CASTELNUOVO–MUMFORD REGULARITY FOR 321-AVOIDING KAZHDAN–LUSZTIG VARIETIES

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Abstract. We introduce an algorithm which combinatorially computes the Castelnuovo–Mumford regularity of 321-avoiding Kazhdan–Lusztig varieties using excited diagrams. This extends earlier work of Rajchgot, Weigandt, and the author (2022) which computes the regularity of Grassmannian Kazhdan–Lusztig varieties. Our results specialize to compute the regularity of all two-sided mixed ladder determinantal varieties in terms of lattice paths.

1. Introduction

Kazhdan–Lusztig varieties are generalized determinantal varieties introduced by A. Woo and A. Yong [39] to study singularities of *Schubert varieties*. *Matrix Schubert varieties*, introduced by W. Fulton [11], and *ladder determinantal varieties*, introduced by S. S. Abhyankar [1], are well-studied families of Kazhdan–Lusztig varieties [24, 25, 6, 7, 8, 14, 15, 16, 20, 30]. Kazhdan–Lusztig varieties indexed by 321-*avoiding* permutations form a large class of Kazhdan–Lusztig varieties with homogeneous defining ideals. As proven by L. Escobar, A. Fink, J. Rajchgot, and A. Woo [10], all *two-sided mixed ladder determinantal varieties* are 321-avoiding Kazhdan–Lusztig varieties.

The *Castelnuovo–Mumford regularity* of a graded module is an invariant used to measure its complexity. In general, regularity may be computed using the minimal free resolution of the module in terms of its Betti numbers. Since Kazhdan–Lusztig varieties are Cohen–Macaulay, one may instead compute their regularities combinatorially in terms of degrees of *unspecialized Grothendieck polynomials*, given by A. Woo and A. Yong [40].

We leverage this fact to provide a combinatorial algorithm that computes the regularity of 321-avoiding Kazhdan–Lusztig varieties. This paper generalizes previous work of J. Rajchgot, A. Weigandt, and the author [35] which gives a tableaux based formula to compute the regularity of Kazhdan–Lusztig varieties indexed by *Grassmannian* permutations.

1.1. Summary of Results.

We give an algorithm to determine the regularity of 321-avoiding Kazhdan–Lusztig varieties using *skew excited Young diagrams.* We construct a diagram $D_{\texttt{zip}}(v, w) \subset [n] \times [n]$, which we decorate to form $D^K_{\texttt{zip}}(v, w)$. This diagram $D^K_{\texttt{zip}}(v, w)$ computes the regularities of 321-avoiding Kazhdan–Lusztig varieties:

Theorem 1.1. *Suppose $v \geqslant w$ are 321-avoiding permutations. Then*

$$\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \#D^K_{\texttt{zip}}(v, w) - \ell(w).$$

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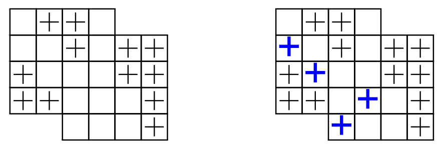
Here $\ell(w)$ denotes the Coxeter length of the permutation w. We define $D_{\mathtt{zip}}(v,w)$ and $D^K_{\mathtt{zip}}(v,w)$ in Section 4.1. Theorem 1.1 is proven in Section 4.3.

Example 1.2. Let $v = (5,8,9,10,1,2,11,3,4,6,7)$ and $w = (1,4,5,8,2,3,9,6,10,11,7)$. The left diagram below is $D_{\mathtt{zip}}(v,w)$. We decorate $D_{\mathtt{zip}}(v,w)$ with bold blue pluses to construct $D^K_{\mathtt{zip}}(v,w)$, the right diagram.



Theorem 1.1 determines that $\mathrm{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = 16 - 12 = 4$. Since $\#D_{\mathtt{zip}}(v,w) = \ell(w)$, $\mathrm{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \#(D^K_{\mathtt{zip}}(v,w) - D_{\mathtt{zip}}(v,w))$, the number of blue pluses. $\diamondsuit$

As with regularity, the *a-invariant* of a module is an invariant providing data that may increase efficiency in computations, see [4] for discussion. Using Theorem 1.1, we compute the a-invariant for 321-avoiding Kazhdan–Lusztig varieties:

Corollary 1.3. *Suppose $v \geqslant w$ are 321-avoiding permutations. Then*

$$a(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \#D^K_{\mathtt{zip}}(v,w) - \ell(v).$$

Corollary 1.3 is proven in Section 4.3.

Example 1.4. Taking v, w as in Example 1.2, Corollary 1.3 computes

$$a(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = 16 - 26 = -10.$$

By construction, $\ell(v)$ is the number of boxes in $D^K_{\mathtt{zip}}(v,w)$. Thus $|a(\mathbb{C}[\mathbf{z}^v]/J_{v,w})|$ counts the empty boxes in $D^K_{\mathtt{zip}}(v,w)$. $\diamondsuit$

In Section 5, we prove specializations of Theorem 1.1 and Corollary 1.3 for two-sided mixed ladder determinantal varieties. These results continue work of J. Rajchgot, A. Weigandt, and the author [35] which gives a combinatorial regularity formula for one-sided ladders.

1.2. **Context in Literature.** The regularity of matrix Schubert varieties is recently well-understood. Initial work of Y. Ren, J. Rajchgot, A. St. Dizier, A. Weigandt, and the author [34] gives a combinatorial formula for the regularity Grassmannian matrix Schubert varieties in terms of integer partitions. The recent work of O. Pechenik, D. Speyer, and A. Weigandt [33] uses poset-theoretic techniques to easily compute the regularity of arbitrary matrix Schubert varieties in terms of permutation statistics.

J. Pan and T. Yu [32] use [33] to give a diagrammatic regularity formula for matrix Schubert varieties. The results in [33] have been re-proven by M. Dreyer, K. Mészáros, and A. St. Dizier [9] using saturated chains in Bruhat order. Formulas for regularities of matrix Schubert varieties for particular cases [19, 35] as well as tangent cones of Schubert varieties [41] have also been studied.

The results of J. Rajchgot, A. Weigandt, and the author [35] give a combinatorial formula to compute the regularity of Grassmannian Kazhdan–Lusztig varieties. Due to a correspondence with matrix Schubert varieties, these results in [35] may be recovered using [33].

In general, 321-avoiding Kazhdan–Lusztig varieties are not isomorphic to matrix Schubert varieties. Thus Theorem 1.1 generalizes [35] in a different direction than [33].

Combinatorial formulas to compute the a-invariant of families of *one-sided ladder determinantal varieties* have been determined [4, 5, 13]. Additionally, S. R. Ghorpade and C. Krattenthaler [13] give an algorithm to compute the a-invariant for a family of two-sided ladder determinantal varieties in terms of lattice paths. Forthcoming work of L. Escobar, A. Fink, J. Rajchgot, and A. Woo [10] shows two-sided ladder determinantal varieties are 321-avoiding Kazhdan–Lusztig varieties. Using this fact in Section 5, we apply Theorem 1.1 and Corollary 1.3 to compute the regularity and a-invariant for two-sided ladder determinantal varieties in terms of lattice paths. To do this, we generalize a bijection of V. Kreiman [26] between lattice paths and excited Young diagrams.

1.3. **Outline.** In Section 2 we establish the combinatorial background. We give the geometric and commutative algebraic background in Section 3. In Section 4 we construct the diagram $D^K_{\texttt{zip}}(v, w)$ and prove our main results, Theorem 1.1 and Corollary 1.3. In Section 5 we specialize our results to two-sided mixed ladder determinantal varieties using lattice paths.

2. Combinatorial Background

For $n \in \mathbb{Z}$, let $[n] := \{i \in \mathbb{Z}_{>0} \mid i \leqslant n\}$.

2.1. **Pipe complexes.** Let S_n denote the symmetric group on n letters. We write $u \in S_n$ in one-line notation and let $u_i := u(i)$ for $i \in [n]$. The **rank function** of $u \in S_n$ is defined as

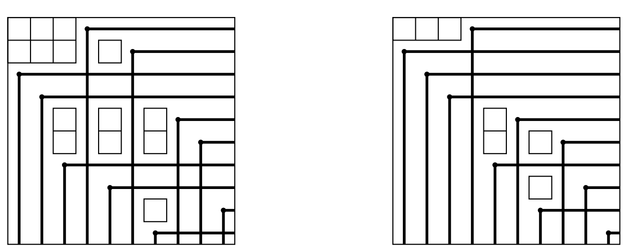
$$\mathsf{rank}_u(i, j) := \#\{(k, u_k) \mid k \in [i], u_k \in [j]\}$$

for $(i, j) \in [n] \times [n]$. The **Rothe diagram** of $u \in S_n$ is the set

$$D(u) := \{(i, j) \in [n] \times [n] \mid u_i > j \text{ and } u_j^{-1} > i\}.$$

We illustrate $D(u)$ as the blank cells in the $n \times n$ grid after placing points in cells (i, u_i) and drawing a line through cells which appear weakly south or weakly east of (i, u_i) for each $i \in [n]$. Let $\ell(u) := \#D(u)$ denote the **Coxeter length** of u. The **Lehmer code** of u is the tuple $\mathsf{code}(u) = (c_1, \ldots, c_n)$ where c_i counts the number of boxes in row i of $D(u)$. Further, $\mathsf{code}(u)$ uniquely encodes u [28, Proposition 2.1.2].

Example 2.1. Below are $D(v)$ and $D(w)$ for $v = 46128935(10)7$ and $w = 412368597(10)$, respectively.



Here $\ell(w) = 7 < \ell(v) = 14$ and $\mathsf{code}(v) = (3, 4, 0, 0, 2, 2, 2, 0, 0, 1, 0)$. ♢

Define the algebra over $\mathbb{Z}$ generated by $\{e_u \mid u \in S_n\}$ with multiplication such that

$$e_u e_{s_i} = \begin{cases} e_{us_i} & \text{if } \ell(us_i) > \ell(u), \text{ and} \\ e_u & \text{otherwise.} \end{cases}$$

Here s_i is the simple transposition which permutes i and $i+1$.

For $u \in S_n$ label the boxes of $D(u)$ along rows so that kth leftmost box in row i is assigned the label $i+k-1$. Given $P \subseteq D(u)$, let $\mathsf{word}(P)$ in $D(u)$ be the sequence formed by reading the labels of P in this labeling of $D(u)$, scanning right to left across rows, from top to bottom. The **Demazure product** $\delta(P)$ of P is the permutation that satisfies

$$e_{s_{i_1}} \cdots e_{s_{i_k}} = e_{\delta(P)},$$

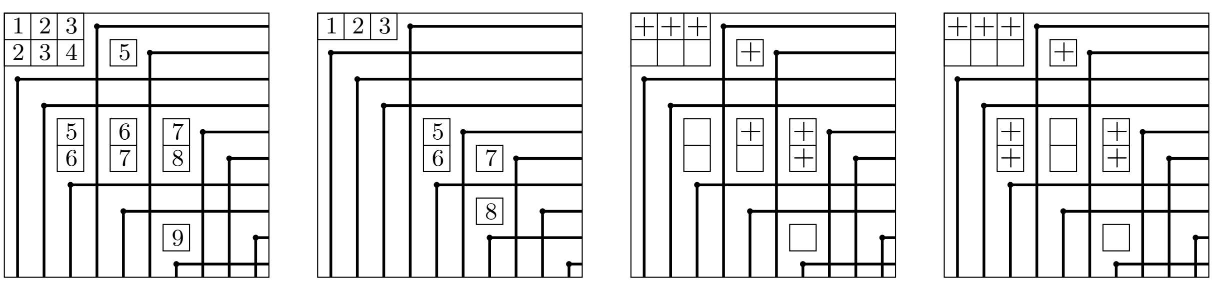
where $\mathsf{word}(P) = (i_1, i_2, \ldots, i_k)$ in $D(u)$.

