COBORDISM OF DOMES OVER CURVES

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Abstract. An integral curve is a closed piecewise linear curve comprised of unit intervals. A dome is a polyhedral surface whose faces are equilateral triangles and whose boundary is an integral curve. Glazyrin and Pak showed that not every integral curve can be domed by analyzing the case of unit rhombi, and conjectured that every integral curve is cobordant to a unit rhombus. We show that this is false for oriented domes, but that every integral curve is cobordant to the union of finitely many unit rhombi.

1. Introduction

Let γ be a closed piecewise linear curve in three dimensional Euclidean space $\mathbb{E}$. We say that γ is integral if all intervals have integer length. Note that we allow γ to have multiple connected components. Now let S be a piecewise linear surface in $\mathbb{E}$ with boundary γ and whose facets are all unit triangles. Note that S need not be embedded or immersed. We say that S is a *dome over* γ, that γ is *spanned* by S, and that γ *can be domed*.

As early as 2005, Kenyon asked if every integral curve can be domed, see [5]. In 2021, this was shown to be false by Glazyrin and Pak, who proved a necessary condition for a unit rhombus to be domed in [4]. Moreover, Glazyrin and Pak conjectured that this is in some sense the only restriction that prevents a general integral curve from being domed.

Conjecture 1.1 ([4, Conj. 5.14]). *For every integral curve γ, there is a unit rhombus ρ and a dome over $\gamma \cup \rho$.*

Formally, we say that two integral curves γ and η are *cobordant* if there is a dome over $\gamma \cup \eta$. The dome will be called a *cobordism* between γ and η. When the cobordism is orientable, we say that γ and η are *orientably cobordant*. Note that if γ and η are cobordant, then γ can be domed if and only if η can be domed, but the converse is not necessarily true. In this context, Glazyrin and Pak asked if every integral curve is cobordant to a unit rhombus. This is false for orientable cobordisms.

Theorem 1.2. *There exist unit rhombi ρ_1, ρ_2 such that $\rho_1 \cup \rho_2$ is not orientably cobordant to any unit rhombus ρ_3.*

Our proof of this negative result is non-constructive, but shows that the statement is true for almost all pairs of unit rhombi ρ_1, ρ_2. Our work builds upon techniques introduced by Anan'in and Korshunov, who gave a second proof that Kenyon's question is false in [1]. Their proof is also non-constructive, and shows that almost all integral curves cannot be domed. They consider a boundary map from the moduli space of domes to the moduli space of integral curves, and prove that its image has measure zero in the orientable case. Our result extends their work by allowing integral curves with multiple components.

We also prove a weaker version of Conjecture 1.1, posed by Glazyrin and Pak in [4, Conj. 5.15], which allows for finitely many rhombi.

Theorem 1.3. *For every integral curve γ, there is a finite set of unit rhombi $\rho_1, \ldots, \rho_k$ such that γ is cobordant to $\rho_1 \cup \ldots \cup \rho_k$. Moreover, for $|\gamma| = n$, it suffices to take $k = n^2 + 2n - 12$ rhombi.*

Our proof is constructive, and uses *rhombus pivots* to reduce a generic integral curve to a planar integral curve. That is, for consecutive vertices u, v, w in γ, we can replace v by some point $v' \in B_1(u) \cap B_1(w)$ by attaching the rhombus $[uvwv']$. Then we prove the theorem directly for planar integral curves. Our approach is similar to ideas introduced in [4, §2].

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Outline of the paper. We prove Theorem 1.3 in Section 2 because it is mostly self-contained. Then in Section 3 we introduce necessary notions to describe the moduli space of domes and curves. This is a generalization of many of the definitions and results given in [1] to allow for domes to bound integral curves with multiple connected components. Then we prove Theorem 1.2 in Section 4 with these techniques. Final remarks are given in Section 5.

Notation. For a list of points $v_1, \dots, v_n \in \mathbb{E}$, let $[v_1 \dots v_n]$ be the integral curve γ with vertices at the given points. For two points v, w, let (v, w) be the line containing the two points, let $[v, w]$ be the line segment connecting the two points, and let $|v, w|$ be the distance between the two points. For an integral curve γ, let $|\gamma|$ be the sum of all edge lengths of γ. Let $\mathcal{M}_n$ denote the set of all integral curves of length n. The set $\mathcal{M}_4$ is important in the proofs and is called the set of unit rhombi.

2. Every integral curve is cobordant to finitely many unit rhombi

In this section we prove Theorem 1.3, that every integral curve $\gamma \in \mathcal{M}_n$ is cobordant to a finite union of rhombi $\rho_1 \cup \ldots \cup \rho_k$.

We say that two integral curves γ and η are *rhombus equivalent* if there exist finitely many unit rhombi $\rho_1, \dots, \rho_k$ and a dome over $\gamma \cup \eta \cup \rho_1 \cup \ldots \cup \rho_k$. Here we say k is the number of *rhombi used* in the rhombus equivalence. This is similar to the definition of *flip equivalence.* (See e.g. [4, § 2.4].) In our definition, we do not need to assume that the added rhombi can be domed, which is necessary for flip equivalence. Clearly if an integral curve γ is rhombus equivalent to an integral curve which can be domed, then γ satisfies the existence condition of Theorem 1.3.

