,2,5,3),w} := [x_1^4 x_2^2 x_3^5 x_4^3]\mathfrak{S}_w = 1$ is witnessed by

- the path $w \xrightarrow{x_4} \bullet \xrightarrow{x_3} \bullet \xrightarrow{x_3} u = 53641278$, and
- the semistandard tableau

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 3 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2}\end{array}$$ of shape $\lambda(u)$, flagged by $\phi(u) = (1, 3, 4, 4)$.

Proposition 5.9 below formalizes a rule for $c_{\alpha,w}$ in terms of such pairs.

5.3. **Proof of #P-ness.** The technical core of our proof of Theorem 1.16 is to show:

Theorem 5.6. *The problem of computing $c_{\alpha,w}$, given input α and* $\mathsf{code}(w)$, *is in* #P.

Define X to be the set consisting of pairs (S, R) where:

(X.1) $S = (s_1, \ldots, s_h)$, $s_t \in [L] \cup \{(x_k, m_t) \;:\; k \in [L], m_t \in \mathbb{Z}_{>0}\}$ such that if $s_t = (x_k, m_t)$ then $s_{t+1} \neq (x_k, m_{t+1})$ for $t < h$, and
(X.2) $R = (r_{ij})_{1 \leq i,j \leq L}$, where $r_{ij} \in \mathbb{Z}_{\geq 0}$.

Fix $w \in S_\infty$ and a vexillary permutation $v \in S_\infty$. A (w, v)-*transition string* is a sequence $S = (s_1, \ldots, s_h)$ satisfying (X.1) such that if we interpret i as $\bullet \xrightarrow{i} \bullet$ and (x_k, m_t) as $\bullet \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} \bullet$ (m_t-times) then S describes a path from w to (a leaf labelled by) v in $\mathcal{T}(w)$. Let $\mathsf{Trans}(w, v)$ be the set of such sequences.

The *deletion weight* of $S \in \mathsf{Trans}(w, v)$ is

$$\mathsf{delwt}(S) = \sum m_t \cdot \vec{e_r},$$

where the summation is over $1 \le t \le h$ such that $s_t = (x_r, m_t) \in S$ for some $r \in [L]$ (depending on t). Here $\vec{e_r} \in \mathbb{Z}^L_{\ge 0}$ is the r-th standard basis vector and L is the length of $\mathsf{code}(w) = (c_1, c_2, \ldots, c_L)$.

Example 5.7. In Figure 1 we read the $(w = 53861247, v = 54631278)$-transition string $S = (2, (x_3, 2))$ as the path $w \xrightarrow{2} \bullet \xrightarrow{x_3} \bullet \xrightarrow{x_3} v$. Here, $\mathsf{delwt}(S) = (0, 0, 2, 0)$.