Take $v, w \in S_n$ where $v \gtrdot w$, *i.e.*, v covers w in Bruhat order. Define

$$\overline{\mathsf{Pipes}}(v,w) := \{P \subseteq D(v) \mid \delta(P) = w\}, \text{ and}$$
$$\mathsf{Pipes}(v,w) := \{P \in \overline{\mathsf{Pipes}}(v,w) \mid \#P = \ell(w)\}.$$

We illustrate $P \subseteq D(v)$ by filling each $(i,j) \in P$ with a $+$ in $D(v)$.

Example 2.2. The left two diagrams are labeled diagrams $D(v)$ and $D(w)$ for v, w as in Example 2.1. This gives $\mathsf{word}(D(w)) = (3,2,1,5,7,6,8)$ in $D(w)$. The third diagram is $P \in \mathsf{Pipes}(v,w)$, and the fourth is some $P' \in \overline{\mathsf{Pipes}}(v,w)$.



$\diamondsuit$

Defined by A. Woo and A. Yong [40], the **unspecialized Grothendieck polynomial** is

$$\mathfrak{G}_{v,w}(\mathbf{t}) := \sum_{P \in \overline{\mathsf{Pipes}}(v,w)} (-1)^{\#P - \ell(w)} \prod_{(i,j) \in P} t_{ij}. \tag{2.1}$$

By setting $v = w_0 \in S_n$ and specializing variables t_{ij}, these unspecialized Grothendieck polynomials recover the double Grothendieck polynomials of [27]. Note that we follow the conventions of [35] for $\mathfrak{G}_{v,w}(\mathbf{t})$, which differ from those in [40].

2.2. **Skew Excited Young Diagrams.** A permutation $u \in S_n$ is 321-**avoiding** if there does not exist a 321 *pattern* in u, *i.e.*, indices $i < j < k$ such that $u_k < u_j < u_i$. For example, $u = 1\underline{7}2\underline{5}83\underline{4}6$ is not 321-avoiding; the underlined entries form a 321 pattern in u. Let $U_n(321) := \{u \in S_n \mid u \text{ is 321-avoiding}\}$. A permutation $u \in S_n$ is **Grassmannian** if there exists at most one $i \in [n-1]$ such that $u_i > u_{i+1}$. Grassmannian permutations form a subset of 321-avoiding permutations.

For $u \in U_n(321)$, let

$$\phi_u : \{P \subseteq D(u)\} \to \{S \subset [n] \times [n]\}$$

be the map which deletes all empty rows and columns of $D(u)$ from $P \subseteq D(u)$, shifting remaining columns left and remaining rows up.

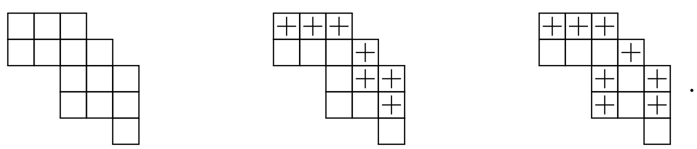
Proposition 2.3. [28, Proposition 2.2.13] *For $u \in U_n(321)$, $\mathcal{R}_u := \phi_u(D(u))$ is a skew Young diagram λ/μ for some partitions $\mu \subseteq \lambda$.*

Our conventions for drawing Young diagrams reflect diagrams in English notation across the y-axis. Throughout this subsection, assume $v \geqslant w$, where $v, w \in U_n(321)$.

Let $D^{NE}(v, w) \subseteq D(v)$ be the boxes corresponding to the earliest subsequence $\mathsf{word}(P)$ of $\mathsf{word}(D(v))$ in $D(v)$ for $P \in \mathsf{Pipes}(v, w)$. Since $w \in U_n(321)$, no braid moves are required to connect reduced words of w, so it is clear $D^{NE}(v, w)$ exists.

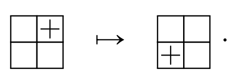
Define $D_{\texttt{top}}(v, w) := \phi_v(D^{NE}(v, w))$. We visualize $D \subseteq \mathcal{R}_v$ by filling $(i, j) \in D$ with $+$'s and call D a **diagram** in $\mathcal{R}_v$.

Example 2.4. Recall v, w as well as P, P' from in Example 2.2. The left picture below is $\mathcal{R}_v$. Note that $P = D^{NE}(v, w)$, so the middle diagram below is $\phi_v(P) = D_{\texttt{top}}(v, w)$. The rightmost diagram is $\phi_v(P')$:



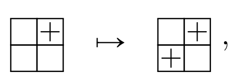
♢

An **excited move** on $D \subseteq \mathcal{R}_v$ is the operation on a 2×2 subsquare of D such that



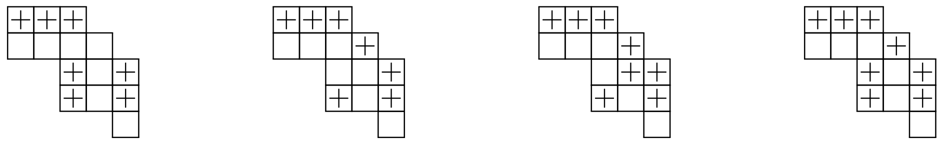
For this move to occur, the subsquare must be contained in $\mathcal{R}_v$. Let $\mathsf{SEYD}(v, w)$ denote the set of $D \subseteq \mathcal{R}_v$ which can be computed through sequential applications of excited moves on $D_{\texttt{top}}(v, w)$. We call a diagram $D \in \mathsf{SEYD}(v, w)$ a **skew excited Young diagram** for v, w. For $v, w \in S_n$ Grassmannian, $\mathsf{SEYD}(v, w)$ are ordinary excited Young diagrams, which arise in the study vexillary matrix Schubert varieties [25] as well as the equivariant cohomology and K-theory of the Grassmannian [18, 21, 26].

A **K-theoretic excited move** on $D \subseteq \mathcal{R}_v$ is the operation on a 2×2 subsquare of D



where all cells pictured are contained in $\mathcal{R}_v$. Write $\overline{\mathsf{SEYD}}(v, w)$ for the set of diagrams obtainable through sequential applications of excited and K-theoretic excited moves on $D_{\texttt{top}}(v, w)$ in $\mathcal{R}_v$. We say a diagram $D \in \overline{\mathsf{SEYD}}(v, w)$ is a **K-theoretic skew excited Young diagram** for v, w. Let $\#D$ denote the number of pluses in D. We say $D \in \overline{\mathsf{SEYD}}(v, w)$ is **maximal** if $D' \in \overline{\mathsf{SEYD}}(v, w)$ implies $\#D' \leqslant \#D$.

Example 2.5. Continuing Example 2.4, the left two diagrams are in $\mathsf{SEYD}(v,w)$. The right two diagrams are maximal diagrams in $\overline{\mathsf{SEYD}}(v,w)$.



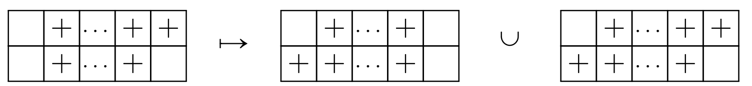
$\diamondsuit$

Proposition 2.6. *For $v \geqslant w$ where $v, w \in U_n(321)$, the map ϕ_v restricted to $\overline{\mathsf{Pipes}}(v,w)$ gives a bijection*

$$\widetilde{\phi}_v : \overline{\mathsf{Pipes}}(v,w) \to \overline{\mathsf{SEYD}}(v,w)$$

such that for $P \in \overline{\mathsf{Pipes}}(v,w)$, $\#P = \#\widetilde{\phi}_v(P)$.

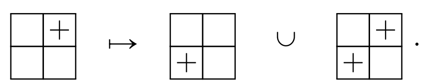
Proof. For $D \subseteq [n] \times [n]$, a ladder move is the operation on a $2 \times k$ strip in D such that



All cells above are contained in $[n] \times [n]$ and $k \geqslant 2$. Let

$$S = \{D \subseteq D(v) \mid D \text{ obtained by applying ladder moves starting from } D^{NE}(v,w)\}.$$

Using [3] and the subword complex interpretation of $\overline{\mathsf{Pipes}}(v,w)$ as given in [40, Section 3], $S = \overline{\mathsf{Pipes}}(v,w)$. By [12, Theorem 4.1] since $w \in U_n(321)$, all ladder moves in this case are of the form



Thus the statement follows by the definition of ϕ_v. $\square$

Corollary 2.7. *Suppose $v \geqslant w$ where $v, w \in U_n(321)$. Then*

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \max\{\#D \mid D \in \overline{\mathsf{SEYD}}(v,w)\}.$$

Proof. This follows by Proposition 2.6 and Equation (2.1). $\square$

Example 2.8. For v, w as in Example 2.5, Corollary 2.7 determines $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = 8$. $\diamondsuit$

3. Castelnuovo–Mumford Regularity of Kazhdan–Lusztig varieties

In this section, we define Castelnuovo–Mumford regularity, a-invariants, and Kazhdan–Lusztig varieties. We then recall results of [35] which relate the Castelnuovo–Mumford regularity of Kazhdan–Lusztig varieties to unspecialized Grothendieck polynomials.

3.1. **Castelnuovo–Mumford Regularity.** Let $S = \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial ring with the standard grading, and let $I \subseteq S$ be a homogeneous ideal. The **Hilbert series** of S/I is a formal power series

$$H(S/I;t) = \sum_{k \in \mathbb{Z}} \dim_{\mathbb{C}}((S/I)_k)t^k = \frac{K(S/I;t)}{(1-t)^n}.$$

The **K-polynomial** of S/I is the numerator $K(S/I;t) \in \mathbb{C}[t^{\pm 1}]$. A minimal free resolution of S/I is the complex

$$0 \to \bigoplus_j S(-j)^{\beta_{l,j}(S/I)} \to \bigoplus_j S(-j)^{\beta_{l-1,j}(S/I)} \to \cdots \to \bigoplus_j S(-j)^{\beta_{0,j}(S/I)} \to S/I \to 0,$$

where $l \leqslant n$ and $S(-j)$ is the free S-module with degree shifted by j. The **Castelnuovo–Mumford regularity** of S/I, written $\operatorname{reg}(S/I)$, is the statistic

$$\operatorname{reg}(S/I) := \max\{j - i \mid \beta_{i,j}(S/I) \neq 0\}.$$

For S/I Cohen–Macaulay,

$$\operatorname{reg}(S/I) = \deg K(S/I;t) - \operatorname{ht}_S I, \tag{3.1}$$

where $\operatorname{ht}_S I$ denotes the height of the ideal I. For more context, consult [2, Lemma 2.5].

The a**-invariant** of S/I, written $a(S/I)$, is the negative of the least degree of a generator of the graded canonical module of S/I, as defined by S. Goto and K. Watanabe [17]. When S/I is Cohen–Macaulay,

$$a(S/I) = \operatorname{reg}(S/I) - d, \tag{3.2}$$

where d is the Krull dimension of S/I.