First, we show that every integral curve is rhombus equivalent to a planar integral curve. An integral curve is *planar* if it lies in a plane $H \subset \mathbb{E}$.

Lemma 2.1. *Every integral curve γ is rhombus equivalent to a planar integral curve. Moreover, for $|\gamma| = n$, it suffices to use $k = \binom{n}{2}$ rhombi.*

Proof. Choose two vertices v and w of γ that are a maximum distance apart, and take any plane H containing the line (v, w). This gives a decomposition of γ as two integral paths containing v and w, and we show that both can be made to lie entirely in H via rhombus equivalence.

Suppose η is an integral path with vertices $v = v_1, \dots, v_m, v_{m+1} = w$. Let h_i denote the distance from v_i to the plane H. (See Figure 1.) Choose v_i maximizing h_i. If the same height is achieved by multiple v_i, choose the one with the smallest index i. Note that this means $h_{i-1} < h_i \geq h_{i+1}$.

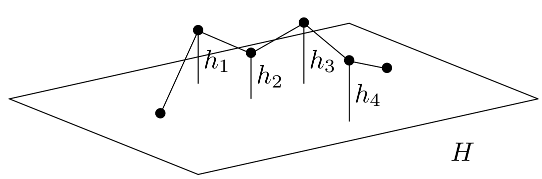

Figure 1. An integral path η above a plane H with heights h_i.

Consider the circle $B_1(v_{i-1}) \cap B_1(v_{i+1})$ which contains v_i. For each point $v_i' \in B_1(v_{i-1}) \cap B_1(v_{i+1})$, we can replace v_i by v_i' in η by rhombus equivalence via $[v_{i-1} v_i v_{i+1} v_i']$. (See Figure 2.) Choose v_i' to minimize h_i', the new distance from v_i' to H. If $h_i' = 0$, then v_i' is in H. Otherwise if $h_i' > 0$, minimizing h_i' corresponds to choosing v_i' so that the slope of the line (v_{i-1}, v_i') is parallel to the slope of the line (v_i, v_{i+1}). Because $h_{i+1} \leq h_i$, we see that $h_i' \leq h_{i-1} < h_i$, so we can always decrease the maximum height h_i by a rhombus equivalence.

Each added rhombus will remove at least one inversion from the sequence $h_1, \dots, h_m$. I.e., $h_{i-1} < h_i$ is replaced with $h_{i-1} \geq h_i'$. Since there can be at most $\binom{m}{2}$ inversions, after at most $\binom{m}{2}$ flips we

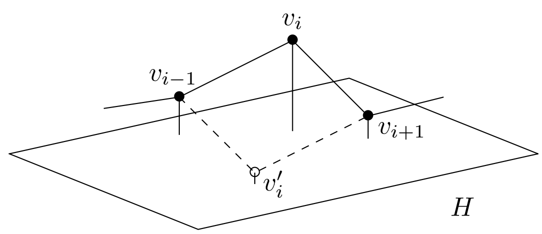

FIGURE 2. Adding a rhombus $[v_{i-1}v_iv_{i+1}v_i']$.

guarantee the integral path η lies in H. Therefore, for the whole integral curve γ, we will need at most $\binom{m}{2} + \binom{n-m}{2} \leq \binom{n}{2}$ flips. □

Now we prove the result of Theorem 1.3 for planar integral curves. Our proof is by induction on $n = |\gamma|$, and divides the integral curves into smaller pieces. We say that an integral curve $\gamma = [v_1 \dots v_n]$ is *ϵ-packing around v_i* if for all j, the length $|v_i, v_j| < \epsilon$. In this case, we say that v_i is the *packing center* of γ, and when the packing center is understood we just say that γ is *ϵ-packing.*

Lemma 2.2. *Every planar integral curve γ is rhombus equivalent to a planar integral curve which is 2-packing. Moreover, if $|\gamma| = n$, then it suffices to use $k = \binom{n}{2}$ rhombi.*

Proof. The existence of a rhombus equivalent 2-packing curve follows from the Steinitz Lemma, which can be stated as follows. (See e.g. [2, Theorem 2.1].) For each dimension $d > 0$, there is a constant B_d such that for unit vectors $u_1, \dots, u_n \in \mathbb{R}^d$ satisfying $u_1 + \dots + u_n = 0$, there is a permutation $\sigma \in S_n$ such that for each $1 \leq i \leq n$,

$$|u_{\sigma(1)} + \dots + u_{\sigma(i)}| \leq B_d$$

For $d = 2$, Bergström showed the optimal value is $B_2 = \sqrt{5}/2$ in [3]. However, we use the constant 2 because it is sufficient for the proof of Theorem 1.3 and gives clearer exposition.

Viewing the integral curve $\gamma = [v_1 \dots v_n]$ as a set of vectors $u_1, \dots, u_n$, with u_i pointing from v_i to v_{i+1}, we can swap a pair of consecutive vectors u_i and u_{i+1}. By adding a rhombus containing the endpoints v_{i-1}, v_i, v_{i+1}, and the point v_i reflected across the line (v_{i-1}, v_{i+1}). This corresponds to the simple transposition $(i, i+1) \in S_n$. And because we can achieve any permutation $\sigma \in S_n$ by the product of simple transpositions, there exists a rhombus equivalent 2-packing planar integral curve around v_1. (See e.g. [6].) Moreover, for every permutation $\sigma \in S_n$, the maximal length of σ in terms of simple transpositions is $\binom{n}{2}$, so this number of rhombi is sufficient. □

Now we are ready to prove that every integral curve is cobordant to finitely many unit rhombi.

