MATH 3C (Spring 2008)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Discussion section in which you are enrolled (Day, time, name of TA, 1A ... 2F):

Using a pen, provide the information asked above, and also write your name on the top of each page of this exam. When the instructions to a question ask you to explain your answer, you should show your work and explain what you are doing carefully; this is then more important than just finding the right answer. Please, write clearly and make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. To cancel anything from your solution, erase it or cross it out. You are not allowed to sit close to students with whom you have studied for this exam, or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7
Score							

Question	8	9	10	11	12	13	14	Total
Score								

1) (10 points) In a group of 11 students, 7 are freshmen and 4 are sophomores. In how many ways can the students in this group form a waiting line, if the only restriction is that the first two in the line should be sophomores? (No need to compute factorials, powers, permutations and combinations.) (No need to show and explain work.)

$$\underbrace{\underset{1st}{\underline{4}} \times \underset{2nd}{\underline{3}}}_{\text{sophomores}} \times \underbrace{\underset{3rd}{\underline{9}} \times \underset{4th}{\underline{8}} \times \underset{5th}{\underline{7}} \times \underset{6th}{\underline{6}} \times \underset{7th}{\underline{5}} \times \underset{8th}{\underline{4}} \times \underset{9th}{\underline{3}} \times \underset{10th}{\underline{2}} \times \underset{11th}{\underline{1}}}_{\text{no restriction}}$$

Answer : $4 \times 3 \times 9! = \boxed{12 \times 9!}$

2) (10 points) A group of 30 people includes Alice and Bob. A team of 3 people and a team of 5 people are selected randomly from this group of 30 people (no one can be selected for both teams). What is the probability that neither Alice nor Bob will [redacted] be chosen for either team? (No need to compute factorials, powers, permutations and combinations.) (No need to show and explain work.)

1st Solution

$$P(\text{neither Alice nor Bob}) = \frac{\binom{28}{3}\binom{25}{5}}{\binom{30}{3}\binom{27}{5}}$$

2nd Solution

$$P(\text{neither Alice nor Bob}) = \frac{\binom{28}{8}}{\binom{30}{8}}$$

Here we are just choosing the 8 people for the two teams.

Note: Simplifying one gets

$$P(\text{neither Alice nor Bob}) = \frac{22 \times 21}{30 \times 29} = \frac{154}{290}$$

3) (10 points) A die is rolled 3 times. Consider the events:

A : The faces shown are all distinct from each other.

B : The faces shown are identical to each other.

Compute $P(A|B^c)$. (Just to be sure: note that, for instance, the outcome 2,2,5 is neither in A nor B.) (Give answer as a fraction or in decimal form.) (No need to show and explain work.)

$$P(A \mid B^c) = \frac{P(AB^c)}{P(B^c)} = \frac{P(A)}{P(B^c)}$$

$$= \frac{P(A)}{1 - P(B)} = \frac{\frac{6\times 5\times 4}{6^3}}{1 - \frac{6}{6^3}}$$

$$= \frac{6\times 5\times 4}{6^3 - 6} = \frac{5\times 4}{6^2 - 1} = \frac{20}{35} = \boxed{\frac{4}{7}}$$

$$= \boxed{.5714}$$

4) (10 points) Roll a die twice. Let S be the sum of the faces shown and D be the difference in absolute value between the faces shown. (For instance, if the faces shown are 2 and 5, then $S = 2 + 5 = 7$, and $D = |2 - 5| = 3$.) Compute the conditional probability

$$P(S = 5|D = 1).$$

(Provide numerical answer as a fraction or in decimal form.) (Show your work.)