3.2. **Kazhdan–Lusztig varieties.** We follow the conventions used in [35]. For $v \in S_n$, define $M^{(v)} = (m_{ij})$ to be the $n \times n$ matrix such that for $i, j \in [n]$,

$$m_{ij} := \begin{cases} 1 & \text{if } v_i = j, \\ z_{ij} & \text{if } (i,j) \in D(v), \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{C}[\mathbf{z}^v] := \mathbb{C}[z_{ij} \mid (i,j) \in D(v)]$. For $v \geqslant w$ where $v, w \in S_n$, the **Kazhdan–Lusztig ideal** $J_{v,w} \subseteq \mathbb{C}[\mathbf{z}^v]$ is defined by

$$J_{v,w} := \langle \mathsf{rank}_w(i,j) + 1 - \text{minors in } M^{(v)}_{[i],[j]} \mid (i,j) \in D(w) \rangle,$$

where $M_{I,J}$ denotes the submatrix of M with row indices in I and column indices in J for $I, J \subseteq [n]$. When $v \in U_n(321)$ $J_{v,w}$ is homogeneous, see [23, Footnote on pg. 25]. Additional cases for which $J_{v,w}$ is homogeneous can be found in [31, Propositions 6.3 and 6.4], but no full characterization is known.

Let $B_+, B_- \subset \mathrm{GL}_n(\mathbb{C})$ denote the Borel and opposite Borel subgroups, respectively. As defined in [39], the **Kazhdan–Lusztig variety** is the intersection of the **Schubert variety** $B_-\backslash\overline{B_- w B_+} \subseteq B_-\backslash\mathrm{GL}_n(\mathbb{C})$ with the **opposite Schubert cell** $B_-\backslash B_- v B_-$. The coordinate ring of this Kazhdan–Lusztig variety is $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$. Using [22, Lemma A.4] and the fact that Schubert varieties are Cohen–Macaulay [11, 24, 36], $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ is Cohen–Macaulay.

As reformulated in [35, Lemma 6.3],

Lemma 3.1. [40, Theorem 4.5] *Let $v, w \in U_n(321)$ where $v \geqslant w$. Then*

$$K(\mathbb{C}[\mathbf{z}^v]/J_{v,w};t) = \sum_{P \in \overline{\mathsf{Pipes}}(v,w)} (-1)^{\#P - \ell(w)} (1-t)^{\#P}.$$

Combining Lemma 3.1 with Equation (3.1) produces the following:

Proposition 3.2. [35, Proposition 6.4] *Let $v, w \in U_n(321)$ where $v \geqslant w$. Then*

$$\deg K(\mathbb{C}[\mathbf{z}^v]/J_{v,w}; t) = \deg \mathfrak{G}_{v,w}(\mathbf{t}).$$

Furthermore, the Castelnuovo–Mumford regularity of $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ is given by

$$\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \deg \mathfrak{G}_{v,w}(\mathbf{t}) - \ell(w).$$

Applying this to a-invariants:

Corollary 3.3. *Let $v, w \in U_n(321)$ where $v \geqslant w$. The a-invariant of $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ is given by*

$$a(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) = \deg \mathfrak{G}_{v,w}(\mathbf{t}) - \ell(v).$$

Proof. This follows by Proposition 3.2 combined with Equation (3.2) since $\mathbb{C}[\mathbf{z}^v]/J_{v,w}$ has dimension $d = \ell(v) - \ell(w)$. □

For S/I Cohen–Macaulay, the a-invariant is the lower bound for when its Hilbert function and Hilbert polynomial agree. Using Equations (3.1) and (3.2), A. Stelzer and A. Yong [37] prove that all homogeneous Kazhdan–Lusztig varieties are Hilbertian, *i.e.*, the Hilbert function and Hilbert polynomial of a Kazhdan–Lusztig variety agree at all non-negative integer values, excepting the $v = w$ case.

4. Main Construction and Proof of Theorem 1.1

Assume $v \geqslant w$ where $v, w \in U_n(321)$. In Section 4.1 we define the diagram $D^K_{\mathtt{zip}}(v, w)$ appearing in Theorem 1.1. Section 4.2 relates $D^K_{\mathtt{zip}}(v, w)$ to $\deg \mathfrak{G}_{v,w}(\mathbf{t})$. Section 4.3 contains the proofs of Theorem 1.1 and Corollary 1.3.

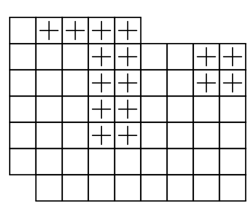
4.1. Main Construction.

We index $\mathcal{R}_v$ using matrix indexing, where its northwest most box corresponds to $(1, 1)$. Suppose $D_{\mathtt{top}}(v, w)$ has connected components $\{C_q\}_{q\in[m]}$. Order components C_q such that the indices increase when scanning components from northwest to southeast. Two boxes sharing only a corner point belong to different components. For a box $\mathsf{b} \in \mathcal{R}_v$, we write $\mathsf{b} = (\mathsf{b}(1), \mathsf{b}(2))$.

Let $q \in [m]$. Define $\psi_E : C_q \to C_q$ such that $\psi_E(\mathsf{b}) = (\mathsf{b}(1), c')$ for $\mathsf{b} \in C_q$, where

$$c' = \max\{k \in [n] \mid (\mathsf{b}(1), k) \in C_q\}.$$

Example 4.1. Consider $D_{\mathtt{top}}(v, w)$ below where $v, w \in U_{16}(321)$[1]. Then $D_{\mathtt{top}}(v, w)$ has connected components C_1 and C_2.



Here $C_1 = (1, 2) \cup (1, 3) \cup \bigcup_{i=1}^{5} \big((i, 4) \cup (i, 5)\big)$ and $C_2 = \{(2, 8), (2, 9), (3, 8), (3, 9)\}$. We see $\psi_E((2, 8)) = (2, 9) = \psi_E((2, 9))$. ♢

[1]Here $v = (6, 11, 12, 13, 14, 15, 1, 16, 2, 3, 4, 5, 7, 8, 9, 10)$, $w = (1, 6, 2, 3, 7, 8, 11, 12, 4, 5, 9, 10, 13, 14, 15, 16)$.

Construction 4.2 (Computing $D^K_{\mathtt{zip}}(v,w)$)**.** Assume $D_{\mathtt{top}}(v,w)$ has connected components $\{C_q\}_{q\in[m]}$. For each $q \in [m]$, define $\mathsf{Diag}_{v,w}(C_q) = \{\mathsf{b}^q_k\}_{k\in[\ell_q]}$ to be the westmost then southmost diagonal of boxes in C_q of maximal length ℓ_q. Boxes in $\mathsf{Diag}_{v,w}(C_q)$ are ordered increasingly northwest to southeast.

For $q \in [m]$ in decreasing order, compute $\mathtt{md}(C_q) = \{\mathsf{d}^q_k\}_{k\in[\ell_q]} \subseteq C_q$ such that $\mathtt{md}(C_q)$ is the westmost then southmost diagonal of length ℓ_q that minimizes

$$\#\big([\|\psi_E(\mathsf{d}^q_{\ell_q})\| + 1] \cap \{\|\mathsf{d}^{q'}_{k'}\|\}_{q'>q,k'\in[\ell_{q'}]}\big).$$

Here $\|\mathsf{b}\| := \mathsf{b}(1)+\mathsf{b}(2)$ for $\mathsf{b} \in D_{\mathtt{top}}(v,w)$. Boxes in $\mathtt{md}(C_q)$ are ordered increasingly northwest to southeast.

We define $D^{(q)}_{\mathtt{zip}}(v,w)$ iteratively for $q \in [m]$. Set $D^{(0)}_{\mathtt{zip}}(v,w) := D_{\mathtt{top}}(v,w)$. Then in $D^{(q-1)}_{\mathtt{zip}}(v,w)$, set

$$S = \{\mathsf{b} \in C_q - \mathtt{md}(C_q) \text{ weakly southwest of } \mathtt{md}(C_q)\}.$$

To each in $\mathsf{b} \in S$, working in order from left to right and bottom to top, let b' be the new position of b after applying as many excited moves as possible to b. Let

$$D^{(q)}_{\mathtt{zip}}(v,w) := D^{(q-1)}_{\mathtt{zip}}(v,w) - S \cup \{\mathsf{b}' \mid \mathsf{b} \in S\}.$$

Define $D_{\mathtt{zip}}(v,w) := D^{(m)}_{\mathtt{zip}}(v,w)$. By construction, $\#D_{\mathtt{zip}}(v,w) = \#D_{\mathtt{top}}(v,w) = \ell(w)$ and $\#\mathcal{R}_v = \ell(v)$. Let $D^K_{\mathtt{zip}}(v,w)$ be the diagram after applying a maximal number of K-theoretic excited moves to each $\mathsf{b} \in \mathtt{md}(C_q)$, for each $q \in [m]$.

For $\mathsf{b} \in \mathtt{md}(C_q)$, define $\mathsf{trail}_{v,w}(\mathsf{b})$ such that

$$\begin{aligned}\mathsf{trail}_{v,w}(\mathsf{b}) := \max\{k \in \{0,1,\ldots,n\} \mid\ & \mathsf{b} + (k',-k'), \mathsf{b} + (k',1-k'),\\ & \mathsf{b} + (k'-1,-k') \in \mathcal{R}_v - D_{\mathtt{zip}}(v,w) \text{ for each } k' \in [k]\}.\end{aligned}$$

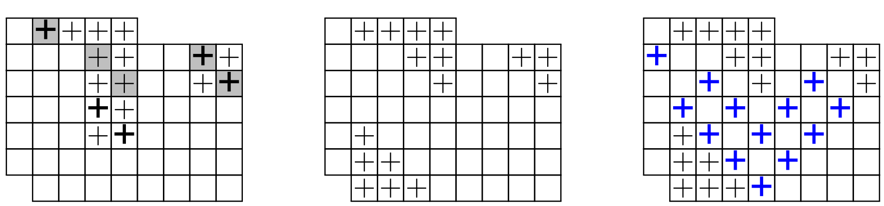
As proven in Section 4.2,

Theorem 4.3. *Suppose $v \geqslant w$, where $v,w \in U_n(321)$. Then*

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \#D^K_{\mathtt{zip}}(v,w) = \ell(w) + \sum_{q\in[m]} \sum_{\mathsf{b}\in\mathtt{md}(C_q)} \mathsf{trail}_{v,w}(\mathsf{b}).$$

Thus $D^K_{\mathtt{zip}}(v,w)$ is maximal in $\overline{\mathsf{SEYD}}(v,w)$.

Example 4.4. We continue with v,w as in Example 4.1. The leftmost diagram is $D_{\mathtt{top}}(v,w)$ with $\mathsf{Diag}_{v,w}(C_q)$ bolded and $\mathtt{md}(C_q)$ shaded for $q \in \{1,2\}$. The middle diagram is $D_{\mathtt{zip}}(v,w)$ and the right diagram is $D^K_{\mathtt{zip}}(v,w)$. The pluses in $D^K_{\mathtt{zip}}(v,w) - D_{\mathtt{zip}}(v,w)$ are drawn bolded in blue in $D^K_{\mathtt{zip}}(v,w)$.