Theorem 1.3. *Every integral curve γ is cobordant to $\rho_1 \cup \dots \cup \rho_k$ for a finite set of unit rhombi $\rho_1, \dots, \rho_k \in \mathcal{M}_4$. Moreover, for $|\gamma| = n$, it suffices to take $k = n^2 + 2n - 12$ rhombi.*

Proof. From Lemma 2.1 and Lemma 2.2, we may assume without loss of generality that γ is planar and 2-packing. This uses $n(n-1)$ rhombi. Our proof is by induction. The base case is when $|\gamma| = 5$.

Suppose $\gamma = [v_1 \dots v_5]$. If the circumradius of the triangle $[v_1v_3v_4]$ is less than 1, then there is a point $z \in B_1(v_1) \cap B_1(v_3) \cap B_1(v_4)$. (See Figure 3, left.) Adding the edges $[z, v_1], [z, v_3]$ and $[z, v_4]$ gives a cobodism between γ and two unit rhombi $[v_1v_2v_3z]$ and $[v_1zv_4v_5]$. If the circumradius of the triangle $[v_1v_3v_4]$ is greater than 1, then we can perform a rhombus equivalence to construct a new integral curve γ' with the desired circumradius. Note that at most 3 rhombi are used.

In the inductive step, for a planar, 2-packing integral curve $\gamma = [v_1 \dots v_n]$ with $n > 5$, there is some point $z \in B_1(v_1) \cap B_1(v_4) \cap H$, where H is the plane containing γ. (See Figure 3, right.) Now $[v_1v_2v_3v_4z]$ is a planar pentagon, so the base case applies, and $[v_1zv_4 \dots v_n]$ is a planar, 2-packing integral curve of length $n-1$, so the inductive hypothesis applies. This divides γ into $n-4$ pentagons, each of which use at most 3 rhombi. So in total at most $n(n-1) + 3(n-4) = n^2 + 2n - 12$ rhombi are used. □

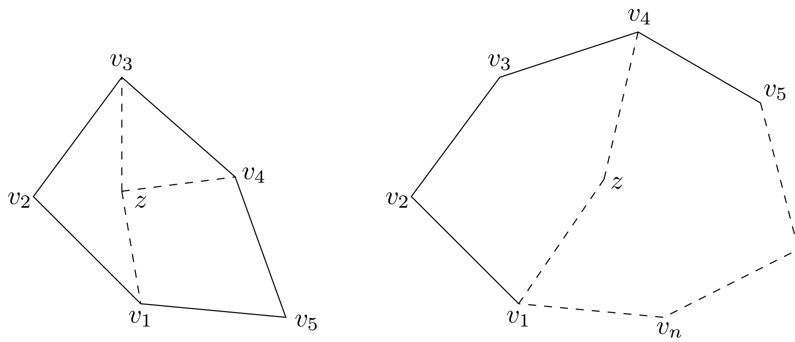

FIGURE 3. The base case and inductive step.

3. Spaces of Polygons and Polyhedra

This section introduces definitions and proves technical results we need to prove Theorem 1.2, that not all integral curves are cobordant to a unit rhombus. Our approach is inspired by techniques introduced by Anan'in and Korshunov, who gave an altenate proof that Kenyon's question is false for almost all integral curves in [1]. Specifically, we generalize Anan'in and Korshunov's results to graph surfaces with multiple boundary components. We state our definitions here. For the original definitions and for a more detailed introduction, see [1, §2, §3].

A *sample polygon* is a finite 1-complex $P = (U, F)$ whose underling space is homeomorphic to S^1. Here U is a set of vertices and $F = \{f_1, \ldots, f_k\}$ is a set of edges oriented and cyclically ordered with respect to the orientation of the circle. F is equipped with an edge length function $\ell : F \to \mathbb{R}_{>0}$ which satisfies a nondegeneracy condition $2\ell(f_i) < \ell(f_1) + \ldots + \ell(f_n)$ for all $1 \leq i \leq n$.

For a sample polygon P, the *space of polygons* $\mathbb{E}^P$ is the set of all continuous maps $P \to \mathbb{E}$ that are isometries on edges. That is, $\mathbb{E}^P$ is the set of all realizations of P is Euclidean space. Let $\mathrm{Isom}^+\mathbb{E}$ be the group of orientation preserving isometries of $\mathbb{E}$. The *moduli space of polygons* $\mathbb{E}^P/\mathrm{Isom}^+\mathbb{E}$ is the set of all realizations up to rigid motions. Let $\mathbb{E}$ be the subgroup of $\mathrm{Isom}^+\mathbb{E}$ of translations, and $SO(3)$ be the subgroup of rotations. Note that $\mathrm{Isom}^+\mathbb{E} \cong \mathbb{E} \rtimes SO(3)$. The *scheme of polygons* $\mathbb{E}^P/\mathbb{E}$ is the set of all realizations up to translations, but not rotations. Ultimately, we need to consider $\mathbb{E}^P/\mathrm{Isom}^+\mathbb{E}$, but in our proof we first consider $\mathbb{E}^P/\mathbb{E}$ and then quotient by the action of $SO(3)$.