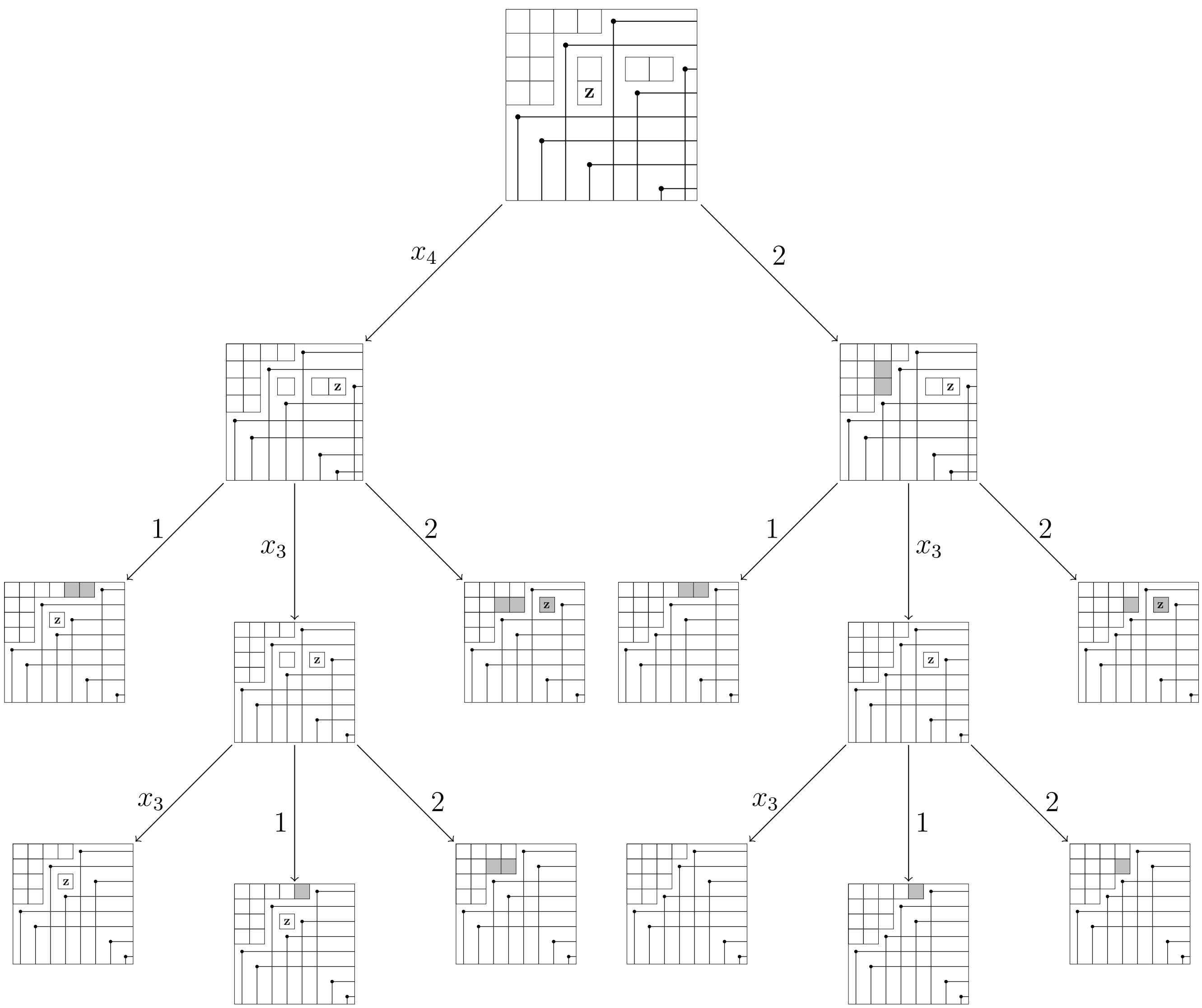


FIGURE 1. $\mathcal{T}(w)$ for $w = 53861247$ where the accessible boxes are marked with z and those boxes of the parent which moved are shaded gray.

Suppose T is a tableau of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_L \geq 0)$, with entries in $[L]$ and weakly increasing along rows. Define

$$R(T) = (r_{ij})_{1 \leq i,j \leq L}$$

to be the $L \times L$ matrix where r_{ij} is the number of j's in row i of T. $R(T)$ encodes T. As pointed out in (a preprint version of) [28], T might have exponentially many (in L) boxes, whereas $R(T)$ is a $O(L^2)$ description of T.

Example 5.8. If $\lambda = (4, 3, 1, 0, 0)$ and

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 4 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array} \longleftrightarrow R(T) = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $X_{\alpha,w} = \{(S, R(T))\} \subseteq X$ such that the following hold:

(X.1′) $S \in \mathsf{Trans}(w, v)$,
(X.2′) $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$, and
(X.3′) $\mathsf{delwt}(S) + \mathsf{content}(T) = \alpha$.

Proposition 5.9. $c_{\alpha,w} = \#X_{\alpha,w}$.

Proof. Iterating (21),

$$\mathfrak{S}_w = \sum_{\text{vexillary } v \in S_\infty} \sum_{S \in \mathsf{Trans}(w,v)} x^{\mathsf{delwt}(S)} \mathfrak{S}_v.$$

Hence

$$c_{\alpha,w} = \sum_{\text{vexillary } v \in S_\infty} \sum_{S \in \mathsf{Trans}(w,v)} [x^\alpha] x^{\mathsf{delwt}(S)} \mathfrak{S}_v. \tag{22}$$