We then compute

$$\sum_{\mathsf{b}\in\mathtt{md}(C_1)} \mathsf{trail}_{v,w}(\mathsf{b}) = 1+2+2=5 \quad\text{and}\quad \sum_{\mathsf{b}\in\mathtt{md}(C_2)} \mathsf{trail}_{v,w}(\mathsf{b}) = 4+4=8.$$

Theorem 4.3 determines

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \#D^K_{\mathtt{zip}}(v,w) = 16+5+8=29.$$

Theorem 1.1 and Corollary 1.3 imply

$$\begin{aligned}\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) &= \#D^K_{\mathtt{zip}}(v,w) - \ell(w) = 29-16=13, \text{ and}\\ a(\mathbb{C}[\mathbf{z}^v]/J_{v,w}) &= \#D^K_{\mathtt{zip}}(v,w) - \ell(v) = 29-58=-29.\end{aligned}$$

We see $\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w})$ counts the blue pluses in $D^K_{\mathtt{zip}}(v,w)$ and $|a(\mathbb{C}[\mathbf{z}^v]/J_{v,w})|$ counts the empty boxes in $D^K_{\mathtt{zip}}(v,w)$. ♢

In Section 5 we discuss cases in which $\mathsf{Diag}_{v,w}(C_q) = \mathtt{md}(C_q)$ for each $q \in [m]$, simplifying computations of $\operatorname{reg}(\mathbb{C}[\mathbf{z}^v]/J_{v,w})$.

4.2. **Proof of Theorem 4.3.** We first prove some key lemmas.

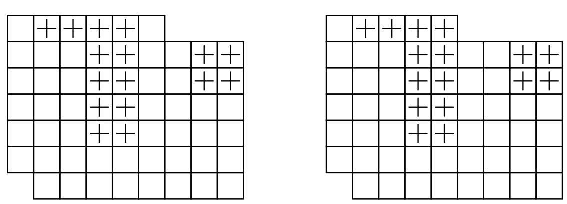
Assume $v \geqslant w$ and $v, w \in U_n(321)$. Let z be the box containing the northmost then eastmost plus in $D_{\mathtt{top}}(v,w)$. Take z' to be the northmost then eastmost box in $\mathcal{R}_v$. Set $i = \mathsf{word}(\phi_v^{-1}(\{\mathsf{z}\}))$ and $i' = \mathsf{word}(\phi_v^{-1}(\{\mathsf{z}'\}))$ in $D(v)$. Define the following:

$$\begin{aligned} w_P &:= s_i w & \qquad w_C &:= w\\ v_P &:= s_{i'} v & \qquad v_C &:= v_P.\end{aligned}$$

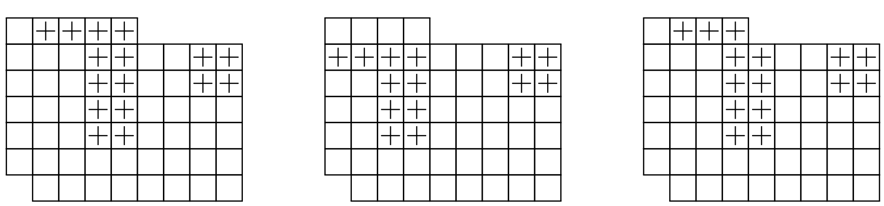
Lemma 4.5. *For $v \geqslant w$ such that $v, w \in U_n(321)$, $\{v_P, w_P, v_C, w_C\} \subseteq U_n(321)$.*

Proof. Since z and z' are northeast most choices, this follows from Proposition 2.3. □

Example 4.6. Let v, w be as in Example 4.1. Below we have $D_{\mathtt{top}}(v', w)$ on the left and $D_{\mathtt{top}}(v'_C, w_C)$ on the right, where $v' = s_6 v$. In this case, $\mathsf{z} = (1,5)$ and $\mathsf{z}' = (1,6)$.



Below are $D_{\mathtt{top}}(v,w)$, $D_{\mathtt{top}}(v_C, w_C)$, and $D_{\mathtt{top}}(v_P, w_P)$, listed from left to right. In this case, $\mathsf{z} = (1,5) = \mathsf{z}'$.



♢

By construction, $\mathcal{R}_{v_C}$ is formed by deleting z' from $\mathcal{R}_v$, up to removing one empty row and column. For simplicity in indexing, we view $\mathcal{R}_{v_C}$ as $\phi_v(D(v_C))$ rather than $\phi_{v_C}(D(v_C))$. Since $v_P = v_C$, we do the same for $\mathcal{R}_{v_P}$.

Lemma 4.7. *Let $v \geqslant w$ and $v, w \in U_n(321)$. Suppose z is the box containing the northmost then eastmost plus in $D_{\mathsf{top}}(v,w)$ and z' is the northmost then eastmost box in $\mathcal{R}_v$. The following hold:*

(1) If $\mathsf{z} \neq \mathsf{z}'$,
$$\overline{\mathsf{SEYD}}(v,w) = \overline{\mathsf{SEYD}}(v_C, w_C).$$
(2) If $\mathsf{z} = \mathsf{z}'$,
$$\overline{\mathsf{SEYD}}(v,w) = \overline{\mathsf{SEYD}}(v_C, w_C) \bigsqcup \Big\{ D \cup \{\mathsf{z}\} \mid D \in \overline{\mathsf{SEYD}}(v_C, w_C) \cup \overline{\mathsf{SEYD}}(v_P, w_P) \Big\}.$$

Proof. First suppose $\mathsf{z} \neq \mathsf{z}'$. By the definition of $D_{\mathsf{top}}(v,w)$, $\mathsf{z}' \notin D$ for any $D \in \overline{\mathsf{SEYD}}(v,w)$. Therefore, $D_{\mathsf{top}}(v,w) = D_{\mathsf{top}}(v_C, w_C)$, so since z' is northeast most, the result follows.

Now suppose $\mathsf{z} = \mathsf{z}'$. Note $\overline{\mathsf{SEYD}}(v_C, w_C)$ may be empty in this case.
($\subseteq$) Fix $D \in \overline{\mathsf{SEYD}}(v,w)$. If $\mathsf{z} \notin D$, then $D \subset \mathcal{R}_{v_C}$ and $\delta({\phi_v}^{-1}(D)) = w = w_C$. Thus $D \in \overline{\mathsf{SEYD}}(v_C, w_C)$ by Proposition 2.6.

Now assume $\mathsf{z} \in D$. If $\delta({\phi_v}^{-1}(D - \{\mathsf{z}\})) = \delta({\phi_v}^{-1}(D))$, then
$$\delta({\phi_v}^{-1}(D - \{\mathsf{z}\})) = \delta({\phi_v}^{-1}(D)) = w = w_C,$$
so $D - \{\mathsf{z}\} \in \overline{\mathsf{SEYD}}(v_C, w_C)$. Alternatively when $\delta({\phi_v}^{-1}(D - \{\mathsf{z}\})) \neq \delta({\phi_v}^{-1}(D))$, then since z is northeast most
$$\delta({\phi_v}^{-1}(D - \{\mathsf{z}\})) = s_i w = w_P$$
so $D - \{\mathsf{z}\} \in \overline{\mathsf{SEYD}}(v_P, w_P)$. Thus the result follows by Proposition 2.6.
($\supseteq$) Fix D' in RHS. If $D' \in \overline{\mathsf{SEYD}}(v_C, w_C)$, then $w = w_C$ and $D' \subseteq \mathcal{R}_{v_C} \subset \mathcal{R}_v$. Thus by Proposition 2.6, $D \in \overline{\mathsf{SEYD}}(v,w)$. Otherwise suppose $D' = D \cup \{\mathsf{z}\}$. Since z is northeast most and $\mathsf{z} = \mathsf{z}'$, i is the first letter of $\mathsf{word}(D(w))$ in $D(v)$. Thus if $D \in \overline{\mathsf{SEYD}}(v_P, w_P)$, the fact that $s_i w_P = w$ implies
$$e_{\delta({\phi_v}^{-1}(D'))} = e_{\delta({\phi_v}^{-1}(\{\mathsf{z}\}))} e_{\delta({\phi_v}^{-1}(D))} = e_{s_i} e_{w_P} = e_w.$$
Then $D \in \overline{\mathsf{SEYD}}(v,w)$ by Proposition 2.6. Alternatively if $D \in \overline{\mathsf{SEYD}}(v_C, w_C)$,
$$e_{\delta({\phi_v}^{-1}(D'))} = e_{\delta({\phi_v}^{-1}(\{\mathsf{z}\}))} e_{\delta({\phi_v}^{-1}(D))} = e_{s_i} e_w = e_w.$$
Thus $D \in \overline{\mathsf{SEYD}}(v,w)$, so the result follows by Proposition 2.6. □

We apply Lemma 4.7 to obtain a recurrence for degrees of unspecialized Grothendieck polynomials.

Corollary 4.8. *Let $v \geqslant w$ and $v, w \in U_n(321)$. Suppose z is the box containing the northmost then eastmost plus in $D_{\mathsf{top}}(v,w)$ and z' is the northmost then eastmost box in $\mathcal{R}_v$. The following hold:*

(1) If $\mathsf{z} \neq \mathsf{z}'$, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \deg(\mathfrak{G}_{v_C, w_C}(\mathbf{t}))$.
(2) If $\mathsf{z} = \mathsf{z}'$, $\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = 1 + \max(\deg(\mathfrak{G}_{v_P, w_P}(\mathbf{t})), \deg(\mathfrak{G}_{v_C, w_C}(\mathbf{t})))$.

Proof. This follows by combining Corollary 2.7 with Lemma 4.7. □

Lastly we establish some technical lemmas for the $\mathsf{z} = \mathsf{z}'$ case.

Lemma 4.9. *Let $v \geqslant w$ and $v, w \in U_n(321)$. Suppose z is the northmost then eastmost box in $\mathcal{R}_v$ and $\mathsf{z} \in C_q$, where C_q is a connected component in $D_{\mathtt{top}}(v,w)$. Then the following hold:*

(1) $D_{\mathtt{top}}(v_P, w_P) = D_{\mathtt{top}}(v,w) - \{\mathsf{z}\}$, and
(2) $D_{\mathtt{top}}(v_C, w_C) = D_{\mathtt{top}}(v,w) - R \cup \widetilde{R}$ when $\overline{\mathsf{SEYD}}(v_C, w_C) \neq \varnothing$,

where $R = \{\mathsf{d} \in C_q \mid \mathsf{d}$ weakly southwest of $\mathsf{z}\}$ and $\widetilde{R} = \{\mathsf{d} + (1,-1) \mid \mathsf{d} \in R\}$.

Proof. (1) follows from the definitions of v_P and w_P since z lies on the northeast boundary of $\mathcal{R}_v$. Assuming $\overline{\mathsf{SEYD}}(v_C, w_C) \neq \varnothing$, $D_{\mathtt{top}}(v_C, w_C) \in \overline{\mathsf{SEYD}}(v,w)$. By construction

$$\delta({\phi_v}^{-1}(D_{\mathtt{top}}(v,w) - R \cup \widetilde{R})) = w = w_C.$$

Since $\overline{\mathsf{SEYD}}(v_C, w_C) \neq \varnothing$, it follows that $D_{\mathtt{top}}(v,w) - R \cup \widetilde{R} \subset \mathcal{R}_{v_C}$. Then

$$D_{\mathtt{top}}(v,w) - R \cup \widetilde{R} \in \overline{\mathsf{SEYD}}(v_C, w_C).$$

No reverse excited moves can be applied in $\mathcal{R}_{v_C}$, so (2) follows. ☐

For brevity, if $\sigma, \rho \in U_n(321)$ such that $\sigma \geqslant \rho$, where $D_{\mathtt{top}}(\sigma,\rho)$ has components $\{C_q\}_{q \in [m]}$ write

$$\Delta_{\sigma,\rho}(q) := \sum_{\mathsf{b} \in \mathtt{md}(C_q)} \mathsf{trail}_{\sigma,\rho}(\mathsf{b}) \quad \text{and} \quad d(\sigma,\rho) := \sum_{q \in [m]} \Delta_{\sigma,\rho}(q).$$

Lemma 4.10. *Let $v \geqslant w$ and $v, w \in U_n(321)$. Suppose z is the northmost then eastmost box in $\mathcal{R}_v$ and $\mathsf{z} \in C_q$, where C_q is a connected component in $D_{\mathtt{top}}(v,w)$. Then the following hold:*

(1) $d(v,w) \geqslant d(v_P, w_P)$, and
(2) $d(v,w) = d(v_P, w_P)$ if $\mathsf{z} \notin \mathtt{md}(C_q)$ or if $\mathsf{z} \in \mathtt{md}(C_q)$ where $\mathsf{trail}_{v,w}(\mathsf{z})) = 0$.