For a vector of polygons $\mathbf{P} = (P_1, \ldots, P_n)$, let $\mathbb{E}^{\mathbf{P}}$ refer to the product $\mathbb{E}^{P_1} \times \ldots \times \mathbb{E}^{P_n}$. Similarly, let $\mathbb{E}^{\mathrm{P}}/\mathrm{Isom}^+\mathbb{E}$ and $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$ refer to the products $\mathbb{E}^{P_1}/\mathrm{Isom}^+\mathbb{E} \times \ldots \times \mathbb{E}^{P_n}/\mathrm{Isom}^+\mathbb{E}$ and $\mathbb{E}^{P_1}/E \times \ldots \times \mathbb{E}^{P_n}/\mathbb{E}$, respectively. As an abuse of terminology, we still refer to these as the *space of polygons*, *moduli space of polygons* and *scheme of polygons* respectively. Note that the group of isometries always acts on each sample polygon separately, see § 5.1. To refer to the polygons explicity, we may write $\mathbb{E}^{P_1,\ldots,P_n}$ rather than $\mathbb{E}^{\mathbf{P}}$. (See Lemma 4.2.) We refer to the sets of edges as $F_1, \ldots, F_n$, individual edges as $f^i_j \in F_i$, and the edge length functions as $\ell_i : F_i \to \mathbb{R}_{>0}$.

As the name suggests, $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$ is indeed a scheme. Note, however, that it may contain singular points. For a single polygon, $p \in \mathbb{E}^P/\mathbb{E}$ is singular if and only if all $p(f_i)$ are parallel. For multiple polygons, $\mathbf{p} \in \mathbb{E}^{\mathbf{P}}/\mathbb{E}$ is singular if and only if some $p_i \in \mathbb{E}^{P_i}/\mathbb{E}$ is singular.

Lemma 3.1. *Let $\boldsymbol{p} \in \mathbb{E}^{\boldsymbol{P}}/\mathbb{E}$ be a singular point. Then $\dim T_{\boldsymbol{p}}(\mathbb{E}^{\boldsymbol{P}}/\mathbb{E}) = m + \dim(\mathbb{E}^{\boldsymbol{P}}/\mathbb{E})$, where m is the number of p_i which are singular in $\mathbb{E}^{P_i}/\mathbb{E}$.*

Proof. Since $\mathbb{E}^{\mathbf{P}}/\mathbb{E} = \mathbb{E}^{P_1}/\mathbb{E} \times \ldots \times \mathbb{E}^{P_n}/\mathbb{E}$, we have

$$\dim T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E}) = \dim T_{p_1}(\mathbb{E}^{P_1}/\mathbb{E}) + \ldots + \dim T_{p_n}(\mathbb{E}^{P_n}/\mathbb{E}).$$

Now for a single P_i, we have $\dim T_{p_i}(\mathbb{E}^{P_i}/\mathbb{E}) = 1 + \dim(\mathbb{E}^{P_i}/\mathbb{E})$ from [1, Lemma 3.2]. Combining this for all P_i in $\mathbf{P}$ gives us the result. □

Consider the following explicit description of the scheme of polygons $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$ for a list of polygons $\mathbf{P}$. A point $\mathbf{p} \in \mathbb{E}^{\mathbf{P}}/\mathbb{E}$ is a collection of maps $p_i : F_i \to \mathbb{E}$ which satisfy $\langle p_i(f^i_j), p_i(f^i_j)\rangle = \ell_i(f^i_j)^2$ for all i, j and $\sum_j p_i(f^i_j) = 0$ for all i. Similarly, for $\mathbf{p} \in \mathbb{E}^{\mathbf{P}}/\mathbb{E}$, a point $\mathbf{t}$ in the tangent space $T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})$ is a collection of maps $t_i : F_i \to \mathbb{E}$ satisfying $\langle t_i(f^i_j), p_i(f^i_j)\rangle = 0$ for all i, j and $\sum_j t_i(f^i_j) = 0$ for all i.

We generalize the definition of a graph surface given in [1] to allow for multiple boundary components as follows. A *genus n graph surface* S is a finite 2-dimensional simplicial complex with nondegenerate triangles and edges contained in a closed surface $\widehat{S}$ such that the complement $\mathcal{D} := \widehat{S} \setminus S$ is homeomorphic to the disjoint union of n disks. This has the following data: a set of vertices V, a set of edges E with orientation, a set of triangles $\mathcal{T}$, and a map $\Phi : E \to E$ defined as $\Phi : e \mapsto -e$ which reverses orientation.

