POWERS OF EDGE IDEALS WITH LINEAR QUOTIENTS

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Abstract. We construct an explicit linear quotient ordering for any power of an edge ideal which admits linear quotients, thus recovering a well-known result of Herzog and Hibi. As a consequence, we give explicit formulas for the projective dimension and Betti numbers of the edge ideals of whisker graphs. We also prove that second and higher powers of the edge ideals of anticycles admit linear quotient orderings, thus resolving an open question of Hoefel and Whieldon in the affirmative.

1. Introduction

In recent years, the problem of describing resolutions of powers of edge ideals has been a topic of intense interest in the field of commutative algebra; see for example [1], [2], [3], [5], [8], [9], [15], [17], [18],[20], [21], and [24]. A central theme in this vast body of work is understanding when such resolutions are linear; that is, when the entries in the matrices representing the differentials are linear forms. Motivated by work of Herzog, Hibi, and Zheng in [15], we approach this problem via the linear quotient property, which is defined as follows.

Definition 1.1. An ideal I of a standard graded polynomial ring $P = \mathrm{k}[x_1, \dots, x_n]$ has *linear quotients* if for some ordering of a minimal set of generators $m_1, \dots, m_r$ of I, each ideal quotient

$$((m_1, ..., m_{i-1}) : (m_i))$$

for $i = 2, \dots, r$, is generated by some subset of the variables $x_1, \dots, x_n$.

The linear quotient property is known to impose strong homological restrictions on the ideal I. In fact, for monomial ideals that admit linear quotients, one can even build a linear resolution explicitly from the successive ideal quotients using the well-known iterated mapping cone construction; see for example [22, Construction 27.3] or [16]. Even more can be said in the case of quadratic monomial ideals (e.g. edge ideals) as we see in the following result of Herzog, Hibi, and Zheng.

Theorem 1.2. *(*[15, Theorem 3.2]*) The following conditions are equivalent for a quadratic monomial ideal I of the standard graded polynomial ring $P = \mathrm{k}[x_1, \dots, x_n]$:*

(a) *I has a linear resolution;*
(b) *I has linear quotients;*
(c) *I^n has a linear resolution for all $n \geq 1$.*

Given this result, a natural question is whether the powers of an ideal having linear quotients have linear quotients as well. The answer to this question is known to be negative for non-quadratic monomial ideals. For example, in [4, Example 4.3], Conca and Herzog provide an example of an ideal I generated by degree 3 monomials that has linear quotients, but such that I^2 does not. For quadratic monomial ideals, however, the question was answered in the affirmative by Herzog and Hibi in [14, Theorems 10.1.9 and 10.2.5]; see also [6, Theorem 2.6]. The theorems above show the existence of such linear quotient orderings, and their proofs rely on finding "nice" enough monomial orderings, satisfying certain conditions.

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In this paper, we construct a new, explicit, and easily implementable linear quotient ordering for any power of an edge ideal which admits linear quotients, recovering the above mentioned result of Herzog and Hibi. Our linear quotient ordering on the powers of an ideal I relies only on the linear quotient ordering of the ideal I itself. A different ordering is given in [11]; see also [1]. We illustrate the utility of our linear quotient ordering by providing explicit formulas for the projective dimension and Betti numbers of the powers of the edge ideal of what we call a *whisker graph*, generalizing work of Ferró, Murgia, and Olteanu in [9] and complementing results such as [7, Corollary 3.6] and [23, Corollary 2.6].

On the other hand, there are interesting edge ideals that do not admit linear quotients, but whose higher order powers have linear resolutions, one example being the edge ideal of the anticycle graph (that is, the complement of a simple cycle) on at least 5 vertices; the former follows from a result of Fröberg in [12], and the latter follows from a result of Banerjee in [1]. Although such edge ideals do not satisfy Theorem 1.2, a natural question is whether their higher order powers admit linear quotients, and relatively little is known in this direction.

Hoefel and Whieldon raised this question in [17] with a focus on the anticyle graph, and provided a linear quotient ordering for the square of its edge ideal. However, they were unable to extend their ordering to higher powers, leaving the open question of whether higher order powers of the edge ideal of the anticycle graph admit linear quotients.

We answer this question in the affirmative; that is, we prove that the second and higher order powers of the edge ideal of the anticycle graph on at least 5 vertices admit linear quotients by constructing an explicit and surprisingly simple linear quotient ordering. This provides a class of edge ideals whose sufficiently high powers have linear quotients, and a first step towards an analogous characterization to Nevo and Peeva's conjectured characterization [21, Open Problem 1.11] of edge ideals whose sufficiently high powers have linear resolutions. For work related to this conjecture see, for example, [1], [8], [19], and [20].

We now outline the contents of this paper. In Section 2 we collect some preliminary definitions and facts regarding the linear quotient property of edge ideals that we will use throughout the paper. In Section 3 we prove in Theorem 3.10 that the powers of an edge ideal with linear quotients also admit linear quotients; our linear quotient ordering appears in Construction 3.3. In Section 4 we use this linear quotient ordering to provide explicit formulas for the projective dimension and Betti numbers of powers of edge ideals of whisker graphs. In Section 5 we prove in Theorem 5.6 that the higher order powers of the edge ideal of the anticycle graph have linear quotients; our ordering appears in Construction 5.2. Finally, in Section 6, we examine the problem of finding linear quotient orderings from a computational perspective using new and original methods on Macaulay2 [13] and highlight some relevant examples that support our work. Our code used to execute these computations as well as relevant documentation can be found in [26].

2. Preliminaries

In this section we collect some preliminary definitions and facts we use throughout the paper, including the definition of an edge ideal and a useful fact for working with linear quotient orderings. We also discuss the iterated mapping cone construction for building free resolutions of monomial ideals and resulting formulas for the projective dimension and Betti numbers of ideals that admit linear quotient orderings.

Establishing some notation to be used throughout the section, let $P = \mathsf{k}[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k. We begin by recalling the notion of an edge ideal.

Definition 2.1. Let $G = (V, E)$ be a simple graph (that is, with no loops nor multiple edges) on vertices $V = \{x_1, \dots, x_n\}$. The *edge ideal* associated to G is the P-ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E).$$

Next we state an observation of Hoefel and Whieldon in [17] on the linear quotient property which we will use throughout this paper.

Lemma 2.2. *Let $I = (m_1, \dots, m_r)$ be a monomial ideal of P. The ordering $m_1, \dots, m_r$ yields linear quotients for I if and only if for all $i, j \in \{1, \dots, r\}$ with $j < i$, there exists some $h < i$ (possibly equal to j), such that*

$$\frac{m_h}{\gcd(m_h, m_i)} = x \quad \text{and} \quad x \,\Big|\, \frac{m_j}{\gcd(m_j, m_i)},$$

for some $x \in \{x_1, \dots, x_n\}$.

Now we recall the iterated mapping cone construction which can be used to build a free resolution of any monomial ideal from its ideal quotients; see for example [22, Construction 27.3] or [7, Construction 2.7] for more details on this construction.

Construction 2.3. Let I be the ideal of P minimally generated by monomials $m_1, \dots, m_r$. Denote by I_i the ideal generated by $m_1, \dots, m_i$. For each $i \geq 1$, we have the following short exact sequence

$$0 \longrightarrow P/(I_i : m_{i+1}) \xrightarrow{m_{i+1}} P/I_i \longrightarrow P/I_{i+1} \longrightarrow 0.$$

Thus, given P-free resolutions G^i of P/I_i and F^i of $P/(I_i : m_{i+1})$, there is a map of complexes $\phi_i : F^i \to G^i$ induced by multiplication by m_{i+1}. Taking the cone of this map, it follows that $F^{i+1} := \operatorname{cone}(\phi_i)$ is a free resolution of P/I_{i+1}. Applying this construction for each $i = 1, \dots, r-1$ to obtain a free resolution of $P/I = P/I_r$ is called the *iterated mapping cone construction.*

For an ideal that has linear quotients, one can obtain explicit formulas for its projective dimension and Betti numbers directly from this construction. The formulas are given in the following result.

Proposition 2.4. *(*[25, Corollary 2.7]*) Let I be a monomial ideal of P with linear quotient ordering $I = (m_1, \dots, m_r)$. Define $\nu_1 = 0$ and let ν_j be the minimal number of generators of the ideal $(m_1, \dots, m_{j-1}) : (m_j)$, for $j = 2, \dots, r$. Then the projective dimension of I is given by*

$$\operatorname{pd}(I) = \max\{\nu_j \mid 1 \leq j \leq r\},$$

and the Betti numbers of I are given by

$$\beta_i(I) = \sum_{j=1}^{r} \binom{\nu_j}{i},$$

for all $0 \leq i \leq \operatorname{pd}(I)$.

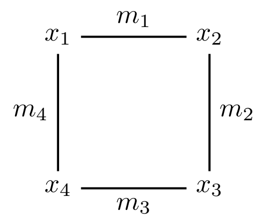
We use Proposition 2.4 to obtain explicit formulas for the projective dimension and the Betti numbers of the powers of the edge ideals of whisker graphs in Section 5.

3. Powers of edge ideals with linear quotients

In this section we prove that the powers of an edge ideal with linear quotients must also admit linear quotients by constructing an explicit linear quotient ordering. Throughout this section, let $P = \mathsf{k}[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k.

For an edge ideal $I(G) = (m_1, \dots, m_r) \subseteq P$ of a simple graph G with vertices $x_1, \dots, x_n$, we are interested in relating the generators of $I(G)^k$ to the generators of $I(G)$ in order to produce a linear quotient ordering of $I(G)^k$. Notice that the set of formal combinations $\mathcal{S}(G)^k = \{m_1^{\alpha_1} \dots m_r^{\alpha_r} \mid \alpha_1 + \cdots + \alpha_r = k\}$ is a generating set for $I(G)^k$, but in many cases contains repetitions and thus is not a minimal generating set. This occurs in the case of cyclic graphs, as we illustrate in the next example.

Example 3.1. Consider the cycle $\mathcal{C}_4$ as pictured below.



The edge ideal $I(\mathcal{C}_4)$ has generators m_1, m_2, m_3, m_4, but the products $m_1m_3 = x_1x_2x_3x_4$ and $m_2m_4 = x_1x_2x_3x_4$ produce the same generator of $I^2_{\mathcal{C}_4}$.

We will need to handle the possibility of repetitions in $\mathcal{S}(G)^k$ carefully in our linear quotient ordering in Construction 3.3. Before this construction, we establish some terminology and notation we will use throughout this section and the rest of the paper.

Notation 3.2. Note that each element of $\mathcal{S}(G)^k$ can also be expressed in terms of the variables:

$$M := m_1^{\alpha_1} \dots m_r^{\alpha_r} = x_1^{a_1} \dots x_n^{a_n}$$

for some list of nonnegative integers $a = (a_1, \dots, a_n)$. Throughout this paper we often refer to $M = x_1^{a_1} \dots x_n^{a_n}$ as the *vertex decomposition* of M and to $M = m_1^{\alpha_1} \dots m_r^{\alpha_r}$ as an *edge decomposition* of M. As we see in Example 3.1 edge decompositions of M need not be unique. For a fixed edge decomposition $M = m_1^{\alpha_1} \dots m_r^{\alpha_r}$, we call m_i a *formal edge* if $\alpha_i > 0$. Furthermore, we say that m_i is an *edge* of M whenever m_i is a formal edge for for at least one edge decomposition of M.

Whenever m_i is formal edge of M, we use the following notation

$$\frac{1}{m_i}M := m_1^{\alpha_1} \dots m_i^{\alpha_i - 1} \dots m_r^{\alpha_r}.$$

Similarly, we denote by m_iM the monomial $m_1^{\alpha_1} \dots m_i^{\alpha_i+1} \dots m_r^{\alpha_r}$. Finally, we say that a variable x_j is *incident to* an an edge m_i whenever x_j divides m_i in the polynomial ring P.

With these considerations in mind, we are now ready to construct an ordering of the generators of $I(G)^k$ for an edge ideal $I(G)$ with linear quotients.

Construction 3.3. Let $I(G) \subseteq P$ be an edge ideal with linear quotient ordering $m_1, \dots, m_r$. For $k \in \mathbb{N}$, consider the set of $\binom{r+k-1}{k}$ formal combinations

$$\mathcal{S}(G)^k = \{m_1^{\alpha_1} \dots m_r^{\alpha_r} \mid \alpha_1 + \dots + \alpha_r = k\}.$$

Next we form a list whose entries are the elements of $\mathcal{S}(G)^k$ ordered according to the *reverse lexicographic (revlex)* ordering with respect to edge decompositions, denoted by

$$\mathcal{R}(G)^k = \left(M_1^k, M_2^k, ..., M^k_{\binom{r+k-1}{k}}\right).$$

More explicitly, $\mathcal{R}(G)^k$ is ordered as follows:

$$\begin{aligned} m_1^{\alpha_1} \dots m_r^{\alpha_r} \text{ precedes } m_1^{\beta_1} \dots m_r^{\beta_r} &\iff \exists\, i \in \{1, \dots, r\} \text{ such that } \alpha_j = \beta_j \text{ for } j > i \text{ and } \alpha_i < \beta_i \\ &\iff \text{the last nonzero entry of the vector } \alpha - \beta \text{ is negative,} \end{aligned}$$

which results in the following ordering:

$$\mathcal{R}(G)^k = \left(m_1^k, m_1^{k-1}m_2, \dots, m_2^k, m_1^{k-1}m_3, m_1^{k-2}m_2m_3, \dots, m_r^k\right).$$

As noted above, $\mathcal{R}(G)^k$ forms a generating set for $I(G)^k$, but is not minimal when it has repetitions. Thus we call $M \in \mathcal{R}(G)^k$ a *representative* if there exists no element M' preceding M in $\mathcal{R}(G)^k$ such that M and M' have the same vertex decompositions (i.e., correspond to the same monomial in the polynomial ring P).