Proof. Let $\{C_q\}_{q \in [m]}$ and $\{\widehat{C}_q\}_{q \in [m']}$ denote the components of $D_{\mathtt{top}}(v,w)$ and $D_{\mathtt{top}}(v_P, w_P)$ respectively. Suppose $\mathsf{z} \in C_q$. We assume $\mathsf{z} \subsetneq C_q$. If $\{\mathsf{z}\} = C_q$ the result follows similarly, so we omit the proof. By Lemma 4.9 if $\mathsf{z} \subsetneq C_q$, $m = m'$. Using Lemma 4.9 and the definition of $\mathtt{md}(\cdot)$, $\mathtt{md}(C_{q'}) = \mathtt{md}(\widehat{C}_{q'})$ for $q' > q$.

Suppose $\mathsf{z} \notin \mathtt{md}(C_q)$. Then $\mathtt{md}(C_{q'}) = \mathtt{md}(\widehat{C}_{q'})$ for $q' \leqslant q$. Since z is northeast most, $\mathsf{trail}_{v,w}(\mathsf{b}) = \mathsf{trail}_{v_P,w_P}(\mathsf{b})$ for each $\mathsf{b} \in \mathtt{md}(C_{q'})$, $q' \in [m]$. Thus $\Delta_{v,w}(q') = \Delta_{v_P,w_P}(q')$ for each $q' \in [m]$, so $d(v,w) = d(v_P, w_P)$.

Now assume $\mathsf{z} \in \mathtt{md}(C_q)$. Then since z is northeast most and $\mathtt{md}(C_q)$ is a westmost diagonal, $\mathtt{md}(C_q) = \mathtt{md}(\widehat{C}_q) \cup \{\mathsf{z}\}$. Since z is a northmost plus and $\mathsf{z} \in \mathtt{md}(C_q)$, $\mathtt{md}(C_{q'}) = \mathtt{md}(\widehat{C}_{q'})$ for $q' < q$. Thus $\mathsf{trail}_{v,w}(\mathsf{b}) = \mathsf{trail}_{v_P,w_P}(\mathsf{b})$ for each $\mathsf{b} \in \mathtt{md}(C_{q'})$, $q \in [m]$ where $\mathsf{b} \neq \mathsf{z}$. Therefore

$$\begin{aligned} \Delta_{v,w}(q') &= \Delta_{v_P,w_P}(q') \text{ for } q' \in [m] - \{q\}, \text{ and} \\ \Delta_{v,w}(q) &= \Delta_{v_P,w_P}(q) + \mathsf{trail}_{v,w}(\mathsf{z}). \end{aligned}$$

Thus $d(v,w) = d(v_P, w_P) + \mathsf{trail}_{v,w}(\mathsf{z})$, so (1) and (2) follow. ☐

Lemma 4.11. *Let $v \geqslant w$ and $v, w \in U_n(321)$. Suppose z is the northmost then eastmost box in $\mathcal{R}_v$ and $\mathsf{z} \in C_q$, where C_q is a connected component in $D_{\mathtt{top}}(v,w)$. Assume $\mathsf{SEYD}(v_C, w_C) \neq \varnothing$. Then the following hold:*

(1) $d(v,w) \geqslant d(v_C, w_C) + 1$, and
(2) $d(v,w) = d(v_C, w_C) + 1$ if $\mathsf{z} \in \mathtt{md}(C_q)$.

Proof. Let $\{C_q\}_{q\in[m]}$ and $\{\widetilde{C}_q\}_{q\in[m']}$ denote the components of $D_{\mathtt{top}}(v,w)$ and $D_{\mathtt{top}}(v_C, w_C)$ respectively. Suppose $\mathsf{z} \in C_q$. By Lemma 4.9, $D_{\mathtt{top}}(v_C, w_C) = D_{\mathtt{top}}(v,w) - R \cup \widetilde{R}$. We assume $\widetilde{R} \subseteq \widetilde{C}_{q-1}$ and $R \subsetneq C_q$. The other cases follow similarly, so we omit their proofs.

When $\widetilde{R} \subseteq \widetilde{C}_{q-1}$ and $R \subsetneq C_q$, $m = m'$. By Lemma 4.9 and the definition of $\mathtt{md}(\cdot)$, $\mathtt{md}(C_{q'}) = \mathtt{md}(\widetilde{C}_{q'})$ for $q' > q$. Partition $\mathtt{md}(C_q) = \{\mathsf{b}_i\}_{i\in[k]} \cup \{\mathsf{b}_i\}_{i\in\{k+1,\ldots,\ell_q\}}$ where

$$\{\mathsf{b}_i\}_{i\in[k]} = \{\mathsf{b}_i \in \mathtt{md}(C_q) \cap R \mid \mathsf{b}_i \text{ lies strictly north of } C_q - R\}.$$

Let $\mathtt{md}(\widetilde{C}_q) = \{\mathsf{c}_i\}_{i\in[\widetilde{\ell}_q]}$. Since $\mathtt{md}(C_q)$ has maximal length, $\widetilde{\ell}_q \leqslant \ell_q - k$. Then since $\mathtt{md}(\widetilde{C}_q)$ and $\mathtt{md}(C_q)$ are southmost choices, $\mathsf{c}_{\widetilde{\ell}_q-i+1}$ and b_{ℓ_q-i+1} lie in the same rows for $i \in [\widetilde{\ell}_q]$. This ensures $\mathsf{trail}_{v,w}(\mathsf{b}) = \mathsf{trail}_{v_C,w_C}(\mathsf{b})$ for each $\mathsf{b} \in C_{q'}$, $q' > q$. Thus

$$\Delta_{v,w}(q') = \Delta_{v_C,w_C}(q') \text{ for } q' > q.$$

Since z is northeast most, by the definition of $\mathtt{md}(\cdot)$, $\mathtt{md}(C_{q-1}) = \mathtt{md}(\widetilde{C}_{q-1}) - \widetilde{R}$. Then $\mathtt{md}(C_{q'}) = \mathtt{md}(\widetilde{C}_{q'})$ for $q' < q-1$, so $\mathsf{trail}_{v,w}(\mathsf{b}) = \mathsf{trail}_{v_C,w_C}(\mathsf{b})$ for each $\mathsf{b} \in C_{q'}$, $q' < q-1$. Therefore

$$\Delta_{v,w}(q') = \Delta_{v_C,w_C}(q') \text{ for } q' < q-1.$$

To prove (1) it remains to show

$$\Delta_{v,w}(q-1) + \Delta_{v,w}(q) \geqslant \Delta_{v_C,w_C}(q-1) + \Delta_{v_C,w_C}(q) + 1. \tag{4.1}$$

Partition $\mathtt{md}(\widetilde{C}_{q-1}) \cap \widetilde{R} = \{\mathsf{a}_i\}_{i\in I_1} \cup \{\mathsf{a}_i\}_{i\in I_2}$, where

$$I_2 := \{i \in [\#(\mathtt{md}(\widetilde{C}_{q-1}) \cap \widetilde{R})] \mid \text{ there exists } \mathsf{c}_j \in \mathtt{md}(\widetilde{C}_q) \text{ strictly north of } \mathsf{a}_i \text{ where } j \in [\widetilde{\ell}_q]\}.$$

Thus we find:

$$\begin{aligned}
\Delta_{v_C,w_C}(q-1) + \Delta_{v_C,w_C}(q) &= \sum_{\mathsf{b}\in\mathtt{md}(\widetilde{C}_{q-1})} \mathsf{trail}_{v_C,w_C}(\mathsf{b}) + \sum_{i\in[\widetilde{\ell}_q]} \mathsf{trail}_{v_C,w_C}(\mathsf{c}_i) \\
&= \Delta_{v,w}(q-1) + \sum_{i\in I_1\cup I_2} \mathsf{trail}_{v_C,w_C}(\mathsf{a}_i) + \sum_{i\in[\widetilde{\ell}_q]} \mathsf{trail}_{v_C,w_C}(\mathsf{c}_i).
\end{aligned}$$

Since $\mathsf{c}_{\widetilde{\ell}_q-i+1}$ and b_{ℓ_q-i+1} lie in the same rows for $i \in [\widetilde{\ell}_q]$, by the definition of I_2 and trail we obtain the first inequality below. Note that $\{\mathsf{a}_i + (-1,1)\}_{i\in I_1\cup I_2}$ lies weakly south of $\{\mathsf{b}_i\}_{i\in[k]}$, and $|I_1 \cup I_2| \leqslant k$ by the definition of $\mathtt{md}(\cdot)$. Combining these with the fact that $\widetilde{\ell}_q \leqslant \ell_q - k$, we obtain the second inequality.

$$\begin{aligned}\sum_{i\in I_1\cup I_2}\mathsf{trail}_{v_C,w_C}(\mathsf{a}_i)+\sum_{i\in[\widetilde{\ell}_q]}\mathsf{trail}_{v_C,w_C}(\mathsf{c}_i) &\leqslant \sum_{i\in I_1}\mathsf{trail}_{v_C,w_C}(\mathsf{a}_i)+\sum_{i\in[\widetilde{\ell}_q]}\mathsf{trail}_{v,w}(\mathsf{b}_{\ell_q-i+1})\\ &\leqslant \sum_{i\in[k]}(\mathsf{trail}_{v,w}(\mathsf{b}_i)-1)+\sum_{i\in\{k+1,\ldots,\ell_q\}}\mathsf{trail}_{v,w}(\mathsf{b}_i)\\ &\leqslant \Delta_{v,w}(q)-1.\end{aligned} \tag{4.2}$$

Thus Equation (4.1) is proven, so (1) follows.

Now suppose $\mathsf{z}\in\mathtt{md}(C_q)$. Then (2) follows once we show equality is attained in Equation (4.1). Since $\widetilde{R}\subseteq\widetilde{C}_{q-1}$, $\mathsf{trail}_{v,w}(\mathsf{z})=1$.

There are no pluses in rows $j<a$ since z is a northmost plus. Since $\mathtt{md}(C_q)$ is westmost in C_q, $\mathsf{z}+(0,-1)\notin D_{\mathtt{top}}(v,w)$. If $\mathsf{z}+(j,-1)\in C_q$ for some $j>0$,

$$\mathsf{z}+(j',-1),\mathsf{z}+(j',0)\in C_q \text{ for all } j'\in[j-1].$$

Therefore $\mathsf{d}\in R$ implies $\mathsf{d}(2)=\mathsf{z}(2)$. Thus $\#(I_1\cup I_2)\leqslant 1$.