A graph surface of genus n has boundary $\partial\mathcal{D} = G_1 \cup \ldots \cup G_n$, where each G_i is a *boundary component* homeomorphic to S^1. Each boundary component G_i can be decomposed into the union $g^i_1 \cup \ldots \cup g^i_{k_i}$ with a cyclic order of the edges $g^i_1, \ldots, g^i_{k_i} \in E$ such that g^i_j and g^i_{j+1} are consecutive for all i, j. Note that the list $g^i_1, \ldots, g^i_{k_i}$ admits repetitions with the same or opposite orientation. As an important special case, note that when a graph surface $S \subset \widehat{S}$ is contained in an orientable surface and contains no triangles, then every edge appears exactly twice, once with each orientation, in the boundary $\partial\mathcal{D}$. See [1, Remark 2.5], and note that each boundary component can be given the same orientation with respect to the oriented surface $\widehat{S}$.

Similarly, we generalize the definition of a *sample polyhedron* to allow for a genus n graph surface instead of just a genus 1 graph surface as in [1, Definition 2.1]. For us, a sample polyhedron will mean a genus n graph surface S equipped with an edge length function $\ell : \mathbb{E} \to \mathbb{R}_{>0}$ satisfying $\ell(e) = \ell(-e)$ for any edge $e \in E$ and $\ell(e_1) + \ell(e_2) > \ell(e_3)$ for any triangle $T \in \mathcal{T}$ with boundary $\partial T = e_1 + e_2 + e_3$. For a given sample polyhedron S, we also consider the *space of polyhedra* $\mathbb{E}^S$, the *moduli space of polyhedra* $\mathbb{E}^S/\mathrm{Isom}^+\mathbb{E}$ and the *scheme of polyhedra* $\mathbb{E}^S/\mathbb{E}$.

An explicit description for the scheme of polyhedra $\mathbb{E}^S/\mathbb{E}$ is given by the set of continuous maps $q : E \to \mathbb{E}$ satisfying

(i) $\langle q(e), q(e)\rangle = \ell(e)^2$ for all $e \in E$,
(ii) $q(-e) = -q(e)$ for all $e \in E$,
(iii) $q(e_1) + q(e_2) + q(e_3) = 0$ for all triangles $T \in \mathcal{T}$ with boundary $\partial T = e_1 + e_2 + e_3$,
(iv) for a set of generators $H \subset H_1(S, \mathbb{Z})$, $\displaystyle\sum_{e \in E} h_e q(e) = 0$ for any generator $\displaystyle\sum_{e \in E} h_e e \in H$.

Similarly, an explicit description for the tangent space $T_q(\mathbb{E}^S/\mathbb{E})$ is given by the set of continuous maps $s : E \to \mathbb{E}$ satisfying

(i') $\langle s(e), q(e)\rangle = 0$ for all $e \in E$,

and the previous conditions (ii) - (iv). See [1, §§2.7-9].

There is a boundary map which connects sample polyhedra and sample polygons. Let S be a sample polyhedron, with boundary components $G_1, \ldots, G_n$ where $G_i = g^i_1 \cup \ldots \cup g^i_{k_i}$ in cyclic order. For each i, j, define an oriented edge f^i_j of length $\ell(g^i_j)$ and glue the edges $f^i_1, \ldots, f^i_{k_i}$ into a sample polygon P_i. We call the polygons $\mathbf{P}$ the *boundary polygons* of S. This defines a map $\delta : F_1 \cup \ldots \cup F_n \to E$ via $\delta : f^i_j \mapsto g^i_j$, and extends to a continuous map $\overline{\delta} : P_1 \cup \ldots \cup P_n \to S$. Thus $\overline{\delta}(f^i_j) = g^i_j$ for all i, j and $\overline{\delta}$ is an isometry on edges. Now $\overline{\delta}(P_i) = G_i$, and $\overline{\delta}(P_1 \cup \ldots \cup P_n) = \partial\mathcal{D}$. We call the map $\overline{\delta}$, or just δ, the *combinatorial boundary map* of the sample polyhedron S.

The boundary map induces a continuous map on schemes $\delta : \mathbb{E}^S/\mathbb{E} \to \mathbb{E}^{\mathbf{P}}/\mathbb{E}$ and a derivative map on the tangent spaces $d\delta : T_q(\mathbb{E}^S/\mathbb{E}) \to T_{q\circ\delta}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})$ for any point $q : E \to \mathbb{E}$ in the scheme of polyhedra

$\mathbb{E}^S/\mathbb{E}$. These maps are from precomposition with δ, see [1, §2.10].

Lastly, we define the collapse of a genus n graph surface S. Let $T \in \mathcal{T}$ be a triangle whose boundary ∂T contains a boundary edge g^i_j. We may collapse S at T. The resulting simplicial complex $S' \subset \widehat{S}$ has the same vertices $V' = V$, one less pair of oriented edges $E' = E \smallsetminus \{g^i_j, -g^i_j\}$, and one less triangle $\mathcal{T}' = \mathcal{T} \smallsetminus \{T\}$. Note that S' is again a genus n graph surface. The boundary component G_i changes from $g^i_1 \cup \ldots \cup g^i_j \cup \ldots \cup g^i_{k_i}$ to $g^i_1 \cup \ldots \cup -e' \cup -e \cup \ldots \cup g^i_{k_i}$, where $\partial T = g^i_j + e + e'$. All other boundary components are unchanged. In terms of a realization $q : E \to \mathbb{E}$ in the scheme of polyhedra $\mathbb{E}^S/\mathbb{E}$, collapse corresponds to restriction.