To complete our construction, we define the ordered list $\widetilde{\mathcal{R}(G)^k}$, which retains the ordering of $\mathcal{R}(G)^k$, but removes all non-representatives, so that the entries of $\widetilde{\mathcal{R}(G)^k}$ form a minimal generating set of $I(G)^k$. We show in Theorem 3.10 that $\widetilde{\mathcal{R}(G)^k}$ is a linear quotient ordering.

We illustrate this construction in the following example.

Example 3.4. Returning to Example 3.1, we have

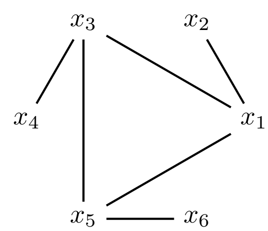
$$\begin{aligned}\mathcal{R}(\mathcal{C}_4)^2 &= \left(M_1^2, M_2^2, M_3^2, M_4^2, M_5^2, M_6^2, M_7^2, M_8^2, M_9^2, M_{10}^2\right)\\ &= \left(m_1^2, m_1m_2, m_2^2, m_1m_3, m_2m_3, m_3^2, m_1m_4, m_2m_4, m_3m_4, m_4^2\right)\\ &= \left(x_1^2x_2^2, x_1x_2^2x_3, x_2^2x_3^2, x_1x_2x_3x_4, x_2x_3^2x_4, x_3^2x_4^2, x_1^2x_2x_4, x_1x_2x_3x_4, x_1x_3x_4^2, x_1^2x_4^2\right).\end{aligned}$$

Since $M_4^2 = m_1m_3 = x_1x_2x_3x_4$ and $M_8^2 = m_2m_4 = x_1x_2x_3x_4$ have the same vertex decompositions (and there are no other repetitions), we have that all elements of $\mathcal{R}(\mathcal{C}_4)^2$ are representatives except for m_2m_4. Thus, according to our construction we remove m_2m_4 to get

$$\widetilde{\mathcal{R}(\mathcal{C}_4)^2} = \left(x_1^2x_2^2, x_1x_2^2x_3, x_2^2x_3^2, x_1x_2x_3x_4, x_2x_3^2x_4, x_3^2x_4^2, x_1^2x_2x_4, x_1x_3x_4^2, x_1^2x_4^2\right).$$

The next example demonstrates that the lexicographic ordering of $\mathcal{S}(G)^k$ with respect to edge decompositions does not, in general, yield a linear quotient ordering.

Example 3.5. Consider the edge ideal $I(G)$ of the graph G pictured below.



It is straightforward to check that the following ordering of the minimal generators of the edge ideal of G is a linear quotient ordering:

$$\begin{aligned}I(G) &= \{m_1, m_2, m_3, m_4, m_5, m_6\}\\ &= \{x_1x_3, x_1x_2, x_3x_4, x_1x_5, x_3x_5, x_5x_6\}.\end{aligned}$$

The lexicographic ordering of the elements of $\mathcal{S}(G)^2$ forms a minimal generating set of $I(G)^2$ (as $\mathcal{S}(G)^2$ has no repetitions) and is given by

$$I(G)^2 = (m_1^2, m_1m_2, m_1m_3, m_1m_4, m_1m_5, m_1m_6, m_2^2, \dots, m_6^2).$$

Observe that this is not a linear quotient ordering. Indeed, $m_1m_6 = x_1x_3x_5x_6$ precedes $m_2m_3 = x_1x_2x_3x_4$, but by quick inspection, the monomials of degree four such that $\frac{M}{\gcd(M,m_2m_3)}$ is linear and divides $\frac{m_1m_6}{\gcd(m_1m_6,m_2m_3)} = x_5x_6$ are precisely:

$$x_1x_2x_3x_5,\ x_1x_2x_4x_5,\ x_1x_3x_4x_5,\ x_2x_3x_4x_5,\ x_1x_2x_3x_6,\ x_1x_2x_4x_6,\ x_1x_3x_4x_6,\ x_2x_3x_4x_6,$$

and none of these precedes $m_2m_3 = x_1x_2x_3x_4$ under the lexicographic ordering.

We aim to prove that the list $\widetilde{\mathcal{R}(G)^k}$ in Construction 3.3 yields a linear quotient ordering of $I(G)^k$; however, to simplify our proof, we reduce to working with $\mathcal{R}(G)^k$ instead. For this, we need a simple lemma and some notation.

Notation 3.6. Adopt notation in Construction 3.3. We use the following notation for the ideal quotients of $I(G)^k$ under the ordering $\mathcal{R}(G)^k$:

$$Q(M_i^k) := ((M_1^k, ..., M_{i-1}^k) : (M_i^k)).$$

Note that for M_i^k a non-representative, we have $Q(M_i^k) = P$. We say that $\mathcal{R}(G)^k$ is a *linear quotient ordering (up to repetition)* if each $Q(M_i^k)$ is either generated by a subset of the variables of P or is equal to the polynomial ring P.

Lemma 3.7. *Adopt notation in Construction 3.3. Then $\widetilde{\mathcal{R}(G)^k}$ is a linear quotient ordering if and only if $\mathcal{R}(G)^k$ is a linear quotient ordering (up to repetition).*

Proof. This follows directly from the definitions of $\widetilde{\mathcal{R}(G)^k}$ and $\mathcal{R}(G)^k$ in Construction 3.3. □

Next we state and prove two technical lemmas about our revlex ordering $\mathcal{R}(G)^k$ which we use in the proof of our main result in this section.

Lemma 3.8. *Adopt Notation 3.6. Let $M_d^k := m_1^{\alpha_1} \dots m_i^{\alpha_i} \dots m_r^{\alpha_r} \in \mathcal{R}(G)^k$, with $\alpha_i > 0$ for some $1 \le i \le r$, and let $M_{d'}^{k-1} := \frac{1}{m_i} M_d^k \in \mathcal{R}(G)^{k-1}$. Then $Q(M_{d'}^{k-1}) \subseteq Q(M_d^k)$.*

Proof. First note that the ideal quotient $Q(M_{d'}^{k-1})$ is generated by the monomials $\frac{M_t^{k-1}}{\gcd(M_t^{k-1},\, M_{d'}^{k-1})}$ for all generators M_t^{k-1} which precede $M_{d'}^{k-1}$ in $\mathcal{R}(G)^{k-1}$. For each such M_t^{k-1}, it remains to show that the generator $m_i M_t^{k-1}$ precedes M_d^k in $\mathcal{R}(G)^k$, since then we have

$$\frac{M_t^{k-1}}{\gcd(M_t^{k-1},\, M_{d'}^{k-1})} = \frac{m_i M_t^{k-1}}{\gcd(m_i M_t^{k-1},\, m_i M_{d'}^{k-1})} = \frac{m_i M_t^{k-1}}{\gcd(m_i M_t^{k-1},\, M_d^k)} \in Q(M_d^k).$$

But, indeed, this follows directly from the definition of the revlex ordering and the fact that M_t^{k-1} precedes $M_{d'}^{k-1}$ in $\mathcal{R}(G)^{k-1}$. □

Lemma 3.9. *Adopt Notation 3.6 and suppose that $M_f^k = m_1^{\alpha_1} \dots m_r^{\alpha_r} = x_1^{a_1} \dots x_n^{a_n}$ precedes $M_s^k = m_1^{\beta_1} \dots m_r^{\beta_r} = x_1^{b_1} \dots x_n^{b_n}$ in $\mathcal{R}(G)^k$, with M_s^k a representative. (As a mnemonic for the reader, the subscripts f and s stand for first and second, respectively.) Let p be the largest index such that $\alpha_p > 0$ and q be the largest index such that $\beta_q > 0$. Then at least one of the following is true:*

(a) *The generators M_f^k and M_s^k share a formal edge; that is, $\alpha_i > 0$ and $\beta_i > 0$, for some $i \in \{1, \dots, r\}$.*
(b) *There is a variable x_i such that $b_i > a_i$ and a formal edge m_j of M_s^k which is incident to x_i, such that for all formal edges m_ℓ of M_f^k, the generator $\frac{1}{m_\ell} M_f^k$ precedes $\frac{1}{m_j} M_s^k$ in $\mathcal{R}(G)^{k-1}$.*
(c) *The generator $\frac{1}{m_q} M_s^k$ divides M_f^k in P, with $p < q$.*

Proof. We prove that if the conditions (a) and (b) are false, then condition (c) must hold.

First note that under the revlex ordering, we have that $p \le q$. Moreover, by our assumption that condition (a) is false, we have that $p < q$, as desired.

Since M_s^k is a representative, we have $M_s^k \neq M_f^k$, and thus $b_i > a_i$, for some $i \in \{1, \dots, r\}$, and in particular x_i must divide some formal edge m_j of M_s^k.

Now we claim that $\beta_q = 1$, because if $\beta_q > 1$, then condition (b) must hold. Indeed, $\beta_q > 1$ implies that m_q is a formal edge of $\frac{1}{m_j} M_s^k$, but for any m_ℓ with $\alpha_\ell > 0$, we have that $\frac{1}{m_\ell} M_f^k$ only has formal edges with index at most p and hence strictly less than q. Thus $\frac{1}{m_\ell} M_f^k$ precedes $\frac{1}{m_j} M_s^k$ in $\mathcal{R}(G)^{k-1}$ under our revlex ordering, and thus the condition (b) holds.

In summary we have $p < q$ and $\beta_q = 1$. Now to show that condition (c) is true, it suffices to show that for all $i \in \{1, \dots, n\}$ and for all $j < q$ with $\beta_j > 0$, we have that $b_i \le a_i$ whenever $x_i | m_j$. For the sake of contradiction, we suppose that there exist indices $i \in \{1, \dots, n\}$ and $j < q$ with $\beta_j > 0$ such that $b_i > a_i$ and $x_i | m_j$. As before, we have that $\frac{1}{m_j} M_s^k$ has the formal edge m_q, but $\frac{1}{m_\ell} M_f^k$ does not for any m_ℓ with $\alpha_\ell > 0$. Thus condition (b) must be true, which is a contradiction to our assumption. Therefore $\frac{1}{m_q} M_s^k$ divides M_f^k, as desired. □

Theorem 3.10. *Let $P = \mathsf{k}[x_1, \ldots, x_n]$ be a standard graded polynomial ring with k a field and let $I(G)$ be an edge ideal that admits linear quotients. Further adopt notation in Construction 3.3. Then $\widetilde{\mathcal{R}(G)^k}$ yields a linear quotient ordering on $I(G)^k$ for all $k \in \mathbb{N}$.*

Proof. By Lemma 3.7 it suffices to show that $\mathcal{R}(G)^k$ yields a linear quotient ordering (up to repetition). We use induction on k. The result holds for $k = 1$ by hypothesis, since $\mathcal{R}(G)^1$ coincides with the linear quotient ordering of $I(G)$.

Assume $\mathcal{R}(G)^{k-1}$ yields a linear quotient ordering (up to repetition). We need to show that $\mathcal{R}(G)^k$ also yields a linear quotient ordering (up to repetition). Suppose M_f^k precedes M_s^k in $\mathcal{R}(G)^k$. Without loss of generality, we may assume that M_s^k is a representative; otherwise, $Q(M_s^k) = P$, as desired. Now we may apply Lemma 3.9 and examine each of the three possibilities separately. As in the lemma, we consider the edge and vertex decompositions $M_f^k = m_1^{\alpha_1} \ldots m_r^{\alpha_r} = x_1^{a_1} \ldots x_n^{a_n}$ and $M_s^k = m_1^{\beta_1} \ldots m_r^{\beta_r} = x_1^{b_1} \ldots x_n^{b_n}$, and let p and q be the largest indices such that $\alpha_p > 0$ and $\beta_q > 0$.

(a) First we assume that M_f^k and M_s^k share a common formal edge m_i and define

$$M_{s'}^{k-1} := \frac{1}{m_i} M_s^k \quad \text{and} \quad M_{f'}^{k-1} := \frac{1}{m_i} M_f^k.$$

Notice $M_{s'}^{k-1}$ is a representative because by Lemma 3.8 we have the containment $Q(M_{s'}^{k-1}) \subseteq Q(M_s^k)$. By the inductive hypothesis and Lemma 2.2, there exists some $x \in Q(M_{s'}^{k-1})$ such that

$$x \,\Big|\, \frac{M_{f'}^{k-1}}{\gcd(M_{f'}^{k-1}, M_{s'}^{k-1})}.$$

Thus by Lemma 3.8 we have $x \in Q(M_{s'}^{k-1}) \subseteq Q(M_s^k)$. Moreover, note that

$$x \,\Big|\, \frac{M_{f'}^{k-1}}{\gcd(M_{f'}^{k-1}, M_{s'}^{k-1})} = \frac{m_i M_{f'}^{k-1}}{\gcd(m_i M_{f'}^{k-1}, m_i M_{s'}^{k-1})} = \frac{M_f^k}{\gcd(M_f^k, M_s^k)}.$$

Now Lemma 2.2 yields the desired result.

