MATH 3C (Spring 2007, Lecture 2)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

2A (T 10, Akemi Kashiwada)

2B (R 10, Akemi Kashiwada)

2C (T 10, Wenhua Gao)

2D (R 10, Wenhua Gao)

2E (T 10, Ilhwan Jo)

2F (R 10, Ilhwan Jo)

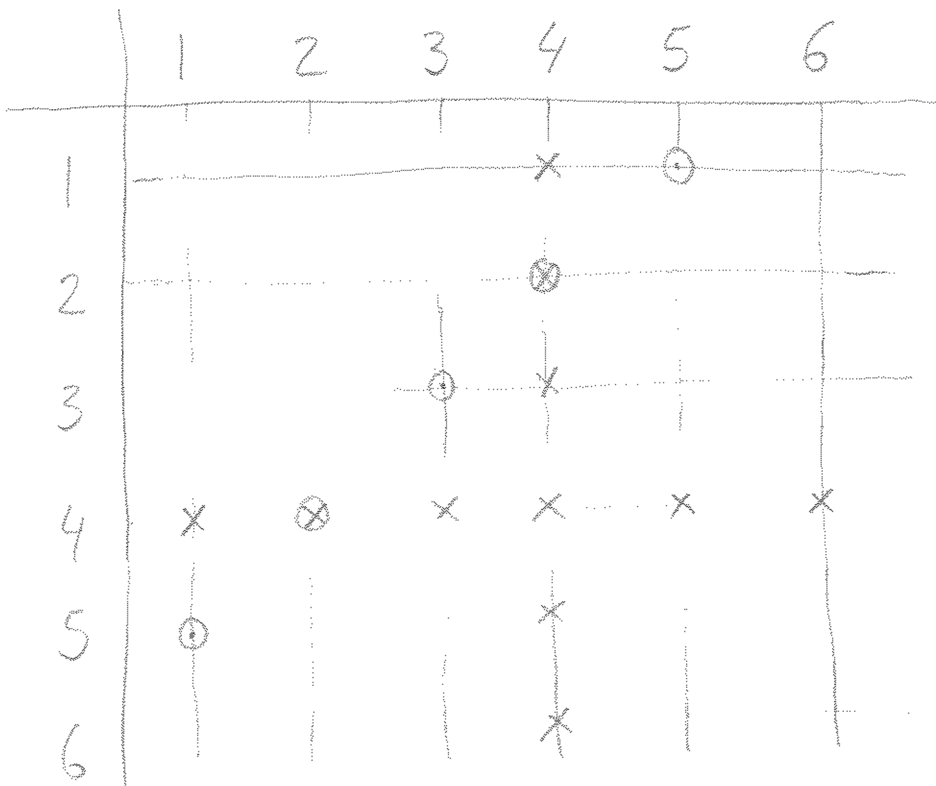
When the instructions to a question ask you to explain your answer, you should show your work and explain what you are doing carefully; this is then more important than just finding the right answer. Please, write clearly and make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. To cancel anything from your solution, erase it or cross it out. You are not allowed to sit close to students with whom you have studied for this exam, or to your friends.

Enjoy the exam, and Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

Question	9	10	11	12	13	14	15	Total
Score								

1) (10 points) Roll a die 2 times. What is the conditional probability that the sum of the faces shown is 6, given that at least one of the faces shown is a 4? (Give answer as a fraction or in decimal form.) (Show and explain your work.)

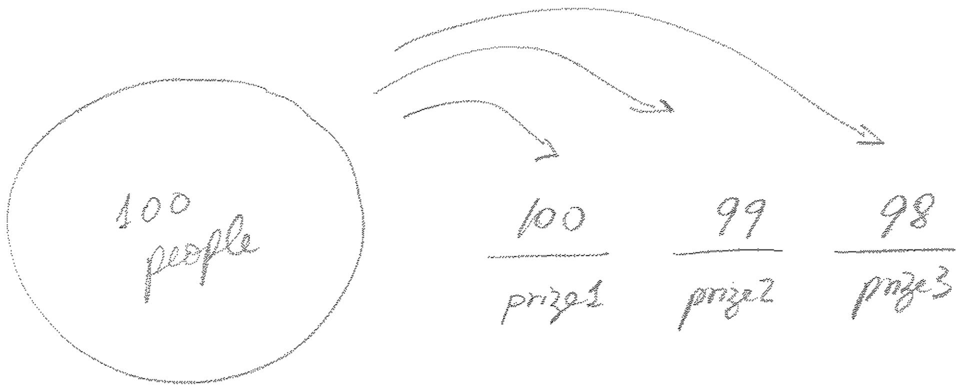


A: sum 6: ⊙

B: at least one face 4: ×

$$\boxed{P(A|B) = \frac{P(AB)}{P(B)} = \frac{2/36}{11/36} = \frac{2}{11} \cong 0.1818}$$

2) (10 points) In how many ways can we give 3 different prizes to a group of 100 people, if each prize can only be given to one person, and each person can receive at most one prize. (Provide a numerical answer.) (No explanation needed, just the answer is enough.)