Lemma 3.2. *Let S' be a sample polyhedra obtained from S by collapse of a triangle, and let $q : E \to \mathbb{E}$ be a realization in the scheme of polyhedra $\mathbb{E}^S/\mathbb{E}$. Denote by $q' = q|_{E'} : E' \to E$ the restriction of q to E'. Then $q' \in \mathbb{E}^{S'}/\mathbb{E}$. Similarly, for $s : E \to \mathbb{E}$ in $T_q(\mathbb{E}^S/\mathbb{E})$, denote by $s' = s|_{E'} : E' \to \mathbb{E}$ the restriction of s to E'. Then $s' \in T_{q'}(\mathbb{E}^{S'}/\mathbb{E})$.*

Proof. This is exactly the statement of [1, Proposition 4.2 (i)]. The same proof follows exactly when we allow for a genus n graph surface. □

Consider the group $SO(3)^{\times n}$ acting on $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$, where each copy of $SO(3)$ acts on the corresponding scheme of polygons $\mathbb{E}^{P_i}/\mathbb{E}$. For $\mathbf{p} \in \mathbb{E}^{\mathbf{P}}/\mathbb{E}$, we need to describe the tangent space $T_{\mathbf{p}}SO(3)^{\times n}\mathbf{p}$. Recall the Lie algebra $\mathfrak{so}_3$ of the Lie group $SO(3)$ with the following description.

$$\mathfrak{so}_3 = \{a \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{E}, \mathbb{E}) \mid \langle a(e), e' \rangle + \langle e, a(e') \rangle = 0 \ \forall e, e' \in E\}.$$

Let $\mathfrak{so}_3^{\times n}$ be the set of vectors $\mathbf{a} = (a_1, \ldots, a_n)$, where each $a_i \in \mathfrak{so}_3$. We have the following description of the tangent space to the $SO(3)^{\times n}$ orbit on $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$.

Lemma 3.3. *Let $\boldsymbol{p} \in \mathbb{E}^{\boldsymbol{P}}/\mathbb{E}$. The tangent space $T_{\boldsymbol{p}}SO(3)^{\times n}\boldsymbol{p}$ to the $SO(3)^{\times n}$-orbit of $\boldsymbol{p}$ is given by*

$$\begin{aligned} T_{\boldsymbol{p}}SO(3)^{\times n}\boldsymbol{p} &= T_{p_1}SO(3)p_1 \times \ldots \times T_{p_n}SO(3)p_n \\ &= \{\boldsymbol{a} \circ \boldsymbol{p} = \{a_i \circ p_i : F_i \to \mathbb{E}\} \mid \boldsymbol{a} \in \mathfrak{so}_3^{\times n}\}. \end{aligned}$$

Proof. The first line is immediate because each copy of $SO(3)$ acts independently on each copy of $\mathbb{E}^{P_i}/\mathbb{E}$. Then for a single $SO(3)$ acting on $p_i \in \mathbb{E}^{P_i}/\mathbb{E}$, we know $T_{p_i}SO(3)p_i = \{a_i \circ p_i : F_i \to \mathbb{E} \mid a_i \in \mathfrak{so}_3\}$ from [1, §3.1]. □

The final ingredient is a symplectic form on the scheme of polygons $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$. Consider the skew symmetric form ω given by the formula

$$\omega_{\mathbf{p}}(\mathbf{t}, \mathbf{t'}) := \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{t_i(f^i_j) \wedge t'_i(f^i_j) \wedge p_i(f^i_j)}{\ell_i(f^i_j)^2 \nu} = \sum_{i=1}^{n} \omega_{p_i}(t_i, t'_i).$$

Here ν is the volume form on $\mathbb{E}$, and $\mathbf{t}, \mathbf{t'} \in T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})$. ω_{p_i} refers to the sum containing all components of P_i, and corresponds to [1, §3.3, Formula V]. We show that the kernel of this form is exactly the tangent space to the $SO(3)^{\times n}$ orbit of a point $\mathbf{p} \in \mathbb{E}^{\mathbf{P}}/\mathbb{E}$. Note that by kernel we are considering $\omega_{\mathbf{p}}$ as a map $T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E}) \to T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})^*$. That is, the kernel is the set of points $\mathbf{t} \in T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})$ such that for all $\mathbf{t'} \in T_{\mathbf{p}}(\mathbb{E}^{\mathbf{P}}/\mathbb{E})$, we have $\omega_{\mathbf{p}}(\mathbf{t}, \mathbf{t'}) = 0$.

Lemma 3.4. *The tangent space $T_{\boldsymbol{p}}(SO(3)^{\times n}\boldsymbol{p})$ to the $SO(3)^{\times n}$ orbit of any point $\boldsymbol{p} \in \mathbb{E}^{\boldsymbol{P}}/\mathbb{E}$ coincides with the kernel of the form $\omega_{\boldsymbol{p}}$ on $T_{\boldsymbol{p}}(\mathbb{E}^{\boldsymbol{P}}/\mathbb{E})$.*

Proof. For a single $p_i \in \mathbb{E}^{P_i}/\mathbb{E}$, we know the kernel of the form ω_{p_i} on $T_{p_i}(\mathbb{E}^{P_i}/\mathbb{E})$ corresponds to $T_{p_i}SO(3)p_i$ from [1, Lemma 3.4]. This immediately shows that $T_{\mathbf{p}}SO(3)^{\times n}\mathbf{p} \subset \ker \omega_{\mathbf{p}}$, as each term in the sum will be 0. For the reverse inclusion, we repeat the dimension counting argument from [1, Lemma 3.4] with our Lemma 3.1 in place of Lemma [1, Lemma 3.2]. This shows the spaces have the same dimension, so they are equal. □

4. Not Every Integral Curve is Cobordant to a Unit Rhombus

This section proves that not every integral curve is orientably cobordant to a unit rhombus. Our method is to generalize [1, Theorem 4.3] to allow for graph surfaces of arbitrary genus. We rephrase a cobordism as a sample polyhedron with multiple boundary components, and project down to smaller moduli space to account for any possible polygon to be chosen for the cobordism.