(b) Next we assume there is a variable x_i such that $b_i > a_i$ and a formal edge m_j of M_s^k which is incident to x_i, such that for all formal edges m_ℓ of M_f^k, the generator $\frac{1}{m_\ell} M_f^k$ precedes $\frac{1}{m_j} M_s^k$ in $\mathcal{R}(G)^{k-1}$. We write $m_j = x_i x_g$ and choose a formal edge m_ℓ of M_f^k as follows. If x_g does not divide M_f^k, select m_ℓ arbitrarily; otherwise, let m_ℓ be any formal edge of M_f^k incident to x_g.

By the same argument as in (a), it suffices to show that

$$\frac{\frac{1}{m_\ell} M_f^k}{\gcd(\frac{1}{m_\ell} M_f^k, \frac{1}{m_j} M_s^k)} \,\Big|\, \frac{M_f^k}{\gcd(M_f^k, M_s^k)}.$$

Equivalently, we need to show that $\gcd(M_f^k, M_s^k) \mid \gcd(M_f^k, \frac{m_\ell}{m_j} M_s^k)$, but this holds since $b_i > a_i$ and since m_ℓ is divisible by x_g whenever x_g divides M_f^k.

(c) Otherwise by Lemma 3.9, we have $p < q$ and $\frac{1}{m_q} M_s^k$ divides M_f^k in P. For convenience, let $M_{\tilde{s}}^{k-1} = \frac{1}{m_q} M_s^k$. Since $M_{\tilde{s}}^{k-1}$ divides M_f^k, the degree of $\frac{M_f^k}{\gcd(M_f^k, M_s^k)}$ is at most 2, and further since M_s^k is a representative, it must be linear or quadratic.

If it is linear, by Lemma 2.2 we are done. So we may assume that

$$\frac{M_f^k}{\gcd(M_f^k, M_s^k)} = x_{c_1} x_{c_2},$$

for some vertices $x_{c_1}, x_{c_2} \in P$; however, $x_{c_1}x_{c_2}$ need not be an edge of $I(G)$. By Lemma 2.2 it suffices to show that $x_{c_1} \in Q(M_s^k)$ or $x_{c_2} \in Q(M_s^k)$.

Let $m_q = x_i x_j$, so that $M_s^k = x_i x_j M_{\tilde{s}}^{k-1}$ and $M_f^k = x_{c_1}x_{c_2}M_{\tilde{s}}^{k-1}$. Note that we must have $x_i, x_j \notin \{x_{c_1}, x_{c_2}\}$, because otherwise $\frac{M_f^k}{\gcd(M_f^k, M_s^k)}$ is not quadratic. Since x_{c_1} divides M_f^k, there must be some formal edge m_ℓ of M_f^k incident to x_{c_1}. Let $m_\ell = x_{c_1}x_{c_3}$, and note that x_{c_3} may possibly equal x_{c_2}.

Since $p < q$, we have that m_ℓ precedes m_q in the linear quotient ordering of $I(G)$, and therefore there is an edge $m_{q'}$ preceding m_q in the linear quotient ordering of $I(G)$ with

$$m_{q'} \in \{x_i x_{c_1}, x_j x_{c_1}, x_i x_{c_3}, x_j x_{c_3}\}.$$

Since $q' < q$, it follows that $m_{q'}M_{\tilde{s}}^{k-1}$ precedes M_s^k in $\mathcal{R}(G)^k$. Therefore, if we have $m_{q'} \in \{x_i x_{c_1}, x_j x_{c_1}\}$, then it follows that

$$\frac{m_{q'}M_{\tilde{s}}^{k-1}}{\gcd(m_{q'}M_{\tilde{s}}^{k-1}, M_s^k)} = x_{c_1} \in Q(M_s^k),$$

and we are done.

Therefore we may assume that $m_{q'} \in \{x_i x_{c_3}, x_j x_{c_3}\}$. Notice that $\frac{m_{q'}}{m_\ell}M_f^k$ precedes M_s^k in $\mathcal{R}(G)^k$ since m_ℓ is a formal edge of M_f^k and $p, q' < q$. Furthermore

$$\frac{\frac{m_{q'}}{m_\ell}M_f^k}{\gcd(\frac{m_{q'}}{m_\ell}M_f^k, M_s^k)} = x_{c_2} \in Q(M_s^k),$$

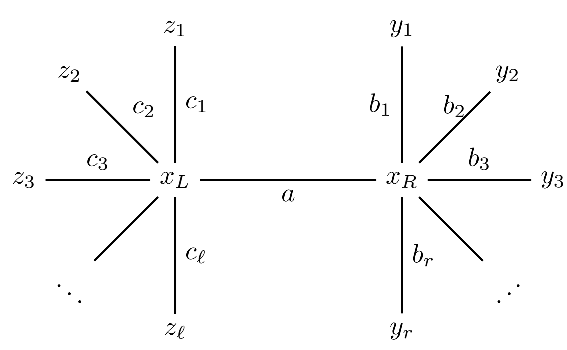
which completes the proof. □

The following corollary, which follows immediately from Theorem 3.10, recovers a well-known result of Herzog and Hibi in [14]; see also [6, Theorem 2.6].

Corollary 3.11. *Let $P = \mathsf{k}[x_1, \dots, x_n]$ be a standard graded polynomial ring with k a field. Then $I(G)$ has linear quotients if and only if $I(G)^k$ has linear quotients for all $k \in \mathbb{N}$.*

4. Powers of edge ideals of whisker graphs

In this section, we illustrate the utility of our linear quotient ordering in Construction 3.3. In particular, we apply Theorem 3.10 to obtain explicit formulas for the projective dimension and the Betti numbers of the powers of the edge ideal of a *whisker graph*; that is, a graph $\mathcal{W}_{r,\ell}$ with $r + \ell + 2$ vertices and $r + \ell + 1$ edges of the following form:



Throughout this section, let $P = \mathsf{k}[x_R, x_L, y_1, \dots, y_r, z_1, \dots, z_\ell]$ be a standard graded polynomial ring over a field k. We begin by giving a linear quotient ordering on the edge ideal $I(\mathcal{W}_{r,\ell})$ of the whisker graph $\mathcal{W}_{r,\ell}$.

Proposition 4.1. *The ideal $I(\mathcal{W}_{r,\ell})$ has linear quotients with respect to the following ordering:*

$$I(\mathcal{W}_{r,\ell}) = (a, b_1, b_2, \dots, b_r, c_1, c_2, \dots, c_\ell),$$

where a denotes the edge x_Rx_L, b_i denotes the edge x_Ry_i for $i \in \{1, 2, ..., r\}$, and c_j denotes the edge x_Lz_j for $j \in \{1, 2, ..., \ell\}$.

Proof. This follows directly from the equalities

$$((a, b_1, \dots, b_{i-1}) : (b_i)) = \{x_L, y_1, \dots, y_{i-1}\}$$
$$((a, b_1, \dots, b_r, c_1, \dots, c_{j-1}) : (c_j)) = \{x_R, z_1, \dots, z_{j-1}\}$$

for $i \in \{1, 2, ..., r\}$ and $j \in \{1, 2, ..., \ell\}$. □

Thus by Theorem 3.10, $I(\mathcal{W}_{r,\ell})^k$ admits linear quotients for all $k \in \mathbb{N}$ under the revlex ordering

$$\widetilde{\mathcal{R}(\mathcal{W}_{r,\ell})^k} = \left(M_1, M_2, ..., M_{\binom{r+\ell+k}{k}}\right)$$

determined by the linear quotient ordering of $I(\mathcal{W}_{r,\ell})$ given in Proposition 4.1; see Construction 3.3. We will adhere to this ordering throughout the remainder of this section. Now we prove a general lemma that when G is a tree, there are no repetitions in $\mathcal{R}(G)^k$; in particular, we have the equality $\widetilde{\mathcal{R}(\mathcal{W}_{r,\ell})^k} = \mathcal{R}(\mathcal{W}_{r,\ell})^k$ since $\mathcal{W}_{r,\ell}$ is a tree.

Lemma 4.2. *Let T be a tree on vertices $x_1, \dots, x_n$ and let $I(T) = (m_1, \dots, m_s)$ be its edge ideal in the standard graded polynomial ring $\mathrm{k}[x_1, \dots, x_n]$. Then for all $k \in \mathbb{N}$, each generator $M \in I(T)^k$ can be expressed uniquely as $M = m_1^{\alpha_1} \dots m_s^{\alpha_s}$, for some list of nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_s)$.*

Proof. We proceed by (strong) induction on the number of vertices n. For the base case, consider the tree T with two vertices, x_1 and x_2. Then $I(T) = (x_1x_2)$, and thus the only generator of $I(T)^k = (x_1^kx_2^k)$ can be represented uniquely, as desired.

Now assume towards induction that the claim is true for any tree on fewer than n vertices, and let T be a tree on n vertices. Then T has at least one leaf; call this vertex x_i and denote by m_j the edge to which it is incident. Let M be a minimal generator of $I(T)^k$. Then any power of x_i dividing M corresponds to a power of the edge m_j. Let

$$M' = \frac{M}{m_j^{\alpha_j}},$$

where α_j is the maximum power of m_j dividing M. Then M' is a generator of $I(T')^{k-\alpha_j}$, where T' is the tree $T\backslash\{x_i\}$. By induction there is a unique list of nonnegative integers $(\alpha_1, \dots, \alpha_{j-1}, \alpha_{j+1}, \dots, \alpha_s)$ such that $M' = m_1^{\alpha_1} \dots m_{j-1}^{\alpha_{j-1}} m_{j+1}^{\alpha_{j+1}} \dots m_s^{\alpha_s}$. Now M is expressed uniquely as $M = m_1^{\alpha_1} \dots m_s^{\alpha_s}$. □

Still adhering to our linear quotient ordering $\mathcal{R}(\mathcal{W}_{r,\ell})^k$ on $I(\mathcal{W}_{r,\ell})^k$, we now establish some additional notation to be used throughout the remainder of the section. As in previous sections, we use the notation for ideal quotients introduced in Notation 3.6 and refer to edge and vertex decompositions of the generators of $I(\mathcal{W}_{r,\ell})^k$ as introduced in Notation 3.2 throughout this section. Note however in this section, by Lemma 4.2, not only the vertex decomposition but also the edge decomposition of a generator of $I(\mathcal{W}_{r,\ell})^k$ is unique. Thus there is no distinction between an edge and a formal edge. Furthermore we introduce the following notation.

Notation 4.3. For any generator $M = a^\alpha b_1^{\beta_1} ... b_r^{\beta_r} c_1^{\gamma_1} ... c_\ell^{\gamma_\ell}$ of $I(\mathcal{W}_{r,\ell})^k$, we define integers

$$B(M) = \begin{cases} \max\{p : \beta_p > 0\}, & \{p : \beta_p > 0\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases} \qquad C(M) = \begin{cases} \max\{p : \gamma_p > 0\}, & \{p : \gamma_p > 0\} \neq \emptyset \\ 0, & \text{otherwise} \end{cases}.$$

In order to obtain formulas for the projective dimension and Betti numbers of $I(\mathcal{W}_{r,\ell})^k$, we need the following technical lemma which describes, for a given generator M of $I(\mathcal{W}_{r,\ell})^k$, when each variable belongs to the corresponding ideal quotient $Q(M)$.

Lemma 4.4. *Adopt Notations 3.6 and 4.3. Fix an integer $k \geq 1$, and let $M = a^{\alpha} b_1^{\beta_1} ... b_r^{\beta_r} c_1^{\gamma_1} ... c_\ell^{\gamma_\ell}$ be a generator of $I(\mathcal{W}_{r,\ell})^k$. The following statements hold:*

1. *For all $1 \leq j \leq r$, $y_j \in Q(M)$ if and only if $B(M) > j$;*
2. *For all $1 \leq j \leq \ell$, $z_j \in Q(M)$ if and only if $C(M) > j$;*
3. *$x_R \in Q(M)$ if and only if $C(M) > 0$;*
4. *$x_L \in Q(M)$ if and only if $B(M) > 0$;*

Proof. Consider the unique vertex decomposition

$$M = y_1^{d_1} \cdot \dots \cdot y_r^{d_r} z_1^{d_{r+1}} \cdot \dots \cdot z_\ell^{d_{r+\ell}} x_R^{d_{r+\ell+1}} x_L^{d_{r+\ell+2}}.$$

Since $\mathcal{W}_{r,\ell}$ is a whisker graph, one can easily check that the following constraints hold on the vertex decomposition:

- Constraint 1: $\begin{cases} d_i > 0 \iff \beta_i > 0, & \text{for } 1 \leq i \leq r \\ d_i > 0 \iff \gamma_{i-r} > 0, & \text{for } r+1 \leq i \leq r+\ell \end{cases}$
- Constraint 2: $d_{r+\ell+1} - \sum_{s=1}^{r} d_s = d_{r+\ell+2} - \sum_{s=1}^{\ell} d_{r+s}$.