	1	2	3	4	5	6
1		X		O		
2	X		X			
3		X		X		
4	O		X		X	
5				X		X
6					X	

X $\{D = 1\}$ O $\{S = 5\}$

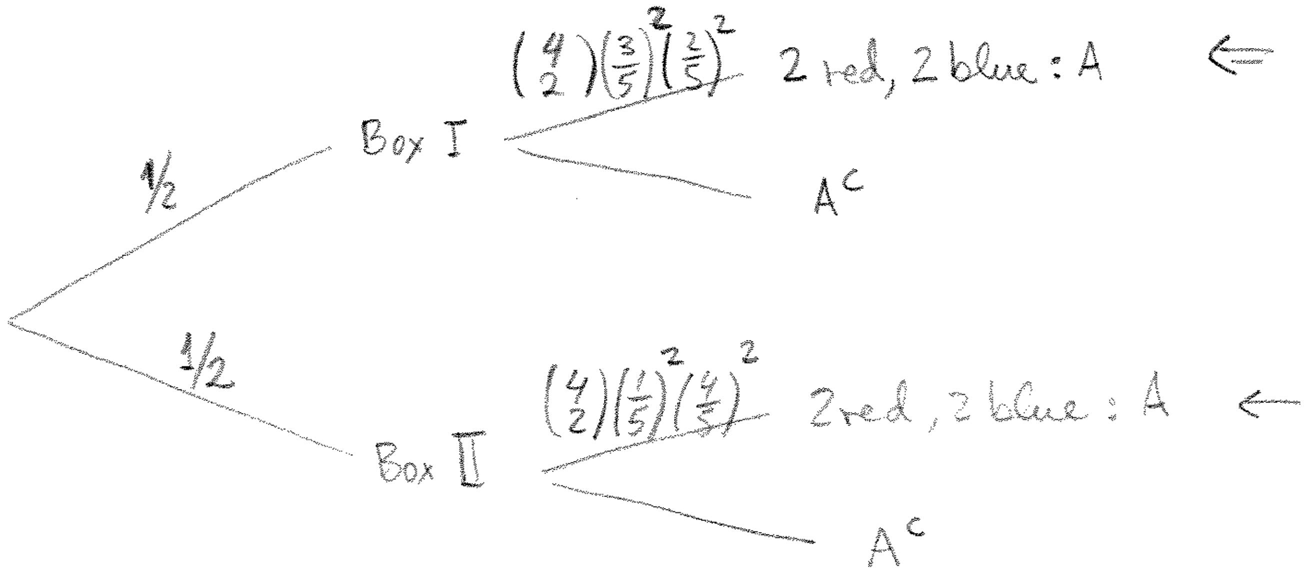
Answer: $\boxed{\frac{2}{10} = \frac{1}{5} = 0.2}$

5) (10 points) A regular deck of cards is missing the ace of spades, so that it has only 51 cards. If 5 cards are taken at random without replacement from this deck, what is the probability that they include three cards of one value and two cards of another value (a hand called "full house", for instance 888JJ, or AAA77)? (No need to compute factorials, powers, permutations and combinations.) (No need to show and explain work.)

no Ace — AAA ?? — ??? AA

$$\frac{12 \times 11 \times \binom{4}{3} \times \binom{4}{2} + 12 \times \binom{3}{3} \times \binom{4}{2} + 12 \times \binom{4}{3} \times \binom{3}{2}}{\binom{51}{5}}$$

6) (10 points) We have 2 boxes. Box I contains 300 red balls and 200 blue balls. Box II contains 100 red ball and 400 blue balls. One box is selected at random. Then 4 balls are taken, with replacement, from the selected box. Compute the conditional probability that the box selected was Box I, given that among the selected balls exactly 2 were red. (Give answer as a fraction or in decimal form.) (Show your work.)



$$P(\text{Box I} \mid A) = \frac{P(\text{Box I}, A)}{P(A)}$$

$$= \frac{\frac{1}{2} \times \binom{4}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2}{\frac{1}{2} \times \binom{4}{2}\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right)^2 + \frac{1}{2} \times \binom{4}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2}$$

$$= \frac{3^2 \times 2^2}{3^2 \times 2^2 + 1^2 \times 4^2} = \frac{9 \times 4}{9 \times 4 + 16} = \frac{9}{9+4} = \boxed{\frac{9}{13} \approx 0.692}$$

7) (10 points) Eggs are sold in boxes that contain 12 eggs each. A supermaket has 100 of these boxes. Suppose that each egg is contaminated with a bacteria, independently of the other eggs, with probability 0.01. Compute the expected number of boxes that contain at least one contaminated egg. (Give answer as a fraction or in decimal form.) (Show your work.)

$X =$ number of boxes with at least one contaminated egg.