Theorem 4.1. *Let $S \subset \widehat{S}$ be a genus n graph surface, with boundary polygons $\boldsymbol{P}$, where $\widehat{S}$ is a closed orientable surface. Then*

$$\delta : \mathbb{E}^S/Isom^+\mathbb{E} \to \mathbb{E}^{\boldsymbol{P}}/Isom^+\mathbb{E},$$

is isotropic. In particular, the rank of $d\delta$ is at most half the dimension of the moduli space of polygons.

Note that the skew symmetric form on $\mathbb{E}^{\mathbf{P}}/\text{Isom}^+\mathbb{E}$ is induced by the skew symmetric form ω on $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$ after taking the quotient by the action of $SO(3)^{\times n}$. We show that the pullback of this skew symmetric form ω' is null. Also note that the orientability hypothesis is necessary here, see § 5.3

Proof. The proof of [1, Theorem 4.3] follows word for word to show that $\delta : \mathbb{E}^S/\mathbb{E} \to \mathbb{E}^{\mathbf{P}}/\mathbb{E}$ is isotropic by a series of collapses. To show that this survives to quotient by $SO(3)^{\times n}$, we use Lemma 3.4. Because $T_{\mathbf{p}}SO(3)^{\times n}\mathbf{p} = \ker \omega_{\mathbf{p}}$, taking the quotient by $SO(3)^{\times n}$ on $\mathbb{E}^{\mathbf{P}}/\mathbb{E}$ gives a new nondegenerate skew-symmetric form ω' on $\mathbb{E}^{\mathbf{P}}/\text{Isom}^+\mathbb{E}$, and the pullback of the form to $\mathbb{E}^S/\text{Isom}^+\mathbb{E}$ is still null. □

Now we can phrase the condition of being cobordant in terms of a sample polyhedron with multiple boundary components.

Lemma 4.2. *Let $S \subset \widehat{S}$ be a sample polyhedron with boundary polygons $P_1, P_2, P_3 \in \mathcal{M}_n$, where $\widehat{S}$ is an orientable closed surface. Let π be the projection from $\mathbb{E}^{P_1,P_2,P_3}/Isom^+\mathbb{E} \to \mathbb{E}^{P_1,P_2}/Isom^+\mathbb{E}$. Then $\pi \circ \delta(\mathbb{E}^S/Isom^+\mathbb{E}) \subset \mathbb{E}^{P_1,P_2}/Isom^+\mathbb{E}$ has measure 0.*

Proof. Take the reduction of the scheme $\mathbb{E}^S/\text{Isom}^+\mathbb{E}$ because we are only interested in the set theoretical image of the space of polyhedra up to sets of measure 0. Consider its smooth locus which is open and of full measure. Now δ is a smooth map of smooth manifolds, and by Theorem 4.1 the rank of its differential is at most half the dimension of the target manifold. Let $\dim \mathbb{E}^{P_i}/\text{Isom}^+\mathbb{E} = m$. Then $\dim \mathbb{E}^{P_1,P_2,P_3}/\text{Isom}^+\mathbb{E} = 3m$, so $d\delta$ has rank $\leq \frac{3}{2}m$. But $\dim \mathbb{E}^{P_1,P_2}/\text{Isom}^+\mathbb{E} = 2m$, and so the map $d\pi \circ d\delta$ has rank $\leq \frac{3}{2}m < 2m$. By Sard's theorem, the image $\pi \circ \delta(\mathbb{E}^S/\text{Isom}^+\mathbb{E})$ has measure 0 in $\mathbb{E}^{P_1,P_2}/\text{Isom}^+\mathbb{E}$. □

We prove the main result of the section by showing that in general we cannot find a cobordism from two unit rhombi down to one unit rhombus.