The result then follows from by (19), (20), and (22) combined. □

Proposition 5.10 (cf. [19]). *Let* $\mathsf{code}(w) = (c_1, \ldots, c_L)$. *Suppose* $D(w')$ *is obtained from* $D(w)$ *using move* (T.1) *and* $D(w'')$ *is obtained from* $D(w)$ *with move* (T.2) *for a pivot in row* i. *There is an* $O(L^2)$*-time algorithm to compute*

(I) $\mathsf{code}(w') = (c_1, \ldots, c_{r-1}, c_r - 1, c_{r+1}, \ldots, c_L)$ *and*
(II) $\mathsf{code}(w'') = (c_1, \ldots, c_{i-1}, c_i + b, c_{i+1}, \ldots, c_{r-1}, c_r - b, c_{r+1}, \ldots, c_L)$, *for some* $b \in \mathbb{Z}_{>0}$.

Proof. By Proposition 3.3, determine $\mathbf{z}_w := (r, c)$ in $O(L^2)$-time.

For (I), $D(w')$ is obtained from $D(w)$ by deleting $\mathbf{z}_w$; so the expression in (I) is clear.

For (II), using Proposition 3.2, we can find $\mathbf{x} = (i, w(i))$, in $O(L^2)$-time; this is our (T.2) pivot. Notice that row r of $D(w) \cap \mathcal{R}$ is nonempty (it contains $\mathbf{z}_w = (r, c)$); let b be the number of boxes in this row. It is straightforward from the graphical description of $\mathcal{R}$ in terms of Rothe diagrams that each row of $D(w) \cap \mathcal{R}$ either has zero boxes or $b > 0$ boxes. Moreover, the d-th box (say, from the left) of each row are in the same column.

Suppose $j_1, \ldots, j_m \in [i + 1, r]$ index the rows where $D(w) \cap \mathcal{R} \neq \emptyset$ (and thus has b boxes). (T.2) moves the b boxes of j_1 to row i and moves the b boxes of j_q to row j_{q-1} for $q = 2, \ldots, m$. As explained above $j_m = r$, so (T.2) moves no boxes into row r. Thus row r of $D(w'') \cap \mathcal{R}$ has zero boxes.

It remains to compute b in $O(L^2)$-time. Using Proposition 3.2 compute, in $O(L^2)$-time,

$$m := \#\{h < r \ : \ w(h) < w(i)\}.$$

Clearly $b = c_r - [(w(i) - 1) - m]$. □

Let $s_t = (x_r, m_t)$, as in (X.1), be a valid (multi)-deletion move on $u \in \mathcal{T}(w)$. Let $u^{\langle m\rangle} \in \mathcal{T}(w)$ be defined by $u \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} u^{\langle m_t\rangle}$ (m_t-times).

Proposition 5.11. *Suppose $u \in \mathcal{T}(w)$ where $\mathsf{code}(u) = (\widetilde{c}_1, \ldots, \widetilde{c}_{L'})$. Let $s_t = (x_k, m_t)$ or $s_t = i$ be as in (X.1). Given input $\mathsf{code}(u)$ and s_t, there is an $O(L^2)$ algorithm to respectively determine if $u \xrightarrow{x_k} \bullet \cdots \bullet \xrightarrow{x_k} u^{\langle m_t\rangle}$ (m_t-times) or $u \xrightarrow{i} u''$ occurs in $\mathcal{T}(w)$ and (if yes) to compute*

- $\mathsf{code}(u^{\langle m_t\rangle})$ *in the case $s_t = (x_k, m_t)$ (a multi-deletion move (T.1)), or*
- $\mathsf{code}(u'')$ *in the case $s_t = i$ (a march move (T.2)).*

Proof. By Proposition 5.10, $L' \leq L$. Thus in our run-time analysis, we replace L' by L.

Proposition 3.3 finds $\mathbf{z}_u := (r, c)$ (or determines it does not exist) in $O(L^2)$-time. If $\mathbf{z}_u$ does not exist then u is dominant and thus vexillary; output s_t is invalid. Thus we assume henceforth that $\mathbf{z}_u$ exists.