$X \sim B(100, p)$

$p = 1 - (1 - 0.01)^{12} = 1 - (.99)^{12} \cong 1 - 0.8864$

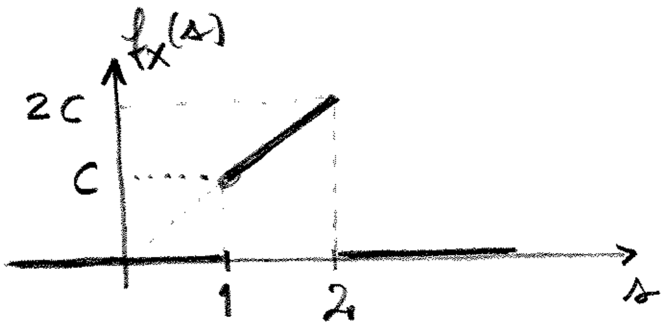
$= 0.1136$

$E(X) = 100p \cong \boxed{11.36}$

8) (10 points) A continuous random variable X has probability density function given by

$$f_X(s) = \begin{cases} 0 & \text{if } s < 1, \\ Cs & \text{if } 1 \le s < 2, \\ 0 & \text{if } s \ge 2. \end{cases}$$

Compute the value of C, the mean μ, and the standard deviation σ of X. (Give answers as fractions or in decimal form.) (Show your work.)



$C = ?$ $\quad \int_1^2 Cs\,ds = 1 \Rightarrow C \frac{s^2}{2}\Big]_1^2 = 1 \Rightarrow \frac{3C}{2} = 1$

$\Rightarrow C = \boxed{\frac{2}{3} = 0.6667}$

$\mu = \int_1^2 s \cdot \frac{2}{3} s\,ds = \frac{2}{3}\frac{s^3}{3}\Big]_1^2 = \boxed{\frac{2\times 7}{9} = \frac{14}{9} = 1.556}$

$E(X^2) = \int_1^2 s^2 \frac{2}{3} s\,ds = \frac{2}{3}\frac{s^4}{4}\Big]_1^2 = \frac{2\times 15}{3\times 4} = \frac{5}{2} = 2.5$

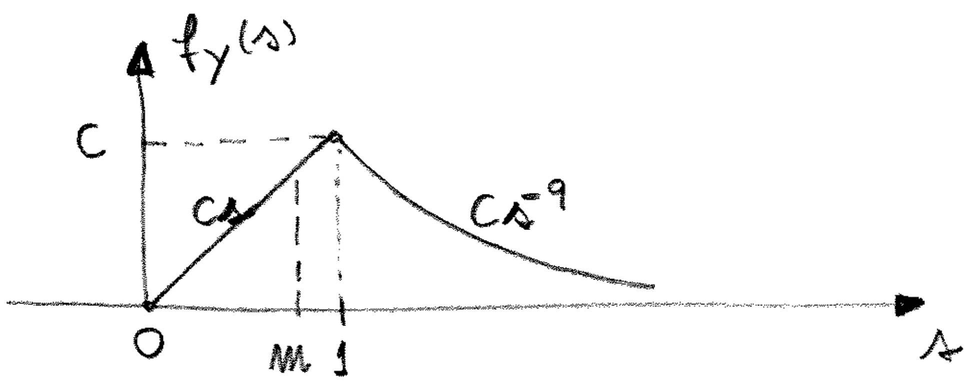
$\mathrm{Var}(X) = E(X^2) - (EX)^2 = 2.5 - (1.556)^2 = 0.07886$

$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{0.07886} = \boxed{0.2808}$

9) (10 points) A continuous random variable Y has probability density function given by

$$f_Y(s) = \begin{cases} 0 & \text{if } s < 0, \\ Cs & \text{if } 0 \le s < 1, \\ Cs^{-9} & \text{if } s \ge 1. \end{cases}$$

Compute the value of C, and find the median m of Y, i.e., find the value of m that makes $P(Y < m) = 1/2$. (Give answers as fractions or in decimal form.) (Show your work.)