Theorem 1.2. *The set of unit rhombi $\rho_1, \rho_2 \in \mathcal{M}_4$ such that $\rho_1 \cup \rho_2$ is orientably cobordant to a third unit rhombus $\rho_3 \in \mathcal{M}_4$ has measure 0.*

Proof. Any dome that forms a cobordism between two unit rhombi $\rho_1, \rho_2 \in \mathcal{M}_4$ and another unit rhombus $\rho_3 \in \mathcal{M}_4$ is exactly a genus 3 graph surface S whose boundary $\delta(\mathbb{E}^S/\text{Isom}^+\mathbb{E}) \subset \mathbb{E}^{\rho_1,\rho_2,\rho_3}/\text{Isom}^+\mathbb{E}$. Since we can take any unit rhombus ρ_3, we care about the projection $\pi\circ\delta(\mathbb{E}^S/\text{Isom}^+\mathbb{E}) \subset \mathbb{E}^{\rho_1,\rho_2}/\text{Isom}^+\mathbb{E}$. This has measure 0 from Lemma 4.2, and there are only countably many domes, so we conclude that the set of pairs of rhombi $\rho_1, \rho_2 \in \mathcal{M}_4$ such that $\rho_1 \cup \rho_2$ is cobordant to a third unit rhombus $\rho_3 \in \mathcal{M}_4$ has measure 0. □

Originally, Glazyrin and Pak only considered integral curves with one connected component. So Theorem 1.2 is not an answer to Conjecture 1.1 in the orientable case, but we give one in the following corollary.

Corollary 4.3. *There exists an integral curve γ with one connected component that is not orientably coboardant to any unit rhombus ρ.*

Proof. Choose a pair of unit rhombi $\rho, \rho' \in \mathcal{M}_4$ such that $\rho \cup \rho'$ is not cobordant to any third unit rhombus $\rho'' \in \mathcal{M}_4$. Let $\rho = [v_1v_2v_3v_4]$ and $\rho' = [v_1'v_2'v_3'v_4']$. Position ρ and ρ' so that $v_1 = v_1'$, and the distances $|v_2, v_2'| = |v_4, v_4'| = 1$. By adding the unit triangles $[v_1v_2v_2']$ and $[v_1v_4v_4']$, we see that $\rho \cup \rho'$ is coboardant to the perimeter $\gamma = [v_4v_3v_2v_2'v_3'v_4']$. Therefore, our choice of ρ, ρ' implies that γ is not cobordant to any unit rhombus, and γ has only one connected component. □

Glazyrin and Pak conjectured that a positive answer to Conjecture 1.1 would involve showing a cobordism between an integral curve and finitely many unit rhombi, and then a cobordism between two unit rhombi and one unit rhombus, see [4, Conj. 5.15, Conj. 5.16]. Thus Theorem 1.2 is a negative answer to [4, Conj. 5.16] and Corollary 4.3 is a negative answer to Conjecture 1.1, both in the orientable case.

5. Final remarks

5.1. The proof of Theorem 1.2 does not account for the fact that the unit rhombi in question can be oriented in some manner with respect to each other. That is, we have chosen to define $\mathbb{E}^{\mathbf{P}}/\mathrm{Isom}^+\mathbb{E}$ with the group of orientation preserving isometries $\mathrm{Isom}^+\mathbb{E}$ acting on each polygon separately. However, we could consider $\mathrm{Isom}^+\mathbb{E}$ as acting on all of the spaces together, so instead of $\mathbb{E}^{P_1}/\mathrm{Isom}^+\mathbb{E} \times \ldots \times \mathbb{E}^{P_n}/\mathrm{Isom}^+\mathbb{E}$ we would consider $\mathbb{E}^{P_1}/\mathrm{Isom}^+\mathbb{E} \times \mathbb{E}^{P_2} \times \ldots \times \mathbb{E}^{P_n}$. This approach could answer more general questions. For example, Glazyrin and Pak conjectured that there are two unit triangles $\Delta_1, \Delta_2 \subset \mathbb{E}$ which are not cobordant, see [4, Conjecture 5.13]. In this question, the relative translation and rotation of the triangles is important, so our current techniques are insufficient.

Additionally, consider Steinhaus' 1957 problem on tetrahedral chains, see [7]. A tetrahedral chain is a polyhedra constructed by attaching regular tetrahedra along faces to form a chain. These can be viewed as cobordisms between two triangles that satisfy stricter conditions than general domes. Steinhaus asked if tetrahedral chains can be closed, and if they are dense in $\mathbb{E}$. In contrast to general domes, the first question was shown to be false by Swierczkowski in [9]. However, the second question remains open. Recently, Stewart showed in [8] that the group generated by reflections across the faces of a regular tetrahedra is dense in $SO(3)$, but this does not resolve Steinhaus' question because it does not consider the translations in the full group $SO(3) \rtimes \mathbb{E}$. I.e., Stewart's approach considers only the relative rotation, not the relative translation of the two triangles.

5.2. The minimal number k of unit rhombi needed for the cobordism in Theorem 1.3 gives a measure of complexity for an integral curve γ. In contrast to Conjecture 1.1 which proposed that $k = 1$ for all γ, we conjecture that $k = \Theta(|\gamma|)$. Theorem 1.3 gives a quadratic upper bound $k = O(|\gamma|^2)$. We do not believe this is optimal. Pak proposes a modified proof of Theorem 1.3 which uses a reduction to generic integral curves rather than planar integral curves to improve to a linear bound.[1] (See [4, §2] for terminology.) We also conjecture that the construction in Corollary 4.3 can be extended to arbitrarily many rhombi to prove that $k = \Omega(|\gamma|)$.

5.3. The orientability hypothesis in Theorem 4.1 is necessary for the cancellation argument, see [1, Remark 2.3] for more detail, and [1, Remark 4.4] for a counterexample with a non-orientable surface. This limits the techniques used in this paper to the orientable case, and new techniques may be necessary to study non-orientable cobordisms.

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