Answer : $100 \times 99 \times 98 = \boxed{970200}$

3) (10 points) You are dealt 6 cards from a standard deck of 52 cards. Compute the probability that among those cards you have: 3 cards of one denomination and 3 cards of another denomination. (No need to compute factorials, powers, permutations and combinations.) (No explanation needed, just the answer is enough.)

2 denominations.

suits of cards of each denomination.

$$\frac{\binom{13}{2} \times \binom{4}{3}^2}{\binom{52}{6}}$$

4) (10 points) Roll a fair die. Consider the events $A = \{1, 3\}$, $B = \{3, 4, 5\}$. Are A and B independent? (Explain your answer carefully.)

$$P(A) = \frac{2}{6} = \frac{1}{3} \qquad P(B) = \frac{3}{6} = \frac{1}{2}$$

$$P(A \cap B) = P(\{3\}) = \frac{1}{6}$$

Since $P(A \cap B) = \frac{1}{6} = \frac{1}{3} \times \frac{1}{2} = P(A) \times P(B)$

we have A and B independent.

5) (10 points) A group of people has 10 adults and 20 children. One of the adults is called Ann and another one Bob. We want to select 3 adults and 7 children to go on a trip. They will go in a vehicle with room for 10, and one of the selected adults has to be the driver. (All the adults are assumed to be potential drivers). Suppose that any possible choice of driver and passengers is equally likely. What is the probability that Ann will be the driver and Bob will be a passenger (not the driver)? (Provide a numerical answer as a fraction or in decimal form.) (Show and explain your work.)

A: Ann is driver

B: Bob is passenger (not driver)

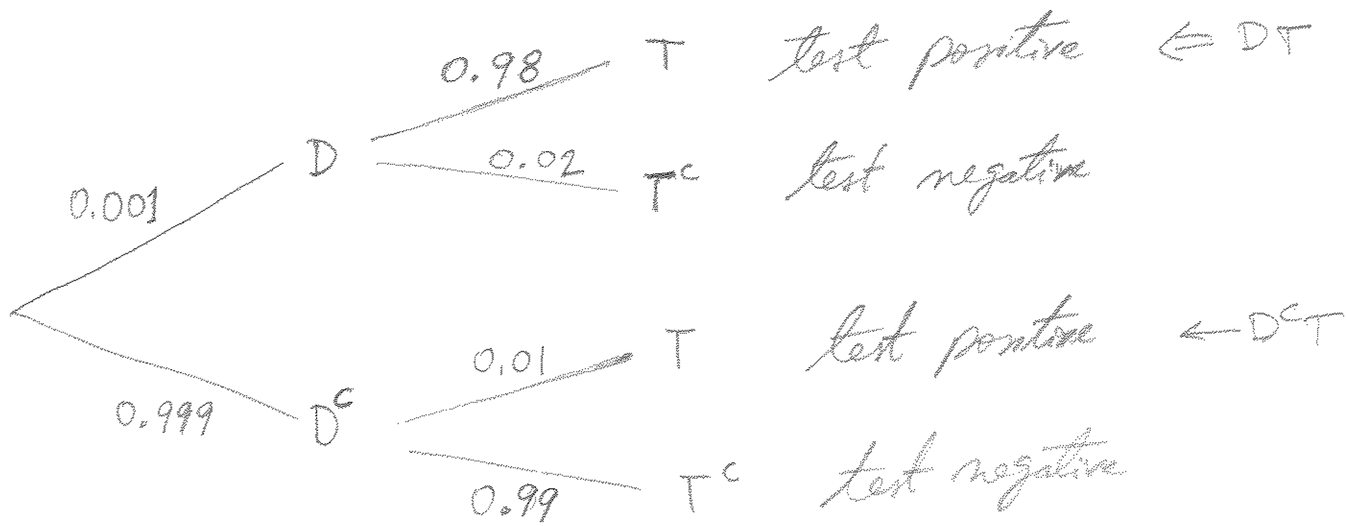
$$\boxed{P(A \cap B)} = P(A) \cdot P(B|A) \overset{\circledast}{=} \frac{1}{10} \cdot \frac{\binom{8}{1}}{\binom{9}{2}}$$

$$= \frac{1}{10} \times \frac{8}{\frac{9 \times 8}{2}} = \frac{2}{90} = \boxed{\frac{1}{45} \cong 0.0222}$$

⊛ Explanation $P(A) = 1/10$ since each adult is equally likely to be the driver.

$P(B|A) = \frac{\binom{8}{1}}{\binom{9}{2}}$ since we have 9 adults (Ann is out) and want to choose 2. Total number of choices $\binom{9}{2}$, choices of Bob and someone else $\binom{8}{1}$.