Now we are ready to prove the desired statements. For ease of notation, in the following arguments we abbreviate $B(M)$ and $C(M)$ to B and C, respectively. Note that the proofs of (2) and (4) are very similar to those of (1) and (3), respectively, but we include them for completeness.

(1) Let $1 \leq j \leq r$. First, assume that $B > j$. Therefore, $b_B = x_R y_B$ is an edge of M, and thus $M \cdot \frac{b_j}{b_B}$ precedes M in the linear quotient ordering. So,

$$y_j = \frac{M \cdot \frac{x_R y_j}{x_R y_B}}{\gcd\left(M \cdot \frac{x_R y_j}{x_R y_B}, M\right)} = \frac{M \cdot \frac{b_j}{b_B}}{\gcd\left(M \cdot \frac{b_j}{b_B}, M\right)} \in Q(M).$$

Conversely, if $y_j \in Q(M)$ then there exists some M' preceding M in the linear quotient ordering such that $(M') : (M) = (y_j)$; that is, such that

$$M' = y_1^{e_1} \cdot \dots \cdot y_r^{e_r} z_1^{e_{r+1}} \cdot \dots \cdot z_\ell^{e_{r+\ell}} x_R^{e_{r+\ell+1}} x_L^{e_{r+\ell+2}} \tag{4.4.1}$$

where for some $w \neq j$, we have equalities $d_n = e_n$ for all $n \in \{1, 2, ..., r+\ell+2\} \backslash \{j, w\}$, $e_j = d_j + 1$, and $e_w = d_w - 1$.

Note that by double application of Constraint 2, we have the following equalities

$$\begin{aligned} d_{r+\ell+1} - \sum_{s=1}^{r} d_s &= d_{r+\ell+2} - \sum_{s=1}^{\ell} d_{r+s} \\ e_{r+\ell+1} - \sum_{s=1}^{r} e_s &= e_{r+\ell+2} - \sum_{s=1}^{\ell} e_{r+s}. \end{aligned} \tag{4.4.2}$$

Since $e_j = d_j + 1$, $e_w = d_w - 1$, and $1 \leq j \leq r$, it follows from these equations that $1 \leq w \leq r$ or $w = r + \ell + 2$. We consider these two cases separately.

If $1 \leq w \leq r$ then we have the following equality

$$M' \cdot x_R y_w = M \cdot x_R y_j.$$

Now by uniqueness of edge decompositions of generators of $I(\mathcal{W}_{r,\ell})^{k+1}$ and since M' precedes M in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})^k$, we must have that $x_R y_j$ precedes $x_R y_w$ in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})$. Thus, $w > j$. Since $e_w = d_w - 1$, we have $d_w > 0$, and thus by Constraint 1 we have $\beta_w > 0$. Therefore $B \geq w > j$, as desired.

On the other hand, if $w = r + \ell + 2$, then we have the following equality

$$M' \cdot x_R x_L = M \cdot x_R y_j,$$

but this is impossible because M' precedes M in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})^k$ and x_Rx_L precedes x_Ry_j in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})$.

(2) Let $1 \leq j \leq \ell$. First, assume that $C > j$. Therefore, $c_C = x_Lz_C$ is an edge of M, and thus $M \cdot \frac{c_j}{c_C}$ precedes M in the linear quotient ordering. So,

$$z_j = \frac{M \cdot \frac{x_Lz_j}{x_Lz_C}}{\gcd\left(M \cdot \frac{x_Lz_j}{x_Lz_C}, M\right)} = \frac{M \cdot \frac{c_j}{c_C}}{\gcd\left(M \cdot \frac{c_j}{c_C}, M\right)} \in Q(M).$$

Conversely, if $z_j \in Q(M)$ then there exists some M' preceding M in the linear quotient ordering such that $(M') : (M) = (z_j)$; that is, M' has the vertex decomposition (4.4.1), where for some $w \neq j+r$, we have equalities $d_n = e_n$ for all $n \in \{1, 2, ..., r+\ell+2\}\backslash\{j+r, w\}$, $e_{j+r} = d_{j+r}+1$, and $e_w = d_w - 1$.

Again by double application of Constraint 2, we have the equalities in (4.4.2). Since we have equalities $e_{j+r} = d_{j+r} + 1$ and $e_w = d_w - 1$, and since $1 \leq j \leq \ell$, it follows that $r + 1 \leq w \leq r + \ell$ or $w = r + \ell + 1$. We consider these two cases separately.

If $r + 1 \leq w \leq r + \ell$, then we have the following equality

$$M' \cdot x_Lz_{w-r} = M \cdot x_Lz_j.$$

Now by uniqueness of edge decompositions of generators in $I(\mathcal{W}_{r,\ell})^{k+1}$ and since M' precedes M in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})^k$, we must have that x_Lz_j precedes x_Lz_{w-r} in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})$. Similarly to part (1), applying Constraint 1 yields $\gamma_{w-r} > 0$. Therefore, $C \geq w - r > j$, as desired.

On the other hand, if $w = r + \ell + 1$, then we have the following equality

$$M' \cdot x_Rx_L = M \cdot x_Lz_j,$$

but this is impossible because M' precedes M in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})^k$ by hypothesis and x_Rx_L precedes x_Lz_j in the linear quotient ordering of $I(\mathcal{W}_{r,\ell})$.

(3) First, assume that $C > 0$. Therefore, $c_C = x_Lz_C$ is an edge of M, and thus $M \cdot \frac{a}{c_C}$ precedes M in the linear quotient ordering. So,

$$x_R = \frac{M \cdot \frac{x_Rx_L}{x_Lz_C}}{\gcd\left(M \cdot \frac{x_Rx_L}{x_Lz_C}, M\right)} = \frac{M \cdot \frac{a}{c_C}}{\gcd\left(M \cdot \frac{a}{c_C}, M\right)} \in Q(M).$$

Conversely, if $x_R \in Q(M)$ then there exists some M' preceding M in the linear quotient ordering such that $(M') : (M) = (x_R)$; that is, such that M' has the vertex decomposition in (4.4.1), where for some $w \neq r + \ell + 1$, we have equalities $d_n = e_n$ for all $n \in \{1, 2, ..., r + \ell + 2\}\backslash\{r + \ell + 1, w\}$, $e_{r+\ell+1} = d_{r+\ell+1} + 1$, and $e_w = d_w - 1$.

Again by double application of Constraint 2, we have the equations in (4.4.2). Since we have equalities $e_{r+\ell+1} = d_{r+\ell+1} + 1$ and $e_w = d_w - 1$, it follows from these equations that $r + 1 \leq w \leq r + \ell$. Since $e_w = d_w - 1$, we have $d_w > 0$, and thus by Constraint 1 we have that $\gamma_{w-r} > 0$. Therefore, $C \geq w - r > 0$, as desired.

(4) First, assume that $B > 0$. Therefore, $b_B = x_Ry_B$ is an edge of M, and thus $M \cdot \frac{a}{b_B}$ precedes M in the linear quotient ordering. So,

$$x_L = \frac{M \cdot \frac{x_Rx_L}{x_Ry_B}}{\gcd\left(M \cdot \frac{x_Rx_L}{x_Ry_B}, M\right)} = \frac{M \cdot \frac{a}{b_B}}{\gcd\left(M \cdot \frac{a}{b_B}, M\right)} \in Q(M).$$

Conversely, if $x_L \in Q(M)$ then there exists some M' preceding M in the linear quotient ordering such that $(M') : (M) = (x_L)$; that is, such that M' has the vertex decomposition in (4.4.1), where for some $w \neq r+\ell+2$, we have equalities $d_n = e_n$ for all $n \in \{1, 2, ..., r+\ell+2\} \backslash \{r+\ell+2, w\}$, $e_{r+\ell+2} = d_{r+\ell+2} + 1$, and $e_w = d_w - 1$.

Again by double application of Constraint 2, we have the equations in (4.4.2). Since we have equalities $e_{r+\ell+2} = d_{r+\ell+2} + 1$ and $e_w = d_w - 1$, it follows that $1 \leq w \leq r$. Similarly to part (3), applying Constraint 1 yields $\beta_w > 0$. Therefore, $B \geq w > 0$, as desired. □

Now we are ready to obtain formulas for the projective dimension and the Betti numbers of the powers of the edge ideals of the whisker graph $\mathcal{W}_{r,\ell}$.

Theorem 4.5. *Let $P = \mathsf{k}[x_R, x_L, y_1, \dots, y_r, z_1, \dots, z_\ell]$ be a standard graded polynomial ring with k a field. Then the projective dimension of the ideal $I(\mathcal{W}_{r,\ell})^k$ is given by*

$$\operatorname{pd}(I(\mathcal{W}_{r,\ell})^k) = \begin{cases} r+\ell & k \geq 2 \\ \max(r,\ell) & k = 1 \end{cases}$$

and its Betti numbers are given by

$$\beta_i(I(\mathcal{W}_{r,\ell})^k) = \begin{cases} \binom{k-2+i}{i}\binom{r+\ell+k}{k+i} + \binom{k-2+i}{i-1}\left[\binom{r+k}{k+i} + \binom{\ell+k}{k+i}\right] & i \geq 1 \\ \binom{r+\ell+k}{k} & i = 0. \end{cases}$$

Proof. By Theorem 3.10 and Proposition 4.1, the powers of $I(\mathcal{W}_{r,\ell})$ have linear quotients with respect to the reverse lexicographic ordering $\mathcal{R}(\mathcal{W}_{r,\ell})^k = \left(M_1, M_2, ..., M_{\binom{r+\ell+k}{k}}\right)$ on the generators of $I(\mathcal{W}_{r,\ell})$; see Construction 3.3. Thus by Proposition 2.4, to calculate projective dimension and Betti numbers, it suffices to calculate the minimal number of generators $\nu(Q(M_j))$ of the ideal quotient $Q(M_j)$, for each $j = 1, \dots, \binom{r+\ell+k}{k}$.

By Lemma 4.4, we have the equality

$$\nu(Q(M_j)) = B(M_j) + C(M_j), \tag{4.5.1}$$

for each $j = 1, \dots, \binom{r+\ell+k}{k}$, where $B(M_j)$ and $C(M_j)$ are defined as in Notation 4.3.

Thus it is easy to see that $\nu(Q(M_j))$ is maximized whenever $B(M_j)$ and $C(M_j)$ are as large as possible. For $k \geq 2$, this is achieved when $B(M_j) = r$ and $C(M_j) = \ell$, and thus by Proposition 2.4 the projective dimension is $r+\ell$. For $k=1$, it is impossible for both $B(M_j)$ and $C(M_j)$ to be positive, so in this case we have that the projective dimension is $\max(r,\ell)$.

Now we calculate the Betti numbers. For ease of notation, we write $\beta_i(I(\mathcal{W}_{r,\ell})^k)$ as β_i throughout the rest of the proof. By Proposition 2.4 we have the following equalities

$$\beta_0 = \sum_{j=1}^{\binom{r+\ell+k}{k}} \binom{\nu(Q(M_j))}{0} = \sum_{j=1}^{\binom{r+\ell+k}{k}} 1 = \binom{r+\ell+k}{k}.$$

Next, we calculate the Betti number β_i in homological degree $i > 0$. For fixed integers $B, C \geq 0$, define $f(B,C)$ to be the cardinality of the following set

$$f(B,C) = \left| \left\{ M \in \mathcal{R}(\mathcal{W}_{r,\ell})^k \;\middle|\; B(M) = B \text{ and } C(M) = C \right\} \right|.$$

Then by Proposition 2.4 and (4.5.1) we have that

$$\beta_i = \sum_{B,C \geq 1} f(B,C)\binom{B+C}{i} + \sum_{\substack{B \geq 1 \\ C=0}} f(B,0)\binom{B}{i} + \sum_{\substack{B=0 \\ C \geq 1}} f(0,C)\binom{C}{i}. \tag{4.5.2}$$

To calculate $f(B,C)$, we first note that $f(B,0)$ is the number of sequences of length $B+1$ of nonnegative integers which sum to k and whose last entry is strictly positive; that is,

$$f(B,0) = \binom{B+k}{k} - \binom{B+k-1}{k} = \binom{B+k-1}{k-1} = \binom{B+k-1}{B}, \tag{4.5.3}$$

where the second equality follows from Pascal's Identity. Similarly

$$f(0,C) = \binom{C+k-1}{C}. \tag{4.5.4}$$

Now for $B, C \geq 1$, $f(B,C)$ is the number of sequences of length $B+C+1$ of nonnegative integers which sum to k whose last two entries (corresponding to the exponents of $b_B = x_R y_B$ and $c_C = x_L z_C$) are strictly positive. By the inclusion-exclusion principle, we see that

$$\begin{aligned} f(B,C) &= \binom{B+C+k}{k} - 2\binom{B+C+k-1}{k} + \binom{B+C+k-2}{k} \\ &= \binom{B+C+k}{k} - \binom{B+C+k-1}{k} - \left[\binom{B+C+k-1}{k} - \binom{B+C+k-2}{k}\right] \\ &= \binom{B+C+k-1}{k-1} - \binom{B+C+k-2}{k-1} \\ &= \binom{B+C+k-2}{k-2} \\ &= \binom{B+C+k-2}{B+C}, \end{aligned} \tag{4.5.5}$$

where the third and fourth equalities follow from Pascal's Identity.