By the definition of $\mathtt{md}(\cdot)$, $\mathsf{z}\in\mathtt{md}(C_q)$ implies

$$\mathtt{md}(\widetilde{C}_q)=\mathtt{md}(C_q)-R=\mathtt{md}(C_q)-\{\mathsf{z}\},$$

so $\mathsf{c}_i=\mathsf{b}_{i+1}$ for $i\in[\ell_q-1]$. Since $\mathtt{md}(\cdot)$ is a southmost choice and $R\subsetneq C_q$, $\mathsf{z}+(1,1)\in C_q$. Since $\mathtt{md}(\cdot)$ is a southwest most choice and $\mathsf{z}\in\mathtt{md}(C_q)$, $\mathsf{z}+(1,1)\in\mathtt{md}(C_q)$. Therefore $I_1=\varnothing$. Then we refine Equation (4.2) to find

$$\begin{aligned}\sum_{i\in I_1\cup I_2}\mathsf{trail}_{v_C,w_C}(\mathsf{a}_i)+\sum_{i\in[\widetilde{\ell}_q]}\mathsf{trail}_{v_C,w_C}(\mathsf{c}_i) &= \sum_{i\in I_2}\mathsf{trail}_{v_C,w_C}(\mathsf{a}_i)+\sum_{i\in\{2,\ldots,\ell_q\}}\mathsf{trail}_{v_C,w_C}(\mathsf{b}_i)\\ &= \mathsf{trail}_{v_C,w_C}(\mathsf{a}_1)+\sum_{i\in\{2,\ldots,\ell_q\}}\mathsf{trail}_{v_C,w_C}(\mathsf{b}_i)\\ &= \sum_{i\in\{2,\ldots,\ell_q\}}\mathsf{trail}_{v,w}(\mathsf{b}_i)\\ &= \Delta_{v,w}(q)-\mathsf{trail}_{v,w}(\mathsf{z})=\Delta_{v,w}(q)-1.\end{aligned}$$

Here if a_1 does not exist, we say $\mathsf{trail}_{v_C,w_C}(\mathsf{a}_1)=0$. Thus (2) is proven. □

Proof of Theorem 4.3: The second equality follows by the definition of $\mathsf{trail}_{v,w}(\mathsf{b})$ where $\mathsf{b}\in\{\mathtt{md}(C_q)\}_{q\in[m]}$ and the fact that $\#D_{\mathtt{zip}}(v,w)=\ell(w)$. If $\ell(w)=0$, $\mathsf{SEYD}(v,w)=\varnothing$, so by Corollary 2.7, the first equality follows.

We prove the remainder of cases for the first equality by induction on $\ell(v)$. For $\ell(v)=0$, the assumption $v\geqslant w$ implies $\ell(w)=0$, which is proven. Suppose the statement holds for v such that $\ell(v)=k-1$ for $k\geqslant 1$.

Consider v such that $\ell(v)=k$ and assume $\ell(w)>0$. Suppose z is the northmost then eastmost plus in $D_{\mathtt{top}}(v,w)$ and z' is the northmost then eastmost box in $\mathcal{R}_v$.

If $\mathsf{z}\neq\mathsf{z}'$, Lemma 4.7 implies $D_{\mathtt{top}}(v,w)=D_{\mathtt{top}}(v_C,w_C)$. Since z' is the northeast most box, $D_{\mathtt{zip}}(v,w)=D_{\mathtt{zip}}(v_C,w_C)$, so $D^K_{\mathtt{zip}}(v,w)=D^K_{\mathtt{zip}}(v_C,w_C)$ again since z' is northeast most. Therefore $\#D^K_{\mathtt{zip}}(v,w)=\#D^K_{\mathtt{zip}}(v_C,w_C)$. By the inductive assumption, we have $\#D^K_{\mathtt{zip}}(v_C,w_C)=\deg(\mathfrak{G}_{v_C,w_C}(\mathbf{t}))$, so the result follows by Corollary 4.8.

Now assume $\mathsf{z} = \mathsf{z}'$. Since $w = w_C = s_i w_P$ where $i \in [n-1]$, $\ell(w) = \ell(w_C)$ and $\ell(w) = \ell(w_P) + 1$. Then by the inductive assumption and Corollary 4.8,

$$\deg(\mathfrak{G}_{v,w}(\mathbf{t})) = \ell(w) + \max(d(v_P, w_P), d(v_C, w_C) + 1).$$

Applying the second equality, it suffices to prove

$$d(v, w) = \max(d(v_P, w_P), d(v_C, w_C) + 1).$$

Suppose $\mathsf{z} \in C_q$, where C_q is a connected component in $D_{\mathtt{top}}(v, w)$. Note that when $\mathsf{z} \in \mathtt{md}(C_q)$, $\mathsf{SEYD}(v_C, w_C) \neq \varnothing$ if and only if $\mathsf{trail}_{v,w}(\mathsf{z})) > 0$. Then the result follows from Lemmas 4.10 and 4.11. ☐

4.3. **Proof of Theorem 1.1 and Corollary 1.3.** Using the results of the previous subsection, we can prove our main result.

Proof of Theorem 1.1: This follows from combining Proposition 3.2 and Theorem 4.3. ☐

Proof of Corollary 1.3: This follows from Corollary 3.3 combined with Theorem 4.3. ☐

5. Regularity of Ladder Determinantal Varieties

In this section we use the result of L. Escobar, A. Fink, J. Rajchgot, and A. Woo [10] which states two-sided ladder determinantal varieties are Kazhdan–Lusztig varieties indexed by particular $v, w \in U_n(321)$. In this setting, Construction 4.2 is simplified. We give specializations of Theorem 1.1 and Corollary 1.3 accordingly.

Lastly in this two-sided ladder case, we reformulate Theorem 1.1 and Corollary 1.3 in terms of lattice paths. This generalizes work of S. R. Ghorpade and C. Krattenthaler [13].

5.1. **Ladder Determinantal Varieties.** A **ladder region** L is a skew Young diagram λ/μ. We assume λ and μ have $\ell(\lambda)$ non-negative parts. For $L = \lambda/\mu$, we define the **perimeter** of L as $2n$, where $n = \lambda_1 + \lambda'_1$, *i.e.*, the number of boxes in the first row plus the number of boxes in the first column of λ.

A ladder region L is equivalently determined by its southwest corners $L^{\mathsf{SW}} = \{\alpha_i\}_{i\in[s]}$ and northeast corners $L^{\mathsf{NE}} = \{\beta_i\}_{i\in[t]}$, with points ordered northwest to southeast. Define $\alpha_0 = (0, 0)$ to be the northwest most corner of L and let α_{s+1} denote the southeast most corner of L. For a point γ in L write $\gamma = (\gamma(1), \gamma(2))$. A box b in L inherits the label of its southeast corner.

Let $\mathcal{M} = \{(\mathsf{p}_i, r_i)\}_{i\in[s']}$ denote a set of marked points along the southwest border of L where $r_i \in \mathbb{Z}_{>0}$. Points in $\mathcal{M}$ are ordered northwest to southeast.

Define $L(z)$ as the filling of each $(i, j) \in L$ with indeterminate z_{ij}. Take $\mathbb{C}[L(z)]$ the polynomial ring generated by entries in $L(z)$. Define the **two-sided mixed ladder determinantal ideal** $I_{L,\mathcal{M}}$:

$$I_{L,\mathcal{M}} := \langle r_i - \text{minors in } L_{[\mathsf{p}_i(1)],[\mathsf{p}_i(2)+1,\alpha_{s+1}(2)]}(z) \mid (\mathsf{p}_i, r_i) \in \mathcal{M}\rangle \subseteq \mathbb{C}[L(z)],$$

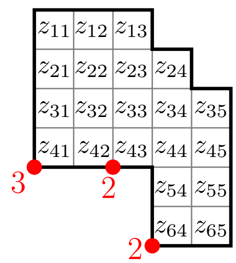
where $L_{I,J}(z)$ denotes the submatrix of $L(z)$ with row indices in I and column indices in J for $I, J \subseteq [n]$. The **two-sided mixed ladder determinantal variety** has coordinate ring $X_{L,\mathcal{M}} := \mathbb{C}[L(z)]/I_{L,\mathcal{M}}$. Taking $L = \lambda$, *i.e.*, when $\mu = \varnothing$, produces a **one-sided mixed ladder determinantal variety**. Define $(L, \mathcal{M})$ to be **minimal ladder** if

(1) each $z_{ij} \in L(z)$ appears in a monomial of a generator in $I_{L,\mathcal{M}}$,

(2) $0 < \mathsf{p}_1(1) - r_1 < \mathsf{p}_2(1) - r_2 < \cdots < \mathsf{p}_{s'}(1) - r_{s'}$, and
(3) $0 < \mathsf{p}_1(2) - r_1 < \mathsf{p}_2(2) - r_2 < \cdots < \mathsf{p}_{s'}(2) - r_{s'}$.

It is straightforward to reduce any two-sided ladder to a minimal two-sided ladder.

Example 5.1. Let $L = \lambda/\mu$, where $\lambda = (5,5,5,5,2,2)$ and $\mu = (2,1,0,0,0,0)$. Then $L^{\mathsf{SW}} = \{(4,0),(6,3)\}$ and $L^{\mathsf{NE}} = \{(0,3),(1,4),(2,5)\}$. Below is $L(z)$ with marked points $\mathcal{M} = \{((4,0),3),((4,2),2),((6,3),2)\}$ drawn in red.



Then $I_{L,\mathcal{M}} = \langle 3 - \text{minors of } L_{[4],[5]}(z), 2 - \text{minors of } L_{[4],\{3,4,5\}}(z), 2 - \text{minors of } L_{[6],\{4,5\}}(z)\rangle$. $\diamondsuit$

5.2. **Two-sided ladders and Kazhdan–Lusztig Varieties.** Let $(L,\mathcal{M})$ be a minimal two-sided ladder where $L = \lambda/\mu$, $\mathcal{M} = \{(\mathsf{p}_i, r_i)\}_{i\in[s']}$, and L has perimeter $2n$. Define $s_v \in \mathbb{Z}^n_{\geqslant 0}$ as the sequence

$$s_v := (\lambda_1 - \mu_1, 0^{\lambda_1-\lambda_2}, \lambda_2 - \mu_2, 0^{\lambda_2-\lambda_3}, \ldots, \lambda_{\ell(\lambda)} - \mu_{\ell(\lambda)}, 0^{\lambda_{\ell(\lambda)}}).$$

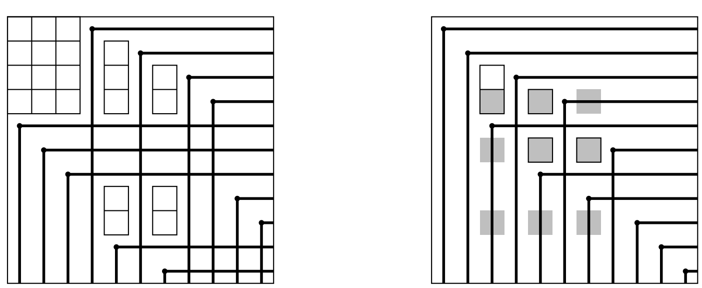
Let $v \in S_n$ be the unique permutation such that $\mathsf{code}(v) = s_v$. Suppose $L^{\mathsf{NE}} = \{\beta_j\}_{j\in[t]}$. Then take $w \in S_n$ to be the minimal length permutation satisfying

$$\mathsf{rank}_w((\|\mathsf{p}_i\|, \|\beta_j\|)) = \min\left(\{\|\mathsf{p}_i\|, \|\beta_j\|, \mathsf{rank}_v((\|\mathsf{p}_i\|, \|\beta_j\|)) + r_i - 1\}\right)$$

for each $i \in [s'], j \in [t]$. Here $\|\gamma\| = \gamma(1)+\gamma(2)$ for a point γ. We define $\mathsf{perm}(L,\mathcal{M}) = (v,w)$.

This formula to compute $\mathsf{perm}(L,\mathcal{M})$ refines the formula in [15, Theorem 4.7.3] for the one-sided ladder case.