Case 1: ($s_t = (x_k, m_t)$.) Proposition 3.2 finds $u(1), \ldots, u(L')$ in $O(L^2)$-time. Determine (taking $O(L^2)$ time) if

$$c_r - \left(\left(\min_{i\in[r]} u(i)\right) - 1\right) \geq m_t, \tag{23}$$

holds. We claim that s_t is valid if and only if (23) holds and $k = r$. Indeed, observe

$$\#\{\text{boxes in row } r \text{ of } \mathsf{Dom}(u)\} = \left(\min_{i\in[r]} u(i)\right) - 1. \tag{24}$$

Thus, (23) is equivalent to the existence of m_t boxes in row r of $D(u) \setminus \mathsf{Dom}(u)$. By (T.1), if $k = r$ this is equivalent to being able to apply $\bullet \xrightarrow{x_r} \bullet$ successively m_t-times.

Finally, if s_t is valid, by m_t applications of Proposition 5.10 (I),

$$\mathsf{code}(u^{\langle m_t\rangle}) = (\widetilde{c}_1, \ldots, \widetilde{c}_{r-1}, \widetilde{c}_r - m_t, \widetilde{c}_{r+1}, \ldots, \widetilde{c}_{L'}). \tag{25}$$

Hence we can output (25) in $O(L^2)$-time.

Case 2: ($s_t = i$.) By Proposition 3.2, determine $u(1), \ldots, u(L')$ from $\mathsf{code}(u)$ in $O(L^2)$-time. In particular this computes $\mathbf{x} := (i, u(i))$ in $O(L^2)$-time. To decide if s_t is valid we must determine if $\mathbf{x} \in \mathsf{Piv}(\mathbf{z}_u)$. To do this, first calculate (in $O(L)$-time)

$$u_{NW}(\mathbf{z}_u) := \{(j, u(j)) \ : \ j < r, u(j) < c\}.$$

By definition,

$$\mathsf{Piv}(\mathbf{z}_u) = \{(j, u(j)) \in u_{NW}(\mathbf{z}_u) \ : \ \nexists (h, u(h)) \in u_{NW}(\mathbf{z}_u) \text{ with } h > j, u(h) > u(j)\}.$$

$\mathsf{Piv}(\mathbf{z}_u)$ takes $O(L)$-time to compute since $\#u_{NW}(\mathbf{z}_u) \leq r - 1 \leq L - 1$. Hence we check if $\mathbf{x} \in \mathsf{Piv}(\mathbf{z}_u)$ in $O(L)$-time. If this is false, we output a rejection. Otherwise, Proposition 5.10 outputs $\mathsf{code}(u'')$ in $O(L^2)$-time. □

Proposition 5.12. *If $S = (s_1, \ldots, s_h) \in \mathsf{Trans}(w, v)$ then $h \leq L^2$.*

Proof. Let $w := w_0 \xrightarrow{s_1} w_1 \xrightarrow{s_2} \ldots \xrightarrow{s_h} w_h = v$ be the path in $\mathcal{T}(w)$ associated to S. By (T.1) and (T.2), $\mathbf{z}_{w_{t+1}}$ is weakly northwest of $\mathbf{z}_{w_t}$. Hence, for any fixed r, those $t \in [0, h-1]$ with $\mathbf{z}_{w_t}$ in row r form an interval $I^{(r)} \subseteq [0, h-1]$. Since $1 \leq r \leq L$, it suffices to prove

$$\#I^{(r)} \leq 2(r-1). \tag{26}$$

By (X.1) the transition moves acting on row r alternate between multi-(T.1) moves (x_r, m_t) and (T.2) moves. Thus to show (26), it is enough to prove

$$\#\{t \in I^{(r)} : w_{t-1} \to w_t \text{ is a (T.2) move}\} \leq r-1. \tag{27}$$

Consider a march move i with $\mathbf{z}_{w_{t-1}} = (r, c)$ and $\mathbf{x} = (i, w_{t-1}(i)) \in \mathsf{Piv}(\mathbf{z}_{w_{t-1}})$. By (T.2), if $(r, c') \in D(w_{t-1})$ is in the same connected component as $\mathbf{z}_{w_{t-1}}$, the move i takes (r, c') strictly north of row r. Thus, each march move strictly reduces the number of components in row r. Let $t_0 = \min\{t \in I^{(r)}\}$. Since there are at most r $\bullet$'s weakly above row r, $D(w_{t_0})$ has at most $r-1$ (non-dominant) components in row r. Hence (27) holds, as desired. □