Substituting (4.5.3), (4.5.4), and (4.5.5) into (4.5.2) and temporarily replacing $A := B+C$ for ease of notation, we obtain

$$\begin{aligned} \beta_i &= \sum_{B,C\geq 1} \binom{A+k-2}{A}\binom{A}{i} + \sum_{\substack{B\geq 1\\ C=0}} \binom{B+k-1}{B}\binom{B}{i} + \sum_{\substack{B=0\\ C\geq 1}} \binom{C+k-1}{C}\binom{C}{i} \\ &= \sum_{B,C\geq 1} \binom{A+k-2}{k-2+i}\binom{k-2+i}{k-2} + \sum_{\substack{B\geq 1\\ C=0}} \binom{B+k-1}{k-1+i}\binom{k-1+i}{k-1} + \sum_{\substack{B=0\\ C\geq 1}} \binom{C+k-1}{k-1+i}\binom{k-1+i}{k-1} \\ &= \binom{k-2+i}{k-2} \sum_{B,C\geq 1} \binom{A+k-2}{k-2+i} + \binom{k-1+i}{k-1}\left[\sum_{\substack{B\geq 1\\ C=0}} \binom{B+k-1}{k-1+i} + \sum_{\substack{B=0\\ C\geq 1}} \binom{C+k-1}{k-1+i}\right], \end{aligned} \tag{4.5.6}$$

where the second equality follows from the combinatorial identity $\binom{x}{y}\binom{y}{z} = \binom{x}{x-y+z}\binom{x-y+z}{x-y}$.

Now we analyze each sum in the equality above separately. By the hockey-stick identity, since B runs from 1 to r, we have the following equality

$$\sum_{\substack{B\geq 1\\ C=0}} \binom{B+k-1}{k-1+i} = \sum_{B=i}^{r} \binom{B+k-1}{k-1+i} = \binom{r+k}{k+i}. \tag{4.5.7}$$

Similarly since C runs from 1 to ℓ, we have the following equality

$$\sum_{\substack{B=0\\ C\geq 1}} \binom{C+k-1}{k-1+i} = \sum_{C=i}^{\ell} \binom{C+k-1}{k-1+i} = \binom{\ell+k}{k+i}. \tag{4.5.8}$$

We use the same identity twice on the remaining sum as follows:

$$\begin{aligned}
\sum_{B,C\geq 1}\binom{A+k-2}{k-2+i} &= \sum_{B\geq 1}\sum_{C\geq 1}\binom{B+C+k-2}{k-2+i}\\
&= \sum_{B\geq 1}\left[\binom{B+\ell+k-1}{k-1+i}-\binom{B+k-1}{k-1+i}\right]\\
&= \left[\binom{r+\ell+k}{k+i}-\binom{\ell+k}{k+i}\right]-\left[\binom{r+k}{k+i}-\binom{k}{k+i}\right]\\
&= \binom{r+\ell+k}{k+i}-\binom{r+k}{k+i}-\binom{\ell+k}{k+i}. \qquad (4.5.9)
\end{aligned}$$

Substituting (4.5.7), (4.5.8), and (4.5.9) into (4.5.6) and simplifying, we have

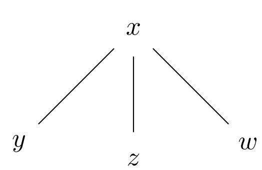
$$\begin{aligned}
\beta_i &= \binom{k-2+i}{k-2}\left[\binom{r+\ell+k}{k+i}-\binom{r+k}{k+i}-\binom{\ell+k}{k+i}\right]+\binom{k-1+i}{k-1}\left[\binom{r+k}{k+i}+\binom{\ell+k}{k+i}\right]\\
&= \binom{k-2+i}{k-2}\binom{r+\ell+k}{k+i}+\left[\binom{k-1+i}{k-1}-\binom{k-2+i}{k-2}\right]\left[\binom{r+k}{k+i}+\binom{\ell+k}{k+i}\right]\\
&= \binom{k-2+i}{k-2}\binom{r+\ell+k}{k+i}+\binom{k-2+i}{k-1}\left[\binom{r+k}{k+i}+\binom{\ell+k}{k+i}\right]\\
&= \binom{k-2+i}{i}\binom{r+\ell+k}{k+i}+\binom{k-2+i}{i-1}\left[\binom{r+k}{k+i}+\binom{\ell+k}{k+i}\right],
\end{aligned}$$

where the third equality follows from Pascal's Identity. This completes the proof. □

This result generalizes the formulas given by Ferra, Murgia, and Olteanu in [9, Corollary 3.4, Remark 3.5] for so-called *star graphs*; that is, whisker graphs with $\ell = 0$.

We finish this section with an example that demonstrates how our linear quotient ordering in Theorem 3.10 can be used to explicitly construct resolutions of powers of edge ideals which have linear quotients. In particular, we construct the resolution of the square of the edge ideal of a star graph via the iterated mapping cone construction in Construction 2.3.

Example 4.6. Let $P = \mathsf{k}[x, y, z, w]$ be a standard graded polynomial ring with k a field and let G be the following graph.



We see that the edge ideal associated with G is $I(G) = (xy,\, xz,\, xw)$ and its square is given by

$$I(G)^2 = (x^2y^2,\, x^2yz,\, x^2z^2,\, x^2yw,\, x^2zw,\, x^2w^2).$$

By Theorem 3.10 this gives a linear quotient ordering. In order to construct the minimal free resolution of $I(G)^2$, we begin by calculating the relevant quotient ideals as follows:

$$\begin{aligned}
((x^2y^2) : (x^2yz)) &= (y)\\
((x^2y^2,\, x^2yz) : (x^2z^2)) &= (y)\\
((x^2y^2,\, x^2yz,\, x^2z^2) : (x^2yw)) &= (y,\, z)
\end{aligned}$$

$$((x^2y^2,\, x^2yz,\, x^2z^2,\, x^2yw) \,:\, (x^2zw)) = (y,\, z)$$
$$((x^2y^2,\, x^2yz,\, x^2z^2,\, x^2yw,\, x^2zw) \,:\, (x^2w^2)) = (y,\, z)$$

Now we use iterated mapping cones as in Construction 2.3 to build the resolution in steps.

Step 1:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} y \end{bmatrix}} & P & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \begin{bmatrix} z \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} x^2yz \end{bmatrix}} & & \\ 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} x^2y^2 \end{bmatrix}} & P & \longrightarrow & 0 \end{array}$$

Step 2:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} y \end{bmatrix}} & P & \longrightarrow & 0 \\ & & & & \Big\downarrow {\scriptstyle \begin{bmatrix} 0 \\ z \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} x^2z^2 \end{bmatrix}} & & \\ 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} z \\ -y \end{bmatrix}} & P^2 & \xrightarrow{\begin{bmatrix} x^2y^2 & x^2yz \end{bmatrix}} & P & \longrightarrow & 0 \end{array}$$

Step 3:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} -z \\ y \end{bmatrix}} & P^2 & \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} & P & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle \begin{bmatrix} -w \\ 0 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} w & 0 \\ 0 & w \\ 0 & 0 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} x^2yw \end{bmatrix}} & & \\ 0 & \longrightarrow & P^2 & \xrightarrow{\begin{bmatrix} z & 0 \\ -y & z \\ 0 & -y \end{bmatrix}} & P^3 & \xrightarrow{\begin{bmatrix} x^2y^2 & x^2yz & x^2z^2 \end{bmatrix}} & P & \longrightarrow & 0 \end{array}$$

Step 4:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} -z \\ y \end{bmatrix}} & P^2 & \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} & P & \longrightarrow & 0 \\ & & & & \Big\downarrow {\scriptstyle \begin{bmatrix} 0 \\ -w \\ 0 \\ 0 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} 0 & 0 \\ w & 0 \\ 0 & w \\ 0 & 0 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} x^2zw \end{bmatrix}} & & \\ 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} -w \\ 0 \\ z \\ -y \end{bmatrix}} & P^4 & \xrightarrow{\begin{bmatrix} z & 0 & w & 0 \\ -y & z & 0 & w \\ 0 & -y & 0 & 0 \\ 0 & 0 & -y & -z \end{bmatrix}} & P^4 & \xrightarrow{\begin{bmatrix} x^2y^2 & x^2yz & x^2z^2 & x^2yw \end{bmatrix}} & P & \longrightarrow & 0 \end{array}$$

Step 5:

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & P & \xrightarrow{\begin{bmatrix} -z \\ y \end{bmatrix}} & P^2 & \xrightarrow{\begin{bmatrix} y & z \end{bmatrix}} & P & \longrightarrow & 0 \\ & & & & \Big\downarrow {\scriptstyle \begin{bmatrix} 0 \\ 0 \\ 0 \\ w \\ -w \\ 0 \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ w & 0 \\ 0 & w \end{bmatrix}} & & \Big\downarrow {\scriptstyle \begin{bmatrix} x^2w^2 \end{bmatrix}} & & \\ 0 & \longrightarrow & P^2 & \xrightarrow{\begin{bmatrix} -w & 0 \\ 0 & -w \\ z & 0 \\ -y & 0 \\ 0 & z \\ 0 & -y \end{bmatrix}} & P^6 & \xrightarrow{\begin{bmatrix} z & 0 & w & 0 & 0 & 0 \\ -y & z & 0 & w & w & 0 \\ 0 & -y & 0 & 0 & 0 & w \\ 0 & 0 & -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 & -y & -z \end{bmatrix}} & P^5 & \xrightarrow{\begin{bmatrix} x^2y^2 & x^2yz & x^2z^2 & x^2yw & x^2zw \end{bmatrix}} & P & \longrightarrow & 0 \end{array}$$

Finally, taking the cone of the diagram above, we get the desired resolution

$$0 \longrightarrow P^3 \xrightarrow{\begin{bmatrix} -w & 0 & 0 \\ 0 & -w & 0 \\ z & 0 & 0 \\ -y & 0 & w \\ 0 & z & -w \\ 0 & -y & 0 \\ 0 & 0 & z \\ 0 & 0 & -y \end{bmatrix}} P^8 \xrightarrow{\begin{bmatrix} z & 0 & w & 0 & 0 & 0 & 0 & 0 \\ -y & z & 0 & w & w & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & -y & -z & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & -y & -z & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & -y & -z \end{bmatrix}} P^6 \longrightarrow I(G)^2 \longrightarrow 0$$

5. Powers of the edge ideal of the anticycle

In this section we provide a linear quotient ordering on the higher order powers of the edge ideal of the anticycle graph (that is, the complement of the simple cycle), despite the fact that the edge ideal of the anticycle itself does not admit linear quotients. This answers in the affirmative a question of Hoefel and Whieldon.

We begin by establishing some notation which we will use throughout this section. Fix integers $n \geq 5$ and $k \geq 2$ and let $P = \Bbbk[x_1, \ldots, x_n]$. Further let $\mathcal{A}_n$ denote the anticycle on vertices $x_1, \ldots, x_n$ with edge ideal

$$I(\mathcal{A}_n) = (x_1x_3, x_1x_4, \ldots, x_1x_{n-1}, x_2x_4, \ldots, x_2x_n, \ldots, x_{n-2}x_n).$$

Note that $I(\mathcal{A}_n)$ is obtained from the edge ideal of the antipath on the same n vertices,

$$I(\mathcal{P}_n) = (x_1x_3, x_1x_4, \ldots, x_1x_n, x_2x_4, \ldots, x_2x_n, \ldots, x_{n-2}x_n)$$

by simply removing the edge x_1x_n. As in previous sections, we refer to edge and vertex decompositions of the generators of $I(\mathcal{A}_n)$ as introduced in Notation 3.2 throughout this section. Furthermore, for a monomial $M \in P$, we denote by $\operatorname{supp} M$ the usual support of a monomial; that is, the set of variables appearing with positive exponent in the vertex decomposition of M. Finally, we update our notation for ideal quotients established in Notation 3.6 to handle the multiple orderings we will utilize throughout this section.

Notation 5.1. Given an ordered list $\mathcal{O}$ of monomials in P and a monomial M from the list, we denote by $Q^{\mathcal{O}}(M)$ the ideal quotient corresponding to the monomial M under the ordering $\mathcal{O}$; that is, if $\mathcal{O} = (m_1, \ldots, m_r)$ then

$$Q^{\mathcal{O}}(m_i) = (m_1, \ldots, m_{i-1}) : (m_i)$$

for all $2 \leq i \leq r$.

Next we construct what we prove in Theorem 5.6 to be a linear quotient ordering on $I(\mathcal{A}_n)^k$.