Example 5.2. Let L and $\mathcal{M}$ be as in Example 5.1. Below are $D(v)$ and $D(w)$ such that $(v,w) = \mathsf{perm}(L,\mathcal{M})$. In $D(w)$, the positions $\{(\|\mathsf{p}_i\|, \|\beta_j\|)\}_{i\in[s'],j\in[t]}$ are shaded.



$\diamondsuit$

One-sided ladder determinantal varieties are isomorphic to vexillary matrix Schubert varieties, see [15, 25]. In general, two-sided ladder determinantal varieties are not isomorphic to matrix Schubert varieties. For example, if $(L,\mathcal{M})$ is as in Example 5.1, $X(L,\mathcal{M})$ is not isomorphic to a matrix Schubert variety. As proven by L. Escobar, A. Fink, J. Rajchgot, and A. Woo, all two-sided ladders can realized as Kazhdan–Lusztig varieties:

Theorem 5.3. [10] *Given $(L, \mathcal{M})$ minimal, suppose $\mathsf{perm}(L, \mathcal{M}) = (v, w)$ and L has perimeter $2n$. Then the following hold:*

(1) $v, w \in U_n(321)$ where $v \geqslant w$, and
(2) $I_{L,\mathcal{M}}$ and $J_{v,w}$ have the same set of generators.

5.3. **Specializing Theorem 1.1.** When $(v, w) = \mathsf{perm}(L, \mathcal{M})$ for $(L, \mathcal{M})$ minimal, diagrams in $\overline{\mathsf{SEYD}}(v, w)$ exhibit additional structure. This allows us to re-frame Theorem 1.1 and Corollary 1.3 in terms of lattice paths in L.

Construction 5.4 (Computing boundary points)**.** Take a minimal two-sided ladder $(L, \mathcal{M})$ where $\mathcal{M} = \{(\mathsf{p}_i, r_i)\}_{i\in[s']}$ and $L^{\mathsf{SW}} = \{\alpha_i\}_{i\in[s]}$. For each $i \in [s-1]$ let

$$r_i^H := \min\{r_{i_j} \mid (\mathsf{p}_{i_j}, r_{i_j}) = ((\alpha_i(1), y)), r_{i_j}) \in \mathcal{M}\}, \text{ and}$$
$$r_i^V := \min\{r_{i_j} \mid (\mathsf{p}_{i_j}, r_{i_j}) = (x, \alpha_i(2)), r_{i_j}) \in \mathcal{M}\}.$$

Initialize $\mathcal{M}' = \mathcal{M}$. For each $i \in [s-1]$, if $((\alpha_i(1), \alpha_{i+1}(2)), r) \notin \mathcal{M}'$ for any $r \in \mathbb{Z}_{>0}$, append

$$\big((\alpha_i(1), \alpha_{i+1}(2)), \min(r_i^H, r_{i+1}^V)\big)$$

to $\mathcal{M}'$. Lastly append $(\alpha_0, 1)$ and $(\alpha_{s+1}, 1)$ to $\mathcal{M}'$. Partition $\mathcal{M}' = \bigcup_{i\in[s]} \mathcal{M}_i^V \cup \mathcal{M}_i^H$, where

$$\mathcal{M}_i^V := \{(\mathsf{p}_{i_j}, r_{i_j}) \mid \mathsf{p}_{i_j} = (x, \alpha_i(2))\}, \text{ and}$$
$$\mathcal{M}_i^H := \{(\mathsf{p}_{i_j}, r_{i_j}) \mid \mathsf{p}_{i_j} = (\alpha_i(1), y)\}.$$

Points in $\mathcal{M}_i^V$ are ordered north to south and those in $\mathcal{M}_i^H$ are ordered east to west.

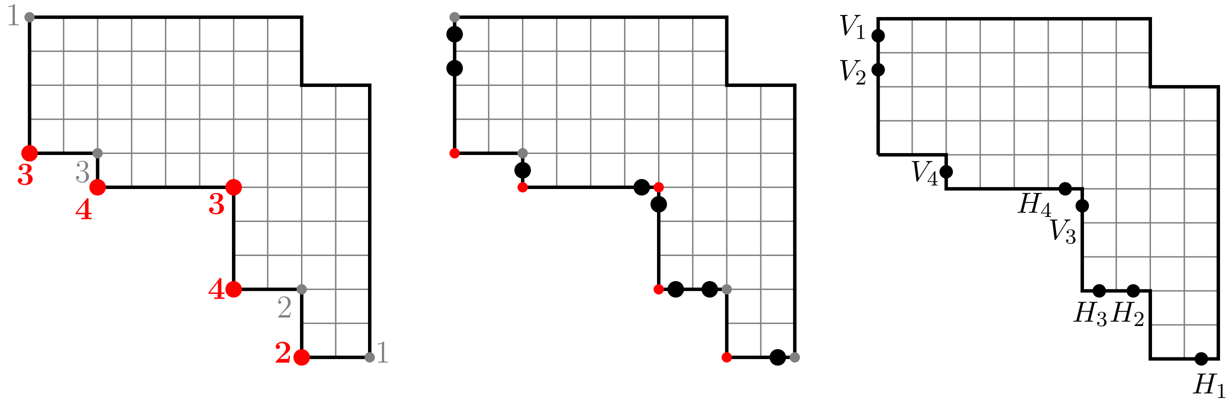
Initialize $V(\mathcal{M}) = \varnothing$ and $H(\mathcal{M}) = \varnothing$. Then iterate the following for each $i \in [s]$:

- For each $j \in [\#\mathcal{M}_i^V - 1]$, take $(\mathsf{p}_{i_j}, r_{i_j}) \in \mathcal{M}_i^V$. If $r_{i_{j+1}} - r_{i_j} = k \geqslant 1$, append $\mathsf{p}_{i_j} + (k' - \frac{1}{2}, 0)$ to $V(\mathcal{M})$ for each $k' \in [k]$.
- For each $j \in [\#\mathcal{M}_i^H - 1]$, take $(\mathsf{p}_{i_j}, r_{i_j}) \in \mathcal{M}_i^H$. If $r_{i_{j+1}} - r_{i_j} = k \geqslant 1$, append $\mathsf{p}_{i_j} + (0, -k' + \frac{1}{2})$ to $H(\mathcal{M})$ for each $k' \in [k]$.

This gives boundary points $V(\mathcal{M})$ and $H(\mathcal{M})$.

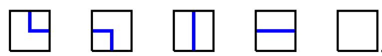
Suppose $\#V(\mathcal{M}) = \ell \in \mathbb{Z}_{\geqslant 0}$. Then $\#H(\mathcal{M}) = \ell$ by construction. For $i \in [\ell]$, label points $H_i \in H(\mathcal{M})$ in increasing order from east to west. For $i \in [\ell]$ in decreasing order, assign label V_i to be the southmost point in $V(\mathcal{M}) - \{V_j \mid \ell \geqslant j > i\}$ that lies northwest of H_i.

Example 5.5. We illustrate Construction 5.4 below. The left diagram draws a ladder L with the original marked points $\mathcal{M}$ bolded in red and $\mathcal{M}' - \mathcal{M}$ in light gray. The middle diagram adds $V(\mathcal{M})$ and $H(\mathcal{M})$ in bold black. The right diagram labels $V(\mathcal{M})$ and $H(\mathcal{M})$.



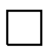
♢

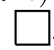
A **lattice path** from H_i to V_i in the region L is a path from H_i to V_i in L consisting north and west steps in L. We visualize lattice paths in L with tiles

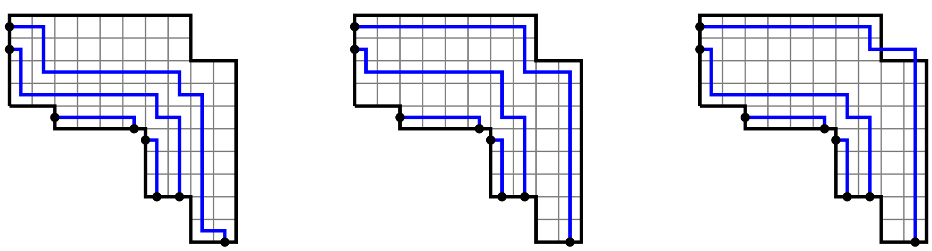


We call the leftmost tile a **SE-elbow** tile and the rightmost tile a **blank** tile. For a box $\mathsf{b} \in L$, let $t(\mathsf{b})$ denote the tile occupying b.

For a minimal two-sided ladder $(L, \mathcal{M})$ where $L = \lambda/\mu$, define $\mathsf{NILP}(L, \mathcal{M})$ to be the set of non-intersecting lattice paths $P = (P_1, \ldots, P_\ell)$ where P_i is a lattice path in L from H_i to V_i, for $H_i \in H(\mathcal{M}), V_i \in V(\mathcal{M})$. A path P_i may occupy box $(x, y) \in \mu$ only if:

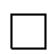
(1) $t(x, y) \neq$ , and
(2) $t(x + k, y - k) \neq$  for any $(x + k, y - k) \in \lambda$ where $k \in \mathbb{Z}_{>0}$.

Example 5.6. For $(L, \mathcal{M})$ as in Example 5.5, the leftmost two diagrams are in $\mathsf{NILP}(L, \mathcal{M})$. The rightmost diagram is not since P_1 occupies $(2, 9) \in \mu$ but $t(2 + 1, 9 - 1) = t(3, 8) =$ .

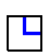


♢

For $P \in \mathsf{NILP}(L, \mathcal{M})$, define

$$\mathsf{blanks}(P) := \Big\{(i, j) \in L \mid t(i, j) = \square\Big\}.$$


Let $\mathtt{wt}(L, \mathcal{M}) = \#L - \#\mathsf{blanks}(P)$, where $\#L$ denotes the number of boxes in L and $P \in \mathsf{NILP}(L, \mathcal{M})$. By the definition of $\mathsf{NILP}(L, \mathcal{M})$, $\#\mathsf{blanks}(P)$ is constant across all $P \in \mathsf{NILP}(L, \mathcal{M})$. For $P \in \mathsf{NILP}(L, \mathcal{M})$, define the **unforced elbows** of P as the set

$$\mathsf{elbows}(P) := \{(i, j) \in L \mid t(i, j) = \text{[SE-elbow tile]} \text{ and } t(i - k, j + k) \in \mathsf{blanks}(P) \text{ for some } k \geqslant 0\}.$$


Define the map

$$\begin{aligned}\psi : \mathsf{NILP}(L, \mathcal{M}) &\to [n] \times [n] \\ P &\mapsto \mathsf{blanks}(P).\end{aligned}$$

For a minimal two-sided ladder $(L, \mathcal{M})$, take $P_{\mathtt{bot}}(L, \mathcal{M}) \in \mathsf{NILP}(L, \mathcal{M})$ to be the non-intersecting lattice path in which each path lies maximally southwest in L.

Ordinary excited Young diagrams naturally biject with non-intersecting lattice paths when $L = \lambda$, as shown by V. Kreiman [26]. We prove the corresponding bijection for skew excited Young diagrams.

Proposition 5.7. *Let $(L, \mathcal{M})$ be a minimal two-sided ladder. Then for $(v, w) = \mathsf{perm}(L, \mathcal{M})$,*

$$\begin{aligned}\psi : \mathsf{NILP}(L, \mathcal{M}) &\to \mathsf{SEYD}(v, w) \\ P &\mapsto \mathsf{blanks}(P)\end{aligned}$$

is a bijective map, where $\psi(P_{\mathtt{bot}}(L, \mathcal{M})) = D_{\mathtt{top}}(v, w)$.