Proposition 5.13. *Let v be vexillary with $\mathsf{code}(v) = (c_1, \ldots, c_{L'})$ and $L' \leq L$. There exists an $O(L^2)$-time algorithm to check if $R = (r_{ij})_{1 \leq i,j \leq L'}$ is $R = R(T)$ for some $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$.*

Proof. Since $L' \leq L$, it is $O(L^2)$-time to calculate $\phi(v), \lambda(v)$. Let

$$\lambda_i := \sum_{j=1}^{L'} r_{ij}, \text{ for } 1 \leq i \leq L'.$$

First verify (in $O(L)$-time) that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i \leq L'-1$. Then $R = R(T)$ where T is the (unique) row weakly increasing tableau of shape λ with r_{ij} many j's in row i.

To verify $T \in \mathsf{SSYT}(\lambda(v), \phi(v))$ we must check that it is (i) is flagged by $\phi(v)$, (ii) has shape $\lambda(v)$, and (iii) is semistandard. For (i), we need

$$r_{ij} = 0 \text{ if } j > \phi(v)_i, \text{ for all } i, j \in [L']. \tag{28}$$

For (ii), we need

$$\lambda_i = \lambda(v)_i \text{ for each } i \in [L']. \tag{29}$$

For (iii), it remains to ensure that T is column strict, i.e.,

$$\sum_{j' \leq j} r_{i+1,j'} \leq \sum_{j'<j} r_{i,j'} \text{ for each } i \in [L'-1], j \in [L']. \tag{30}$$

We found the inequalities (29) and (30) from a (preprint) version of [28]. The inequalities (28), (29), and (30) can be checked in $O(L^2)$-time since $i, j \in [L'] \subseteq [L]$. □

The following completes our proof that we can check that $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time.

Proposition 5.14. *Given $(S, R) \in X$ and $(\mathsf{code}(w), \alpha)$, one can determine if $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time.*

Proof. By Propositions 5.11 and 5.12 combined, one determines in $O(L^4)$-time if S encodes a path $w := w_0 \xrightarrow{s_1} w_1 \xrightarrow{s_2} \cdots \xrightarrow{s_h} w_h = v$ in $\mathcal{T}(w)$. If so, the length of $\mathsf{code}(v)$ is at most L. Thus, using Theorem 5.1, one checks v is vexillary in $O(L^3)$-time. This decides if S satisfies (X.1′). Proposition 5.13 checks R satisfies (X.2′) in $O(L^2)$-time. Finally since $h \leq L^2$, computing $\mathsf{delwt}(S)$ takes $O(L^2)$-time. Hence (X.3′) is checkable in $O(L^2)$ time. □

Proof of Theorem 5.6: By Proposition 5.9, $\#X_{\alpha,w} = c_{\alpha,w}$. By Proposition 5.12, $(S, R) \in \#X_{\alpha,w}$ only if the list S has at most L^2 elements. Assuming this, we check (S, R) satisfies (X.1) and (X.2) in $O(L^2)$-time. Using Proposition 5.14, we can verify $(S, R) \in X_{\alpha,w}$ in $L^{O(1)}$-time. Thus, given input α and $\mathsf{code}(w)$, computing $c_{\alpha,w}$ is in $\#\mathsf{P}$. $\square$

5.4. **Hardness, and the conclusion of the proof of Theorem 1.16.** A permutation w is *grassmannian* if it has at most one *descent* i, i.e., where $w(i) > w(i + 1)$. Given a partition $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_L > 0)$ define a grassmannian permutation w_λ by setting

$$w_\lambda(i) = i + \lambda_{L-i+1} \text{ for } 1 \leq i \leq L.$$

It is well-known (see, e.g., [25]) that

$$\mathsf{code}(w_\lambda) = (\lambda_L, \lambda_{L-1}, \ldots, \lambda_1). \tag{31}$$