Construction 5.2. Denote by $\mathcal{O}_n^{(k)}$ the following ordering of the minimal generators of $I(\mathcal{A}_n)^k$:

(1) First, order all the generators divisible by x_n according to the lexicographic ordering by:

$$x_n > x_2 > x_3 > \cdots > x_{n-1} > x_1;$$

that is, $x_n^{\alpha_n}x_2^{\alpha_2}x_3^{\alpha_3}\ldots x_{n-1}^{\alpha_{n-1}}x_1^{\alpha_1}$ precedes $x_n^{\beta_n}x_2^{\beta_2}x_3^{\beta_3}\ldots x_{n-1}^{\beta_{n-1}}x_1^{\beta_1}$ whenever the first nonzero entry in the vector $(\alpha_n, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}, \alpha_1) - (\beta_n, \beta_2, \beta_3, \ldots, \beta_{n-1}, \beta_1)$ is positive.

(2) Next, order the remaining generators according to the lexicographic ordering by

$$x_1 > x_2 > x_3 > \cdots > x_{n-1};$$

that is, $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_{n-1}^{\alpha_{n-1}}$ precedes $x_1^{\beta_1}x_2^{\beta_2}\ldots x_{n-1}^{\beta_{n-1}}$ whenever the first nonzero entry in the vector $(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) - (\beta_1, \beta_2, \ldots, \beta_{n-1})$ is positive.

(3) Finally, move the generator $(x_1x_{n-1})^k$ immediately after the generator

$$D := (x_1x_{n-1})^{k-1}(x_2x_{n-1}).$$

We call D the *distinguished generator.*

Note that $\mathcal{O}_n^{(k)}$ is a concatenation of two orderings F and S, where F is the sub-ordering containing all generators divisible by x_n as described in (1) and S is the sub-ordering containing all generators not divisible by x_n as described in (2) and (3).

Example 5.3. Let $n = 5$ and $k = 2$. The ordering constructed above is $\mathcal{O}_n^{(k)} = (F, S)$, where

$$F = (x_2^2x_5^2,\ x_2x_3x_5^2,\ x_3^2x_5^2,\ x_2^2x_4x_5,\ x_2x_3x_4x_5,\ x_1x_2x_3x_5,\ x_1x_2x_4x_5,\ x_1x_3^2x_5,\ x_1x_3x_4x_5)$$
$$S = (x_1^2x_3^2,\ x_1^2x_3x_4,\ x_1x_2x_3x_4,\ x_1x_2x_4^2,\ x_1^2x_4^2,\ x_2^2x_4^2).$$

Remark 5.4. Note that the concatenation of two linear quotient orderings need not be a linear quotient ordering. For instance, (x_1x_2, x_2x_3) and (x_3x_4, x_4x_5, x_1x_5) are both linear quotient orderings, but their concatenation is not; indeed, the simple cycle $\mathcal{C}_5$ admits no linear quotient ordering.

On the other hand, the concatenation of two orderings being a linear quotient ordering does not necessarily imply that the second ordering is itself a linear quotient ordering. As an example, (x_2x_3, x_1x_2, x_3x_4) is a linear quotient ordering of the simple path $\mathcal{P}_4$, but (x_1x_2, x_3x_4) is not a linear quotient ordering.

Surprisingly, however, we show in Theorem 5.6 that F and S as defined in Construction 5.2 interact "nicely" with each other, in that their concatenation is a linear quotient ordering and furthermore they are both themselves linear quotient orderings.

Before we prove that $\mathcal{O}_n^k$ is a linear quotient ordering, we illustrate why some natural simplifications of the ordering $\mathcal{O}_n^{(k)}$ do not produce linear quotient orderings.

Remark 5.5. (1) The lexicographic ordering of all of the generators by

$$x_n > x_2 > x_3 > \cdots > x_{n-1} > x_1$$

does not yield a linear quotient ordering of $I(\mathcal{A}_n)^k$ because in this case if $M_1 = x_nx_1x_3^2(x_1x_3)^{k-2}$ and $M_2 = x_n^2x_2^2(x_1x_3)^{k-2}$, there is no generator M_3 preceding M_2 such that

$$\frac{M_3}{\gcd(M_3, M_2)} \in \{x_1, x_2 \dots, x_n\} \quad \text{and} \quad \frac{M_3}{\gcd(M_3, M_2)} \Big| \frac{M_1}{\gcd(M_1, M_2)}.$$

This illustrates the importance of our choice in Construction 5.2(2).

(2) The lexicographic ordering of the generators divisible by x_n by

$$x_n > x_2 > x_3 > \cdots > x_{n-1} > x_1$$

followed by the lexicographic ordering of the remaining generators by

$$x_1 > x_2 > x_3 > \cdots > x_{n-1}$$

does not yield a linear quotient ordering of $I(\mathcal{A}_n)^k$, because in this case if $M_1 = x_2x_n(x_1x_{n-1})^{k-1}$ and $M_2 = (x_1x_{n-1})^k$, there is no generator M_3 preceding M_2 such that

$$\frac{M_3}{\gcd(M_3, M_2)} \in \{x_1, x_2 \dots, x_n\} \quad \text{and} \quad \frac{M_3}{\gcd(M_3, M_2)} \Big| \frac{M_1}{\gcd(M_1, M_2)}.$$

This illustrates the final move in Construction 5.2 (3), which surprisingly requires no further changes.

Theorem 5.6. *Let $P = k[x_1, \dots, x_n]$ be a standard graded polynomial ring with $n \geq 5$, and let $I(\mathcal{A}_n) \subseteq P$ be the edge ideal of the anticycle graph $\mathcal{A}_n$ on n vertices. Then for all integers $k \geq 2$, the ordering $\mathcal{O}_n^{(k)}$ defined in Construction 5.2 is a linear quotient ordering of $I(\mathcal{A}_n)^k$.*

Proof. Let M_1 and M_2 be part of a minimal generating set for $I(\mathcal{A}_n)^k$, with M_1 preceding M_2 in $\mathcal{O}_n^{(k)}$. Assume the following vertex decompositions:

$$M_1 = x_1^{\alpha_1} \dots x_n^{\alpha_n} \quad \text{and} \quad M_2 = x_1^{\beta_1} \dots x_n^{\beta_n},$$

and define vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$. By Lemma 2.2, it suffices to show that there exists some M_3 preceding M_2 such that

$$\frac{M_3}{\gcd(M_3, M_2)} \in \{x_1, x_2 \dots, x_n\} \quad \text{and} \quad \frac{M_3}{\gcd(M_3, M_2)} \Big| \frac{M_1}{\gcd(M_1, M_2)}.$$

We separate our proof into three main cases depending on whether M_1 and M_2 are in F or S. Note that since M_1 precedes M_2 in $\mathcal{O}_n^{(k)}$, the case where M_1 is in S and M_2 is in F is impossible.

Case 1: In this case we assume that $M_1 \in F$ and $M_2 \in S$, and proceed with two subcases.

Subcase 1.1: First we assume that $M_2 = (x_1 x_{n-1})^k$. Since $M_1 \in F$, there is some $i \in \{2, \dots, n-2\}$ such that $x_i x_n$ is an edge of M_1. We choose a generator M_3 of $I(\mathcal{A}_n)^k$ as follows:

$$M_3 := \begin{cases} M_2 \cdot \frac{x_i x_{n-1}}{x_1 x_{n-1}}, & \text{if } i = 2 \\ M_2 \cdot \frac{x_1 x_i}{x_1 x_{n-1}}, & \text{if } i > 2 \end{cases}$$

In either case, M_3 precedes M_2 in $\mathcal{O}_n^{(k)}$. Indeed, when $i > 2$ this follows from the lex ordering on S in Construction 5.2(2), and otherwise M_3 is the distinguished generator and this follows from Construction 5.2(3). Furthermore we have:

$$\frac{M_3}{\gcd(M_3, M_2)} = x_i \quad \text{and} \quad x_i \Big| \frac{M_1}{\gcd(M_1, M_2)},$$

where the equality follows directly from the definitions of M_2 and M_3 and the divisibility statement follows from the fact that x_i divides M_1, but not M_2. This completes this subcase.

Subcase 1.2: Now we assume that $M_2 \neq (x_1 x_{n-1})^k$. Since $M_1 \in F$ and $M_2 \in S$, it follows that x_n divides M_1 but not M_2, and thus

$$x_n \Big| \frac{M_1}{\gcd(M_1, M_2)}.$$

Now it suffices to find a generator M_3 preceding M_2 in $\mathcal{O}_n^{(k)}$ such that

$$\frac{M_3}{\gcd(M_3, M_2)} = x_n. \tag{5.6.1}$$

By assumption, M_2 must have an edge $x_a x_b$ such that $1 \leq a < b \leq n-1$ and $x_a x_b \neq x_1 x_{n-1}$. In particular, if $a = 1$, then $1 < b < n-1$, and thus $x_n x_b$ is an edge of $\mathcal{A}_n$ in this case. Therefore, we may choose a generator M_3 of $I(\mathcal{A}_n)^k$ as follows:

$$M_3 := \begin{cases} M_2 \cdot \frac{x_n x_b}{x_a x_b}, & \text{if } a = 1 \\ M_2 \cdot \frac{x_a x_n}{x_a x_b}, & \text{if } a > 1 \end{cases}$$

In either case, it follows from Construction 5.2(1) that M_3 precedes M_2 in $\mathcal{O}_n^{(k)}$ and satisfies (5.6.1), as desired.

Case 2: In this case we assume that $M_1, M_2 \in S$. Note that it suffices to show S is itself a linear quotient ordering. Since S is an ordering of the minimal generators of $I(\mathcal{A}_n)^k$ not divisible by x_n, it follows that S is an ordering of the minimal generators of $I(P_{n-1})^k$. Furthermore, by [17, Proposition 3.1], the lex ordering $x_1 > x_2 > x_3 > \cdots > x_{n-1}$, which we denote by L, yields a linear quotient ordering on $I(P_{n-1})^k$. Thus it remains only to show that our choice to move $(x_1 x_{n-1})^k$ immediately after the distinguished generator as described in Construction 5.2(3) does not disturb the linear quotient property.

Observe that if M_2 precedes $(x_1 x_{n-1})^k$ in L or if M_2 succeeds the distinguished generator $D := (x_1 x_{n-1})^{k-1}(x_2 x_{n-1})$ in L, then $Q^L(M_2) = Q^S(M_2)$. Thus, since L is a linear quotient ordering, we may assume without loss of generality that M_2 lies between $(x_1x_{n-1})^k$ and D (inclusive) in L. We proceed with three subcases.

Subcase 2.1: First we assume that M_2 lies strictly between $(x_1x_{n-1})^k$ and D in L. Thus by the lex ordering L, we have that $\beta_1 = k-1$, and so x_1^{k-1} divides M_2. Further since M_2 precedes D in L, we have $\beta_2 \geq 1$, implying that x_2 divides M_2. Thus $x_1^{k-1}x_2$ divides M_2, while x_1^k cannot divide M_2. It follows that

$$\frac{M_2}{\gcd(M_2,(x_1x_{n-1})^k)} \notin \{x_1, x_2 \ldots, x_n\}.$$

Indeed, we have that x_2 divides $\frac{M_2}{\gcd(M_2,(x_1x_{n-1})^k)}$, but equality would imply $M_2 = D$, which is impossible in our case. This implies that

$$M := \frac{(x_1x_{n-1})^k}{\gcd((x_1x_{n-1})^k, M_2)} \notin \{x_1, x_2 \ldots, x_n\}$$

as well by degree considerations

Since $Q^L(M_2)$ is generated by a subset of the variables $\{x_1, \ldots, x_n\}$ and since $M \notin \{x_1, \ldots, x_n\}$, M is not part of a minimal generating set for $Q^L(M_2)$. It follows that $Q^S(M_2) = Q^L(M_2)$, and thus $Q^S(M_2)$ is generated by a subset of the variables, as desired.

Subcase 2.2: Next we assume that $M_2 = (x_1x_{n-1})^k$. By the proof of Subcase 1.1, it follows that $(x_2, x_3, \ldots, x_{n-2})$ is contained in the ideal quotient $Q^S((x_1x_{n-1})^k)$. Pairing this with the fact that M_1 is in S, and so x_n cannot divide M_1, we may assume without loss of generality that

$$\operatorname{supp}\left(\frac{M_1}{\gcd(M_1, M_2)}\right) \subseteq \{x_1, x_{n-1}\}.$$

However, this is a clear contradiction since $M_2 = (x_1x_{n-1})^k$.

Subcase 2.3: Finally we assume that $M_2 = D$ is the distinguished generator. First note that $(x_3, \ldots, x_{n-2})$ is contained in the ideal quotient $Q^S(M_2)$. Indeed, each generator

$$N_i := M_2 \cdot \frac{x_1x_i}{x_1x_{n-1}},$$

for $i \in \{3, \ldots, n-2\}$, precedes M_2 in S. Thus

$$\frac{N_i}{\gcd(N_i, M_2)} = x_i \in Q^S(M_2)$$

for all $i \in \{3, \ldots, n-2\}$, as claimed.