$$C = ? \qquad \int_0^1 Cs\,ds + \int_1^\infty Cs^{-9}\,ds = 1$$

$$\Rightarrow \quad C\, s^2/2 \Big]_0^1 + C \frac{s^{-8}}{-8}\Big]_1^\infty = 1$$

$$\Rightarrow \quad C \cdot \frac{1}{2} + C \cdot \frac{1}{8} = 1 \Rightarrow 5C = 8 \Rightarrow \boxed{C = \frac{8}{5} = 1.6}$$

Want m s.t. $P(Y < m) = 1/2$. Note $P(Y<1) = C/2 = .8 > 1/2$

So $m \in [0,1]$: $\displaystyle\int_0^m Cs\,ds = \frac{1}{2} \Rightarrow \frac{Cm^2}{2} = \frac{1}{2}$

$$\Rightarrow m^2 = \frac{1}{C} = \frac{5}{8} \Rightarrow m = \sqrt{\frac{5}{8}} = \boxed{0.791}$$

10) (10 points) Compute the probability that a standard normal distribution is within 1/2 standard deviation from its mean. (Provide numerical answer in decimal form.) (Show your work.)

$$P(-1/2 < Z < 1/2)$$

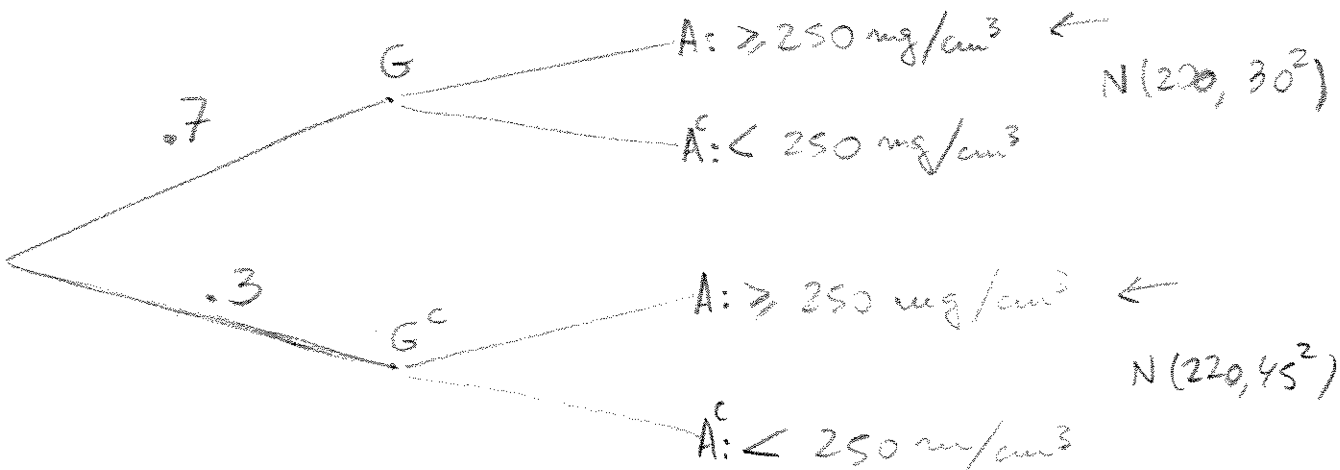
$$= \Phi(1/2) - \Phi(-1/2)$$

$$= \Phi(1/2) - (1 - \Phi(1/2))$$

$$= 2\,\Phi(1/2) - 1 = 2\,\Phi(0.5) - 1 = 2 \times .6915 - 1$$

$$= \boxed{.383}$$

11) (10 points) Substance S in the blood is crucial for good health; too much or too little of it causes health problems. A simple blood test gives the amount of substance S in the blood in mg/cm^3. People that have a certain gene G have an amount of substance S in their blood that is well described by a normal distribution with mean 200 and standard deviation 30. People that do not have that gene G, have an amount of substance S in their blood that is well described by a normal distribution with mean 220 and standard deviation 45. In a certain population, 70 % of people have gene G, while the remaining 30 % do not have it. Find the probability that a person taken at random from that population will have more than 250 mg/cm^3 of substance S in his/her blood. (Provide numerical answers in decimal form.) (Show your work.)