Moreover,

$$\mathfrak{S}_{w_\lambda} = s_\lambda(x_1, \ldots, x_L) = \sum_\alpha K_{\lambda,\alpha} x^\alpha, \tag{32}$$

where $K_{\lambda,\alpha}$ is (as in Example 1.5) the *Kostka coefficient*. This number counts semistandard tableaux of shape λ with content α. By (32),

$$c_{\alpha,w_\lambda} = K_{\lambda,\alpha}. \tag{33}$$

By Theorem 5.6, counting $c_{\alpha,w}$ is in $\#\mathsf{P}$. Suppose there is an oracle to compute $c_{\alpha,w}$ in polynomial time in the input length of $(\mathsf{code}(w), \alpha)$. This input length is the same as for the input λ, α for $K_{\lambda,\alpha}$. Hence (31) and (33) combined imply a polynomial-time counting reduction from Kostka coefficients to $\{c_{\alpha,w}\}$. Now H. Narayanan [28] proved that counting $K_{\lambda,\alpha}$ is a $\#\mathsf{P}$-complete problem. Thus counting $c_{\alpha,w}$ is a $\#\mathsf{P}$-complete problem. $\square$

Remark 5.15. Suppose the input for counting $c_{\alpha,w}$ is (α, w) where $w \in S_n$ (in one-line notation). Then the above counting reduction is not polynomial time in the input length of the Kostka problem. For example, suppose $\lambda = \alpha = (2^L, 2^L, \ldots, 2^L)$ (L-many). Then the input length of this instance of the Kostka problem is $2L^2 \in O(L^2)$. On the other hand, $w_\lambda \in S_{L+2^L}$. Therefore, a polynomial time algorithm for the Schubert coefficient problem in n would have $\Omega(2^L)$ run time for the Kostka problem.

It seems unlikely that there is a polynomial-time reduction under this input assumption. This is our justification to encode w via $\mathsf{code}(w)$ rather than one line notation. $\square$

Acknowledgments

AY was supported by an NSF grant, a Simons Collaboration grant and a UIUC Campus Research Board grant. CR was supported by an NSF Graduate Research Fellowship. We acknowledge Mathoverflow, S. Kintali's blog "My brain is open" and R. O'Donnell's Youtube videos (from his class at Carnegie Mellon) for background. We thank Philipp Hieronymi, Cara Monical, Erik Walsberg and Alexander Woo for helpful conversations.

References

[1] D. Aharonov and O. Regev, *Lattice problems in* $\mathsf{NP} \cap \mathsf{coNP}$, J. ACM 52(5): 749–765 (2005)

[2] M. Agrawal, N. Kayal, and N. Saxena, *PRIMES is in* P, Ann. of Math. (2) 160 (2004), no. 2, 781–793.

[3] S. Billey, W. Jockusch and R. P. Stanley, *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2**(1993), no. 4, 345–374.