So we may assume without loss of generality that

$$\operatorname{supp}\left(\frac{M_1}{\gcd(M_1, M_2)}\right) \subseteq \{x_1, x_2, x_{n-1}\},$$

where, as before, we can disregard x_n since $M_1 \in S$. Therefore, since x_1x_2 is not an edge in $\mathcal{A}_n$, we have

$$M_1 = (x_1x_{n-1})^a(x_2x_{n-1})^b = x_1^a x_2^b x_{n-1}^k,$$

for some nonnegative integers a and b with $a + b = k$. In particular we have $a \leq k$. Furthermore since M_1 precedes $D = x_1^{k-1}x_2x_{n-1}^k$ in S, it follows that $a \geq k-1$. Thus $a \in \{k-1, k\}$ yielding

$$M_1 = x_1^{k-1}x_2x_{n-1}^k \quad \text{or} \quad M_1 = x_1^k x_{n-1}^k.$$

However, both of these cases are impossible, as the former implies $M_1 = M_2 = D$, which contradicts the assumption that M_1 precedes M_2, and the latter implies $(x_1x_{n-1})^k$ precedes the distinguished generator D in S, which contradicts Construction 5.2(3).

Case 3: In this case we assume that $M_1, M_2 \in F$. The vertex decompositions of M_1 and M_2 imply

$$\operatorname{supp}\left(\frac{M_1}{\gcd(M_1, M_2)}\right) = \{x_i \mid \alpha_i > \beta_i\}.$$

We now proceed in four subcases depending on the possibilities for the vectors α and β.

Subcase 3.1: First we assume that $\alpha_n > \beta_n$. It follows that

$$x_n \Big| \frac{M_1}{\gcd(M_1, M_2)},$$

and so it suffices to find a generator M_3 preceding M_2 in F such that

$$\frac{M_3}{\gcd(M_3, M_2)} = x_n. \tag{5.6.2}$$

If there is some edge $x_a x_b$ of M_2 with $1 \leq a < b \leq n-1$ and $x_a x_b \neq x_1 x_{n-1}$, then we may proceed precisely as in Subcase 1.2 to find the desired M_3, and we are done.

Thus we may assume without loss of generality that all edges of M_2 are either equal to $x_1 x_{n-1}$ or incident to x_n. Since $\alpha_n > \beta_n$, it follows that at least one edge of M_2 is not incident to x_n, and thus $x_1 x_{n-1}$ is an edge of M_2. This implies that $\{x_2, \ldots, x_{n-2}\} \subseteq Q^F(M_2)$; indeed, each generator

$$N_i := \begin{cases} M_2 \cdot \frac{x_i x_{n-1}}{x_1 x_{n-1}}, & \text{if } i = 2 \\ M_2 \cdot \frac{x_1 x_i}{x_1 x_{n-1}}, & \text{if } 2 < i < n-1 \end{cases}$$

for $i \in \{2, \ldots, n-2\}$ precedes M_2 in F by Construction 5.2(1), and thus

$$\frac{N_i}{\gcd(N_i, M_2)} = x_i \in Q^F(M_2)$$

for all $i \in \{2, \ldots, n-2\}$, as claimed.

So we may assume without loss of generality that $\beta_i \geq \alpha_i$ for all $i \in \{2, \ldots, n-2\}$. Notice however that since M_2 has only the edges $x_1 x_{n-1}$ and edges incident to x_n, it follows that $\beta_1 + \beta_{n-1} + 2\beta_n = 2k$, which yields the following inequalities

$$\beta_1 + \beta_{n-1} + \beta_n = 2k - \beta_n > 2k - \alpha_n = \alpha_1 + \cdots + \alpha_{n-1} \geq \alpha_1 + \alpha_{n-1} + \alpha_n, \tag{5.6.3}$$

where the strict inequality follows from our assumption in this subcase and the other inequality follows directly from the structure of the anticycle graph (x_n is only connected by edges to vertices $x_2, \ldots, x_{n-2}$). The strict inequality $\beta_1 + \beta_{n-1} + \beta_n > \alpha_1 + \alpha_{n-1} + \alpha_n$ in (5.6.3) together with the assumption that $\beta_i \geq \alpha_i$ for all $i \in \{2, \ldots, n-2\}$ yields the inequality

$$2k = \sum_{i=1}^{n} \beta_i > \sum_{i=1}^{n} \alpha_i = 2k,$$

which is a clear contradiction. This completes this subcase.

Subcase 3.2: Next we assume that $\alpha_n = \beta_n$ and $\alpha_2 > \beta_2$. It follows that

$$x_2 \Big| \frac{M_1}{\gcd(M_1, M_2)},$$

and so it suffices to find a generator M_3 preceding M_2 in F such that

$$\frac{M_3}{\gcd(M_3, M_2)} = x_2. \tag{5.6.4}$$

If there is some edge $x_a x_b$ of M_2 such that $3 \leq a < b \leq n-1$, then the generator

$$M_3 = M_2 \cdot \frac{x_2 x_b}{x_a x_b}$$

precedes M_2 in F and satisfies (5.6.4), as desired.

Thus we may assume that all edges of M_2 are incident to x_1, x_2, or x_n. If x_1x_b is an edge of M_2 for some $4 \le b \le n-1$, we find that the generator

$$M_3 = M_2 \cdot \frac{x_2x_b}{x_1x_b}$$

precedes M_2 in F and satisfies (5.6.4), as desired. Similarly, if x_ax_n is an edge of M_2 for some $3 \le a \le n-2$, then choosing

$$M_3 = M_2 \cdot \frac{x_2x_n}{x_ax_n}$$

will complete our argument.

Thus we may assume that every edge of M_2 is either incident to x_2 or equal to x_1x_3. Since $\alpha_2 > \beta_2$, it follows that at least one edge of M_2 is not incident to x_2, and thus x_1x_3 is an edge of M_2. Furthermore, since $M_2 \in F$, it follows that x_2x_n is also an edge of M_2. Thus the generator

$$M := M_2 \cdot \frac{(x_2x_4)(x_3x_n)}{(x_1x_3)(x_2x_n)}$$

precedes M_2 in F, yielding

$$\frac{M}{\gcd(M, M_2)} = x_4 \in Q^F(M_2).$$

Furthermore, we have that $\{x_5, \dots, x_n\} \subseteq Q^F(M_2)$. Indeed, each generator

$$N_i = M_2 \cdot \frac{x_3x_i}{x_1x_3}$$

for $i \in \{5, \dots, n\}$ precedes M_2 in F by Construction 5.2(1), and thus

$$\frac{N_i}{\gcd(N_i, M_2)} = x_i \in Q^F(M_2).$$

Therefore we may further assume that $\beta_i \ge \alpha_i$, for all $i \in \{4, \dots, n\}$. However, by the same argument as in Subcase 3.1, the equality $\beta_1+\beta_3+2\beta_2 = 2k$ implies that $\beta_1+\beta_2+\beta_3 > \alpha_1+\alpha_2+\alpha_3$, which coupled with the fact that $\beta_i \ge \alpha_i$ for all $i \in \{4, \dots, n\}$, produces the same contradiction as in Subcase 3.1 and completes this subcase.

Subcase 3.3: Next we assume that $\alpha_n = \beta_n$, $\alpha_i = \beta_i$, for all $i \in \{2, \dots, j-1\}$, and $\alpha_j > \beta_j$, for some $j \in \{3, \dots, n-2\}$. It follows that

$$x_j \Big| \frac{M_1}{\gcd(M_1, M_2)},$$

so it suffices to find a generator M_3 preceding M_2 in F such that

$$\frac{M_3}{\gcd(M_3, M_2)} = x_j. \tag{5.6.5}$$

If there is an edge x_ax_b of M_2 such that $1 \le a \le j-2$ and $j+1 \le b \le n-1$, then the generator

$$M_3 = M_2 \cdot \frac{x_ax_j}{x_ax_b}$$

precedes M_2 in F and satisfies (5.6.5), as desired. Similarly, if there is an edge x_ax_b of M_2 such that $j+1 \le a < b \le n$, then choosing

$$M_3 = M_2 \cdot \frac{x_jx_b}{x_ax_b}$$

will complete the argument.

Also note that if x_1x_c is an edge of M_2 for some $3 \le c \le j-2$, then choosing

$$M_3 = M_2 \cdot \frac{x_cx_j}{x_1x_c}$$

will complete the argument.

So in summary, we may assume that all edges of M_2 are incident to x_{j-1} or x_j, are of the form $x_c x_n$, for some $2 \le c \le j-2$, or are of the form $x_a x_b$ with $2 \le a < b \le j-2$. By our hypothesis for this subcase, there is an integer $r \in \{1\} \cup \{j+1, \dots, n-1\}$ such that $\alpha_r < \beta_r$, and in particular, $\beta_r > 0$. Thus by the preceding description of edges of M_2, it follows that $x_r x_z$ is an edge of M_2 for some $z \in \{j-1, j\}$, and furthermore that every edge of M_2 incident to x_r must also be incident to x_{j-1} or x_j. This allows us to rule out edges of the form $x_a x_b$ with $2 \le a < b \le j-2$ in M_2. Indeed, suppose $x_a x_b$ is such an edge. Since $2 \le a < j-2$ and $2 < b \le j-2$, we have that $x_a x_z, x_b x_r \in I(\mathcal{A}_n)$. Thus M_2 has the following edge decomposition:

$$M_2 = M_2 \cdot \frac{(x_a x_z)(x_b x_r)}{(x_a x_b)(x_r x_z)},$$

which implies that $x_b x_r$ is an edge of M_2, but this contradicts the fact that all edges of M_2 incident to x_r must also be incident to x_{j-1} or x_j.

Now we may assume that all edges of M_2 are incident to x_{j-1} or x_j, or are of the form $x_c x_n$ for some $2 \le c \le j-2$. In fact, there must be an edge of the third type. Otherwise, if all edges of M_2 are incident to x_j or x_{j-1}, then our hypotheses in this subcase imply

$$k = \beta_{j-1} + \beta_j = \alpha_{j-1} + \beta_j < \alpha_{j-1} + \alpha_j,$$

and it follows that the degree of M_1 is at least $2(\alpha_{j-1} + \alpha_j) > 2k$, which is a contradiction. Thus there must be an edge of M_2 of the form $x_c x_n$ for some $2 \le c \le j-2$.

Next recall that $x_r x_z$ is an edge of M_2 for some $r \in \{1\} \cup \{j+1, \dots, n-1\}$ and $z \in \{j-1, j\}$. In fact, we must have $r = 1$ and $c = 2$. Otherwise, if $r \ne 1$ or $c \ne 2$, then M_2 has the following edge decomposition

$$M_2 = M_2 \cdot \frac{(x_c x_r)(x_z x_n)}{(x_c x_n)(x_r x_z)},$$

which implies that $x_c x_r$ is an edge of M_2, but this contradicts our description of the edges of M_2 given that $c, r \notin \{j-1, j, n\}$. Thus $r = 1$ and $c = 2$.

Thus we may assume that all edges of M_2 are incident to x_{j-1} or x_j, or are of the form $x_2 x_n$, and that $x_2 x_n$ and $x_1 x_z$ with $z \in \{j-1, j\}$ are edges of M_2. Now it follows that $j > 3$. Indeed, if $j = 3$, then $j - 1 = 2$, and it follows that all edges of M_2 are incident to x_{j-1} or x_j, which we have already shown is impossible.

So we may choose the generator

$$M_3 = M_2 \cdot \frac{(x_2 x_j)(x_z x_n)}{(x_1 x_z)(x_2 x_n)},$$

which precedes M_2 in F and satisfies (5.6.5), as desired.

Subcase 3.4: Finally we assume that $\alpha_n = \beta_n$, $\alpha_i = \beta_i$ for all $i \in \{2, \dots, n-2\}$, and $\alpha_{n-1} > \beta_{n-1}$. Note that this is indeed the final case because if $\alpha_n = \beta_n$ and $\alpha_i = \beta_i$ for all $i \in \{2, \dots, n-1\}$, then $M_1 = M_2$, which contradicts our assumption that M_1 precedes M_2 in F.

In this case, notice that since M_1 and M_2 both have degree $2k$, our assumptions in this subcase imply that $\beta_1 > \alpha_1$. In particular we have $\beta_1 > 0$. Since $\alpha_{n-1} > \beta_{n-1}$, it also follows that

$$x_{n-1} \Big| \frac{M_1}{\gcd(M_1, M_2)},$$

and so it suffices to find a generator M_3 preceding M_2 in F such that

$$\frac{M_3}{\gcd(M_3, M_2)} = x_{n-1}. \tag{5.6.6}$$

Suppose for the sake of contradiction that there is no such generator M_3. Since $\beta_1 > 0$, we have that x_1x_z is an edge of M_2 for some $z \in \{3, \dots, n-1\}$. In fact we must have $z \in \{n-2, n-1\}$, because otherwise the generator

$$M_3 = M_2 \cdot \frac{x_z x_{n-1}}{x_1 x_z}$$

precedes M_2 in F and satisfies (5.6.6), yielding a contradiction. In particular, every edge incident to x_1 must also be incident to x_{n-2} or x_{n-1}.

Now we claim that there are no edges x_ax_b of M_2 with $2 \leq a < b \leq n-3$. This is indeed the case because otherwise, M_2 has the following edge decomposition

$$M_2 = M_2 \cdot \frac{(x_1x_b)(x_ax_z)}{(x_ax_b)(x_1x_z)},$$

and thus x_1x_b is an edge of M_2 with $b \notin \{n-2, n-1\}$, which is impossible by the argument above. Thus our claim holds, and since every edge incident to x_1 must also be incident to x_{n-2} or x_{n-1}, it follows that every edge of M_2 is incident to x_{n-2}, x_{n-1}, or x_n.