$$P(A) = .7 \times P\left(Z \geq \frac{250-200}{30}\right) + .3 \times P\left(Z \geq \frac{250-220}{45}\right)$$

$$= .7 \times P(Z \geq 1.67) + .3 \times P(Z \geq 0.67)$$

$$= .7 \times (1 - .9525) + .3 \times (1 - .7486)$$

$$= .7 \times 0.0475 + .3 \times 0.2514 = 0.10867$$

12) (10 points) The radius (measured in mm) of the spheres produced by a machine are well described by a normal distribution, with mean 100 and standard deviation 20. What is the probability that the volume of a sphere produced by this machine will be less than 6000000 mm^3? (Recall that the volume of a sphere of radius R is given by $V = 4\pi R^3/3$.) (Give answer in decimal form.) (Show ~~and explain~~ your work.)

$$P(V < 6000000) = P\left(\frac{4\pi R^3}{3} < 6000000\right)$$

$$= P\left(R < \sqrt[3]{\frac{6000000 \times 3}{4\pi}}\right) = P(R < 112.73)$$

$$= P\left(Z < \frac{112.73 - 100}{20}\right)$$

$$= P(Z < .637) = \Phi(.637) = \boxed{.7389}$$

13) (10 points) Suppose S_n is binomially distributed with parameters $n = 100$, $p = .7$. Use the central limit theorem to compute approximately the value of $P(S_n \le 73)$. (Do not use the histogram correction.) (Provide a numerical answer in decimal form.) (Show ~~and explain~~ your work.)

$$P(S_n \le 73) = P(\bar{X}_n \le .73)$$

$$= P\left(Z \le \frac{.73 - .7}{\sqrt{\frac{.7 \times .3}{100}}}\right) = P\left(Z \le \frac{.3}{\sqrt{.21}}\right)$$

$$= \Phi(0.65) = \boxed{.7422}$$

Since

$$\bar{X}_n = \frac{S_n}{n} \overset{\text{approx.}}{\sim} N\left(p, \left(\sqrt{\frac{p(1-p)}{n}}\right)^2\right)$$

14) (10 points) A die is rolled 120 times. Use the central limit theorem to compute the approximate value of the probability that the number of times the face 6 shows is at least 20. (Do not use the histogram correction.) (Provide a numerical answer in decimal form.) (Show your work.)

$$X_i = \begin{cases} 1 & \text{if i-th roll of die gives face "6"} \\ 0 & \text{otherwise} \end{cases}$$

$$S_n = X_1 + X_2 + \dots + X_n \qquad \bar{X}_n = S_n / n$$

$$\bar{X}_n \overset{\text{approx.}}{\sim} \text{Normal}$$

$$\mu_{\bar{X}_n} = \mu_{X_1} = 1/6$$

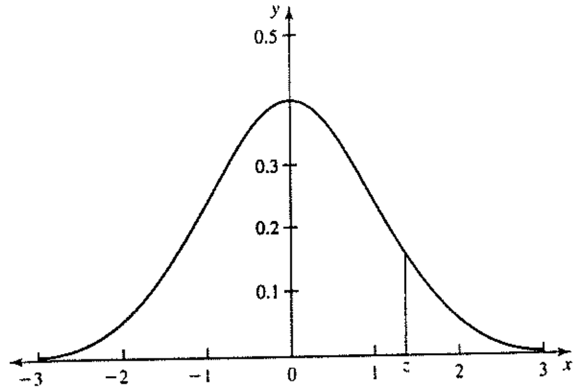
$$\sigma_{\bar{X}_n} = \frac{\sigma_{X_1}}{\sqrt{n}} = \frac{\sqrt{1/6 \times 5/6}}{\sqrt{120}} = .03402$$

$$P(S_n \geq 20) = P\left(\bar{X}_n \geq \frac{20}{120}\right) = P\left(\bar{X}_n \geq \frac{1}{6}\right)$$

$$\cong P\left(Z \geq \frac{1/6 - 1/6}{.03402}\right) = P(Z \geq 0) = \boxed{0.5}$$

TABLE OF THE STANDARD NORMAL DISTRIBUTION

Areas under the Standard Normal Curve from $-\infty$ to z (see Figure B.1).



◀ **Figure B.1**

z	0	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5754
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7258	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7518	.7549
0.7	.7580	.7612	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7996	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986