Proof. We first show $\psi(P_{\mathtt{bot}}(L, \mathcal{M})) = D_{\mathtt{top}}(v, w)$. Suppose $L = \lambda/\mu$. Without loss of generality, assume λ, μ are such that connected components in λ/μ share corners.

We proceed by induction on $|\mu|$. When $|\mu| = 0$, L is a one-sided ladder. This implies v, w are Grassmannian, as proven in [15, Theorem 4.7.3]. This case is proven in [26, Section 5]. See [29] and [38, Section 7.3] for additional discussion.

Suppose the result holds for $|\mu| = k - 1 \geqslant 0$. Consider $L = \lambda/\mu$ such that $|\mu| = k$. Take $(a, b) \in L$ to the northmost box in L such that $(a + 1, b + 1) \in L$ and $(a, b + 1) \notin L$. Let $L' := L \cup (a, b + 1)$.

By the inductive assumption since $(L', \mathcal{M})$ is also minimal, the result holds for $(L', \mathcal{M})$. Let $(v', w') = \mathsf{perm}(L', \mathcal{M})$. Since $L' = L \cup (a, b + 1)$, this determines

$$(L')^{\mathsf{NE}} = L^{\mathsf{NE}} - \{(a - 1, b), (a, b + 1)\} \cup \{(a - 1, b + 1)\}. \tag{5.1}$$

Then by definition of v and choice of (a, b), $v = s_i v'$, where $i = \mathsf{word}((a, b + 1))$ in $D(v')$.

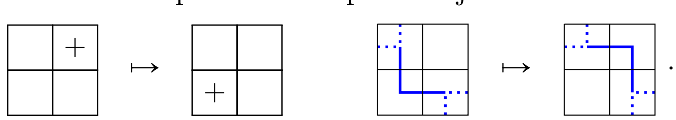
Suppose $t(a, b + 1) = \square$ in $P_{\mathtt{bot}}(L', \mathcal{M})$. Using the inductive assumption, we know that $(a, b + 1) \in D_{\mathtt{top}}(v', w')$. By the definition of $\mathsf{perm}(L, \mathcal{M})$ and Equation (5.1), it is straightforward to check through case analysis on $L^{\mathsf{NE}} \cap \{(a-1, b), (a, b+1)\}$ that $w = s_i w'$. Thus $D_{\mathtt{top}}(v, w) = D_{\mathtt{top}}(v', w') - \{(a, b + 1)\}$. Since

$$\mathsf{blanks}(P_{\mathtt{bot}}(L, \mathcal{M})) = \mathsf{blanks}(P_{\mathtt{bot}}(L', \mathcal{M})) - \{(a, b + 1)\},$$

we find $\psi(P_{\mathtt{bot}}(L, \mathcal{M})) = D_{\mathtt{top}}(v, w)$.

Otherwise $t(a, b + 1) \neq \square$ in $P_{\mathtt{bot}}(L', \mathcal{M})$. By the inductive assumption, we know that $(a, b + 1) \notin D_{\mathtt{top}}(v', w')$. By the definition of $\mathsf{perm}(L, \mathcal{M})$ and Equation (5.1), it is straightforward to check through case analysis on $L^{\mathsf{NE}} \cap \{(a - 1, b), (a, b + 1)\}$ that $w = w'$. Thus $D_{\mathtt{top}}(v, w) = D_{\mathtt{top}}(v', w')$. Since $\mathsf{blanks}(P_{\mathtt{bot}}(L, \mathcal{M})) = \mathsf{blanks}(P_{\mathtt{bot}}(L', \mathcal{M}))$, we have $\psi(P_{\mathtt{bot}}(L, \mathcal{M})) = D_{\mathtt{top}}(v, w)$.

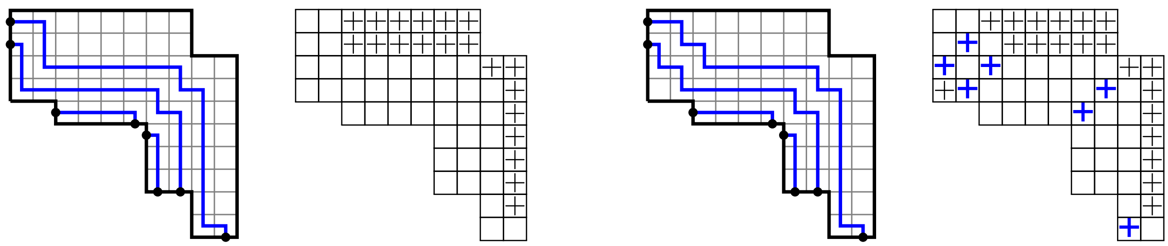
We see excited moves and droops of lattice paths biject with each other:



Therefore since $\mathcal{R}_v = \lambda/\mu$, the result follows. $\square$

Then for $(v, w) = \mathsf{perm}(L, \mathcal{M})$, let $P_{\texttt{zip}}(L, \mathcal{M}) := \psi^{-1}(D_{\texttt{zip}}(v, w))$.

Example 5.8. Take $(L, \mathcal{M})$ as in Example 5.5. To the left are $P_{\texttt{bot}}(L, \mathcal{M})$ and $D_{\texttt{top}}(v, w)$. To the right we have $P_{\texttt{zip}}(L, \mathcal{M})$ and $D^K_{\texttt{zip}}(v, w)$ where pluses in $D^K_{\texttt{zip}}(v, w) - D_{\texttt{zip}}(v, w)$ are drawn in bold blue. Note that $\mathsf{elbows}(P_{\texttt{zip}}(L, \mathcal{M}))$ coincides with $D^K_{\texttt{zip}}(v, w) - D_{\texttt{zip}}(v, w)$.



$\diamondsuit$

Lemma 5.9. *Let $(L, \mathcal{M})$ be a minimal two-sided ladder. Suppose $\mathsf{perm}(L, \mathcal{M}) = (v, w)$ and $D_{\texttt{top}}(v, w)$ has connected components $\{C_q\}_{q \in [m]}$. Then for $q \in [m]$, $\texttt{md}(C_q) = \mathsf{Diag}_{v,w}(C_q)$.*

Proof. Define $w^{(q)} = \delta(\phi_v^{-1}(C_q))$ for $q \in [m]$. Then $w^{(q)} \in U_n(321)$ by Proposition 2.3.
We claim the following:

$$\mathsf{SEYD}(v, w) = \mathsf{SEYD}(v, w^{(1)}) \times \mathsf{SEYD}(v, w^{(2)}) \times \cdots \times \mathsf{SEYD}(v, w^{(m)}).$$

If this fails, there exists $(x, y) \in C_q$ and $(x, y + k) \in C_{q+1}$ for some $q \in [m - 1]$, $k > 1$ such that $(x, y + k - 1) \notin D_{\texttt{top}}(v, w)$. By Proposition 5.7, $\psi(P_{\texttt{bot}}(L, \mathcal{M})) = D_{\texttt{top}}(v, w)$, so in $P_{\texttt{bot}}(L, \mathcal{M})$, $t(x, y + k - 1) \neq \square$. Since $D_{\texttt{top}}(v, w)$ is northeast most, this implies that some path P_j in $P_{\texttt{bot}}(L, \mathcal{M})$ occupies tile $(x, y + k - 1)$. This forces P_j to pass north of C_q, violating condition (2) for lattice paths, a contradiction. Therefore the result follows by the definition of $\texttt{md}(\cdot)$. $\square$

This gives the following:

Corollary 5.10. *Suppose $(L, \mathcal{M})$ is a minimal two-sided ladder where $\mathsf{perm}(L, \mathcal{M}) = (v, w)$. Then if $D_{\texttt{top}}(v, w)$ has connected components $\{C_q\}_{q \in [m]}$,*

$$\operatorname{reg}(X(L, \mathcal{M})) = \#D^K_{\texttt{zip}}(v, w) - \ell(w) = \sum_{q \in [m]} \sum_{b \in \mathsf{Diag}_{v,w}(C_q)} \mathsf{trail}_{v,w}(\mathsf{b}), \textit{ and}$$

$$a(X(L, \mathcal{M})) = \#D^K_{\texttt{zip}}(v, w) - \ell(v) = \Big(\sum_{q \in [m]} \sum_{b \in \mathsf{Diag}_{v,w}(C_q)} \mathsf{trail}_{v,w}(\mathsf{b}) \Big) - \ell(v) + \ell(w).$$

Proof. By Theorem 1.1 and Corollary 1.3 combined with Proposition 5.3 and Theorem 4.3:

$$\operatorname{reg}(X(L, \mathcal{M})) = \sum_{q \in [m]} \sum_{b \in \texttt{md}(C_q)} \mathsf{trail}_{v,w}(\mathsf{b}), \text{ and}$$

$$a(X(L, \mathcal{M})) = \ell(w) - \ell(v) + \sum_{q \in [m]} \sum_{b \in \texttt{md}(C_q)} \mathsf{trail}_{v,w}(\mathsf{b}).$$

Then the result follows by the above equation combined with Lemma 5.9. $\square$

In [13, Lemma 14] S. R. Ghorpade and C. Krattenthaler give an algorithm to compute $a(X(L,\mathcal{M}))$ for two-sided ladders generated by k-minors with additional marked points (p, r), where $r \in [k]$ and p lies on the northmost vertical edge or eastmost horizontal edge of L. This algorithm computes $a(X(L,\mathcal{M}))$ by maximizing $\#\mathsf{elbows}(P)$ for $P \in \mathsf{NILP}(L,\mathcal{M})$. We extend this to all minimal two-sided ladders:

Corollary 5.11. *Let $(L,\mathcal{M})$ be a minimal two-sided ladder. Then*

$$\operatorname{reg}(X(L,\mathcal{M})) = \#\mathsf{elbows}(P_{\mathtt{zip}}(L,\mathcal{M})), \text{ and}$$
$$a(X(L,\mathcal{M})) = \#\mathsf{elbows}(P_{\mathtt{zip}}(L,\mathcal{M})) - \mathtt{wt}(L,\mathcal{M}).$$

Proof. From Proposition 5.7 and the definition of $\mathsf{elbows}(P)$ where $P \in \mathsf{NILP}(L,\mathcal{M})$, it is straightforward to see

$$\mathsf{elbows}(P_{\mathtt{zip}}(L,\mathcal{M})) = \mathsf{elbows}(\psi^{-1}(D_{\mathtt{zip}}(v,w))) = D^K_{\mathtt{zip}}(v,w) - D_{\mathtt{zip}}(v,w).$$

Since $\ell(w) = \#D_{\mathtt{zip}}(v,w)$, the first result follows. By combining this with Corollary 5.10 and the fact that $\mathtt{wt}(L,\mathcal{M}) = \ell(v) - \ell(w)$, the second result follows. $\square$

Example 5.12. Let $(L,\mathcal{M})$ be as in Example 5.8. By Corollary 5.11,

$$\operatorname{reg}(X(L,\mathcal{M})) = \#\mathsf{elbows}(P_{\mathtt{zip}}(L,\mathcal{M})) = 7, \text{ and}$$
$$a(X(L,\mathcal{M})) = \#\mathsf{elbows}(P_{\mathtt{zip}}(L,\mathcal{M})) - \mathtt{wt}(L,\mathcal{M}) = 7 - (60 - 20) = -33. \qquad \diamondsuit$$

In general, the lattice path constructed for Corollary 5.11 differs from the outputted lattice path in [13, Lemma 14] that maximizes unforced elbows. Applying Construction 5.4, it is straightforward to extend [13] to all minimal two-sided ladders.

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