Next we rule out x_1x_{n-2} as an edge of M_2. Supposing to the contrary that x_1x_{n-2} is an edge of M_2, we claim that in this case x_ax_n is not an edge of M_2 for $2 \leq a \leq n-3$. Indeed, if x_ax_n is an edge of M_2 for some $3 \leq a \leq n-3$, then M_2 has the following edge decomposition

$$M_2 = M_2 \cdot \frac{(x_1x_a)(x_{n-2}x_n)}{(x_1x_{n-2})(x_ax_n)},$$

and thus x_1x_a is an edge of M_2 with $a \notin \{n-2, n-1\}$, which is impossible. On the other hand, if x_2x_n is an edge of M_2, then the generator

$$M_3 = M_2 \cdot \frac{(x_2x_{n-1})(x_{n-2}x_n)}{(x_1x_{n-2})(x_2x_n)}$$

precedes M_2 in F and satisfies (5.6.6), yielding another contradiction. This proves the claim, and it follows that any edge of M_2 incident to x_n must also be incident to x_{n-2}. Thus every edge of M_2 must be incident to x_{n-2} or x_{n-1}. However, since $\alpha_{n-2} = \beta_{n-2}$ and $\alpha_{n-1} > \beta_{n-1}$, this implies that

$$k = \beta_{n-1} + \beta_{n-2} < \alpha_{n-1} + \alpha_{n-2},$$

which is impossible given the structure of $\mathcal{A}_n$. Therefore, x_1x_{n-2} is not an edge of M_2, as desired, and in particular x_1x_{n-1} must be an edge of M_2.

Finally suppose that M_2 has edges x_ax_{n-2} and x_bx_n, for some $a, b \in \{1, 2, \dots, n-3\}$. Then we have that the generator

$$M_3 = M_2 \cdot \frac{(x_{n-2}x_n)(x_ax_{n-1})(x_bx_{n-1})}{(x_ax_{n-2})(x_bx_n)(x_1x_{n-1})}$$

precedes M_2 in F and satisfies (5.6.6), which yields a contradiction. It follows that either all edges of M_2 incident to x_{n-2} must also be incident to x_n, or all edges of M_2 incident to x_n must also be incident to x_{n-2}. Using the fact that every edge of M_2 must be incident to x_{n-2}, x_{n-1}, or x_n, the former case implies that all edges of M_2 are incident to x_{n-1} or x_n, and the latter case implies that all edges of M_2 are incident to x_{n-1} or x_{n-2}. In either case, we have a contradiction to our hypotheses that $\alpha_{n-1} > \beta_{n-1}$, $\alpha_n = \beta_n$, and $\alpha_{n-2} = \beta_{n-2}$. This completes the proof.

□

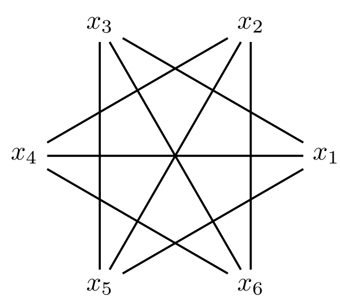
6. Linear quotient computations

In this section we examine the linear quotient property from a computational perspective using new methods we constructed on Macaulay2. In particular, we introduce three methods: `isLinear`, `getQuotients`, and `findLinearOrderings`. To describe these methods and provide examples, we adopt Notation 5.1 for ideal quotients to be used throughout this section.

Our first two methods take as an input an ordered list $\mathcal{O} = (m_1, \dots, m_r)$ whose entries minimally generate an ideal I of a standard graded polynomial ring. The method `isLinear` tests whether $\mathcal{O}$ is a linear quotient ordering for I, and the method `getQuotients` outputs the ordered list of ideal quotients $\big(Q^{\mathcal{O}}(m_1), Q^{\mathcal{O}}(m_2), \dots, Q^{\mathcal{O}}(m_r)\big)$ corresponding to the ordering $\mathcal{O}$. The method `findLinearOrderings` takes an ideal I as an input and returns an ordered list $\mathcal{O}$ of the generators of I that is what we call *most linear*; see Definition 6.2. The code for each of these methods and further documentation is in [26].

Our methods complement those in [10]; the method `isLinear` offers similar capabilities, although `getQuotients` and `findLinearOrderings` offer new capabilities beyond those in [10]. In particular, `findLinearOrderings` provides useful homological information about edge ideals, even those that do not admit linear quotients; see Definition 6.2, Example 6.3, and Example 6.4 for a precise description of its capabilities.

To illustrate our results in Section 5, we focus on powers of the anticycle on six vertices $\mathcal{A}_6$, pictured below, first using our methods `isLinear` and `getQuotients`. We input the list of generators ordered according to the linear quotient ordering in Construction 5.2 (see also Theorem 5.6) and verify that indeed the given ordering yields linear quotients.



Example 6.1. (`getQuotients` and `isLinear` with $I(\mathcal{A}_6)^2$)

We start by considering the ordering $\mathcal{O}_6^{(2)}$ defined in Construction 5.2; this is a linear quotient ordering by Theorem 5.6. We first define our polynomial ring in six variables, and then create the list $\mathcal{O}_6^{(2)}$ by defining two sublists, F and S, where F consists of all generators divisible by x_6, and S consists of all generators not divisible by x_6, as is described in Construction 5.2.

```
i1 : Q = QQ[x_1..x_6];

i2 : F = {x_6^2*x_2^2, x_6^2*x_2*x_3, x_6^2*x_2*x_4, x_6^2*x_3^2, x_6^2*x_3*x_4,
          x_6^2*x_4^2, x_6*x_2^2*x_4, x_6*x_2^2*x_5, x_6*x_2*x_3*x_4,
          x_6*x_2*x_3*x_5, x_6*x_2*x_3*x_1, x_6*x_2*x_4^2, x_6*x_2*x_4*x_5,
          x_6*x_2*x_4*x_1, x_6*x_2*x_5*x_1, x_6*x_3^2*x_5, x_6*x_3^2*x_1,
          x_6*x_3*x_4*x_5, x_6*x_3*x_4*x_1, x_6*x_3*x_5*x_1, x_6*x_4^2*x_1,
          x_6*x_4*x_5*x_1};

i3 : S = {x_1^2*x_3^2, x_1^2*x_3*x_4, x_1^2*x_3*x_5, x_1^2*x_4^2, x_1^2*x_4*x_5,
          x_1*x_2*x_3*x_4, x_1*x_2*x_3*x_5, x_1*x_2*x_4^2, x_1*x_2*x_4*x_5,
          x_1*x_2*x_5^2, x_1^2*x_5^2, x_1*x_3^2*x_5, x_1*x_3*x_4*x_5,
          x_1*x_3*x_5^2, x_2^2*x_4^2, x_2^2*x_4*x_5, x_2^2*x_5^2,
          x_2*x_3*x_4*x_5, x_2*x_3*x_5^2, x_3^2*x_5^2};
```

Now we concatenate these two lists to get $\mathcal{O}_6^{(2)}$, and pass it as input into `getQuotients`:

```
i4 : getQuotients (F | S)

o4 = {{x }, {x , x }, {x }, {x , x }, {x , x }, {x }, {x , x }, {x , x },
        2     3   2     2     3   2     3   2     6     6   4     6   2
     --------------------------------------------------------------------------
      {x , x , x }, {x , x , x }, {x , x , x }, {x , x , x , x },
        6   4   2     6   5   4     6   3   2     6   4   3   2
     --------------------------------------------------------------------------
      {x , x , x , x , x }, {x , x , x }, {x , x }, {x , x , x }, {x , x , x },
        6   5   4   3   2     4   3   2     6   2     6   5   2     6   3   2
     --------------------------------------------------------------------------
      {x , x , x , x }, {x , x , x }, {x , x , x }, {x , x , x }, {x }, {x , x },
        6   5   3   2     4   3   2     6   3   2     4   3   2     6     6   3
     --------------------------------------------------------------------------
      {x , x , x }, {x , x }, {x , x , x }, {x , x }, {x , x , x }, {x , x , x },
        6   4   3     6   3     6   4   3     6   1     6   4   1     6   3   1
     --------------------------------------------------------------------------
      {x , x , x , x }, {x , x , x }, {x , x , x }, {x , x , x },
        6   4   3   1     6   4   3     4   3   2     6   2   1
     --------------------------------------------------------------------------
      {x , x , x , x }, {x , x , x , x , x }, {x , x }, {x , x , x },
        6   3   2   1     6   4   3   2   1     6   1     6   4   1
     --------------------------------------------------------------------------
      {x , x , x }, {x , x , x }, {x , x , x , x }, {x , x , x }}
        6   4   1     6   2   1     6   4   2   1     6   2   1

o4 : List
```

Inspecting the output, we notice that each quotient is generated by vertices in $\mathcal{A}_6$, and thus $\mathcal{O}_6^{(2)}$ is a linear quotient ordering as we proved in Theorem 5.6. To verify this result computationally, we use the method `isLinear`:

```
i5 : isLinear (F | S, n = 6)

o5 = true
```

Next we introduce the notion of a *most linear ordering* for an ideal I.

Definition 6.2. Given an ordered list $\mathcal{O} = (m_1, \ldots, m_r)$ whose entries minimally generate an ideal I of a standard graded polynomial ring, we say that $\mathcal{O}$ is *linear up to n* if its ideal quotients $Q^{\mathcal{O}}(m_i)$ are linear for all $i \leq n$. If $\mathcal{O}$ is linear up to n, and no permutation of $\mathcal{O}$ is linear up to $j > n$, then we call $\mathcal{O}$ a *most linear ordering* for the ideal I.

To find a linear quotient ordering of an edge ideal $I(G)$, or a *most linear* ordering in the case that $I(G)$ does not admit linear quotients, we use the method `findLinearOrderings`.

Example 6.3. (`findLinearOrderings` with $I(\mathcal{A}_6)$)

```
i6 : antiCycleSix = ideal (x_1*x_3, x_1*x_4, x_1*x_5, x_2*x_4, x_2*x_5,
                           x_2*x_6, x_3*x_5, x_3*x_6, x_4*x_6);

i7 : findLinearOrderings (antiCycleSix, 6)
```

```
No linear ordering found; returning most linear ordering as a list.
    -- Elapsed time: .16227 seconds.

o7 = {x x , x x , x x , x x , x x , x x }
       4 6   3 6   2 6   3 5   2 5   2 4

o7 : List
```

As expected from [12] (see also [17]), we observe that $I(\mathcal{A}_6)$ does not admit linear quotients. Furthermore, we find a most linear ordering for $\mathcal{A}_6$ and consequently, a subgraph G of $\mathcal{A}_6$ of largest size whose edge ideal $I(G) = (x_4x_6, x_3x_6, x_2x_6, x_3x_5, x_2x_5, x_2x_4)$ admits linear quotients. This gives some indication of how close $I(\mathcal{A}_6)$ is to admitting linear quotients.

In the next example we consider the second power of the anticycle on six vertices, which admits linear quotients as proved in Theorem 5.6. We use `findLinearOrderings` to exhibit a linear quotient ordering.

Example 6.4. (`findLinearOrderings` with $I(\mathcal{A}_6)^2$)

```
i8 : findLinearOrderings (trim antiCycleSix^2, 6)

Linear ordering found, returning as a list.
    -- Elapsed time: 1.54978 seconds.

       2 2      2       2   2 2      2   2 2                     2
o8 = {x x , x x x , x x x , x x , x x x , x x , x x x x , x x x x , x x x ,
       4 6   3 4 6   2 4 6   3 6   2 3 6   2 6   3 4 5 6   2 4 5 6   3 5 6
     ------------------------------------------------------------------------
                         2                   2
      x x x x , x x x x , x x x , x x x x , x x x , x x x x , x x x x ,
       2 3 5 6   1 3 5 6   2 5 6   1 2 5 6   2 4 6   2 3 4 6   1 3 4 6
     ------------------------------------------------------------------------
        2               2                   2              2 2      2
      x x x , x x x x , x x x , x x x x , x x x , x x x x , x x , x x x ,
       1 4 6   1 4 5 6   2 4 6   1 2 4 6   1 3 6   1 2 3 6   3 5   2 3 5
     ------------------------------------------------------------------------
         2    2 2      2                       2                 2      2 2
      x x x , x x , x x x , x x x x , x x x x , x x x , x x x x , x x x , x x ,
       1 3 5   2 5   1 2 5   2 3 4 5   1 3 4 5   2 4 5   1 2 4 5   1 4 5   1 5
     ------------------------------------------------------------------------
        2               2      2 2      2   2 2            2      2 2
      x x x , x x x x , x x x , x x , x x x , x x , x x x x , x x x , x x }
       1 3 5   1 2 3 5   1 3 5   2 4   1 2 4   1 4   1 2 3 4   1 3 4   1 3

o8 : List
```

As desired we find a linear quotient ordering of $I(A_6)^2$. Moreover, we note that this ordering is quite different from $O_6^{(2)}$ (see Example 6.1), demonstrating that linear quotient orderings need not be unique.

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