[4] A. Condon, *The Complexity of Stochastic Games*, Inf. Comput. 96(2): 203–224 (1992).
[5] G. B. Dantzig, *Reminiscences about the origins of linear programming*, Technical report of the Systems Optimization Laboratory (Stanford U.) 81-5 (April 1981).
[6] L. Escobar and A. Yong, *Newton polytopes and symmetric Grothendieck polynomials*, C. R. Math. Acad. Sci. Paris 355 (2017), no. 8, 831–834.
[7] A. Fink, K. Mészáros, and A. St. Dizier. Schubert polynomials as integer point transforms of generalized permutahedra. *Preprint*, arXiv:1706.04935, 2017.
[8] S. Fomin and C. Greene, *Noncommutative Schur functions and their applications*, Discrete Math. **193**(1998), 179–200, Selected papers in honor of Adriano Garsia (Taormina, 1994).
[9] S. Fomin, C. Greene, V. Reiner, and M. Shimozono, *Balanced labellings and Schubert polynomials*, European J. Combin. 18 (1997), no. 4, 373–389.
[10] S. Fomin and A. N. Kirillov, *Grothendieck polynomials and the Yang-Baxter equation*, Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique, 183–189, DIMACS, Piscataway, NJ.
[11] W. Fulton, *Young tableaux. With applications to representation theory and geometry*. London Mathematical Society Student Texts, 35. Cambridge University Press, Cambridge, 1997.
[12] K. Gödel, *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I.,* Monatshefte für Mathematik und Physik (1931) 38, 173–198.
[13] O. Goldreich, *P, NP, and NP-completeness: The Basics of Computational Complexity*, Cambridge University Press, 2010. 155 pp.
[14] J. Haglund, M. Haiman and N. Loehr, *A combinatorial formula for Macdonald polynomials*, J. Amer. Math. Soc. 18 (2005), no. 3, 735–761.
[15] I. Holyer, *The NP-completeness of some edge-partition problems*, SIAM J. Comput. 10 (1981), no. 4, 713–717.
[16] M. Jurdziński, *Deciding the winner in parity games is in* UP ∩ co-UP, Information Processing Letters, Elsevier, 1998, 68 (3): 119–124
[17] A. Knutson and A. Yong, *A formula for K-theory truncation Schubert calculus*, Int. Math. Res. Not. 2004, no. 70, 3741–3756.
[18] A. Lascoux, *Transition on Grothendieck polynomials*, Physics and combinatorics, 2000 (Nagoya), World Sci. Publishing, River Edge, NJ, 2001, pp. 164–179.
[19] A. Lascoux and M. -P. Schützenberger, *Schubert polynomials and the Littlewood-Richardson rule*, Letters in Math. Physics 10 (1985), 111–124.
[20] ———, *Polynômes de Schubert*, C. R. Acad. Sci. Paris Sér. I Math. **294** (1982), 447–450.
[21] K. Mészáros and A. St. Dizier, *From generalized permutahedra to Grothendieck polynomials via flow polytopes*, preprint, 2017. arXiv:1705.02418
[22] K. D. Mulmuley, *The GCT program toward the* P *vs.* NP *problem*, CACM, vol. 55, issue 6, June 2012, pp. 98-107.
[23] K. D. Mulmuley and M. Sohoni, *Geometric Complexity Theory I: An Approach to the P vs. NP and Related Problems*, SIAM J. Comput. 31(2), 496–526, 2001.
[24] K. D. Mulmuley, H. Narayanan, M. Sohoni, *Geometric complexity theory III: on deciding nonvanishing of a Littlewood-Richardson coefficient*, J. Algebraic Combin. 36(1) (2012), 103–110.
[25] L. Manivel, *Symmetric functions, Schubert polynomials and degeneracy loci*. Translated from the 1998 French original by John R. Swallow. SMF/AMS Texts and Monographs, American Mathematical Society, Providence, 2001.
[26] C. Monical, *Polynomials in algebraic combinatorics*, PhD thesis, University of Illinois at Urbana-Champaign, 2018.
[27] C. Monical, N. Tokcan and A. Yong, *Newton polytopes in algebraic combinatorics*, preprint, 2017. arXiv:1703.025843
[28] H. Narayanan, *On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients*, J. Alg. Comb., Vol. 24, N. 3, 2006, 347–354.
[29] C. H. Papadimitriou and K. Steiglitz, Kenneth, *Combinatorial optimization: algorithms and complexity.* Corrected reprint of the 1982 original. Dover Publications, Inc., Mineola, NY, 1998. xvi+496 pp.
[30] V. Pratt, *Every Prime Has a Succinct Certificate*, SIAM J. Comput. 4, 214–220, 1975.
[31] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & sons, 1998.
[32] R. P. Stanley, *On the number of reduced decompositions of elements of coxeter groups*, European Journal of Combinatorics **5** (1984), no. 4, 359 – 372.

[33] ———, *A symmetric function generalization of the chromatic polynomial of a graph*, Adv. Math. **111**(1995), no. 1, 166–194.
[34] ———, *Enumerative combinatorics*, Vol. 2. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999. xii+581 pp.
[35] ———, *Some Schubert shenanigans*, preprint, 2017. arXiv:1704.00851
[36] A. Turing, *On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction.*, Proc. London Math. Soc. (2) 43 (1937), no. 7, 544–546.
[37] L. G. Valiant, *The complexity of computing the permanent*, Theoret. Comput. Sci., 8(2):189–201, 1979.
[38] A. Weigandt, *Schubert polynomials, 132-patterns, and Stanley's conjecture*, preprint, 2017. arXiv:1705.02065

UCLA, LOS ANGELES, CA 90095
E-mail address: aadve@g.ucla.edu

DEPT. OF MATHEMATICS, U. ILLINOIS AT URBANA-CHAMPAIGN, URBANA, IL 61801, USA
E-mail address: cer2@illinois.edu, ayong@illinois.edu