6) (10 points) A screening test for a disease shows a false positive with probability 1% and a false negative with probability 2%. In the population 0.1% of people have that disease. Given that someone tested positive for the disease, what is the probability that he/she has the disease? (Provide a numerical answer as a fraction or in decimal form.) (Show and explain your work.)

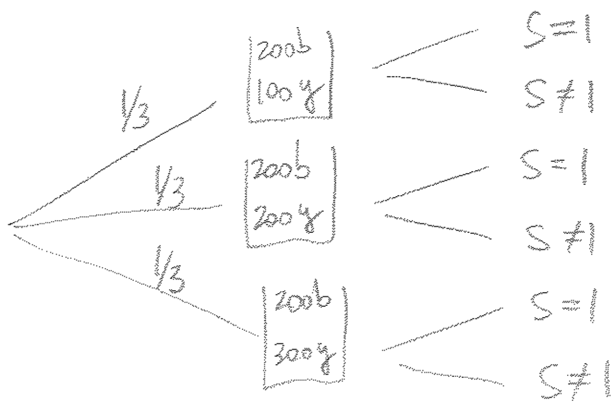


$$P(D \mid T) = \frac{P(DT)}{P(T)} = \frac{0.001 \times 0.98}{0.001 \times 0.98 + 0.999 \times 0.01}$$

$$= \frac{0.00098}{0.00098 + 0.00999} = \frac{0.00098}{0.01097}$$

$$\simeq 0.0893 = \boxed{8.93\%}$$

7) (10 points) You have three boxes. The first box contains 200 blue balls and 100 yellow balls. The second box contains 200 blue balls and 200 yellow balls. The third box contains 200 blue balls and 300 yellow balls. A box is selected at random, and from this box 5 balls are selected without replacement. Let S be the random variable that gives the number of yellow balls among the selected ones. Find $P(S \leq 1)$. (No need to compute factorials, powers, permutations and combinations.) (No explanation needed, just the answer is enough.)



$$P(S=1) = \frac{1}{3} \times \frac{\binom{100}{1}\times\binom{200}{4}}{\binom{300}{5}} + \frac{1}{3} \times \frac{\binom{200}{1}\times\binom{200}{4}}{\binom{400}{5}} + \frac{1}{3} \times \frac{\binom{300}{1}\times\binom{200}{4}}{\binom{500}{5}}$$

8) (10 points) Suppose that the independent events A and B have probabilities $P(A) = .6$ and $P(B) = .5$. Compute $P(A \cup B)$. (Provide a numerical answer in decimal form.) (Show and explain your work.)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A)P(B) \quad \text{by independence}$$

$$= .6 + .5 - .6 \times .5$$

$$= .6 + .5 - .3$$

$$= \boxed{.8}$$

9) (10 points) The mean deviation of a discrete random variable X is defined as $\sum_n |n - \mu_X| p_X(n)$, where $p_X(n) = P(X = n)$ is the probability mass function of X, and the sum is over all the values that X can take. Compute the mean deviation of a random variable X that has binomial distribution with 4 attempts and probability $1/2$ of success each time, i.e., $X \sim \mathrm{B}(4, 1/2)$. (Give answer as a fraction or in decimal form.) (Show and explain your work.)

$$p_X(0) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$\mu_X = np = 4 \times \frac{1}{2} = 2$$

$$p_X(1) = 4 \times \left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$p_X(2) = \binom{4}{2}\left(\frac{1}{2}\right)^4 = \frac{4 \times 3}{2} \times \frac{1}{16} = \frac{3}{8}$$

$$p_X(3) = \binom{4}{3}\left(\frac{1}{2}\right)^4 = 4 \times \left(\frac{1}{2}\right)^4 = \frac{1}{4}$$

$$p_X(4) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$$\boxed{\text{mean deviation} = \sum_n |n - \mu_x| p_x(n)}$$

$$= \frac{1}{16} \times |0-2| + \frac{1}{4} \times |1-2| + \frac{3}{8} \times |2-2| + \frac{1}{4} \times |3-2| + \frac{1}{16} \times |4-2|$$

$$= \frac{2}{16} + \frac{1}{4} + 0 + \frac{1}{4} + \frac{2}{16} = \frac{2+4+0+4+2}{16} = \frac{12}{16} = \boxed{\frac{3}{4} = .75}$$

10) (10 points) Suppose that X and Y are independent random variables with probability mass functions given by: $p_X(-1) = .3$, $p_X(0) = .4$, $p_X(1) = .2$, $p_X(2) = .1$, and $p_Y(1) = .2$, $p_Y(2) = .8$. Compute $P(X + Y = 2)$. (Give answer in decimal form.) (Show and explain your work.)

Y \ X	-1	0	1	2	
1	.06	.08	.04	.02	.2
2	.24	.32	.16	.08	.8
	.3	.4	.2	.1	

Due to independence $P(X=k, Y=\ell) = P(X=k)P(Y=\ell)$

$$P(X+Y=2) = P(X=1, Y=1) + P(X=0, Y=2)$$

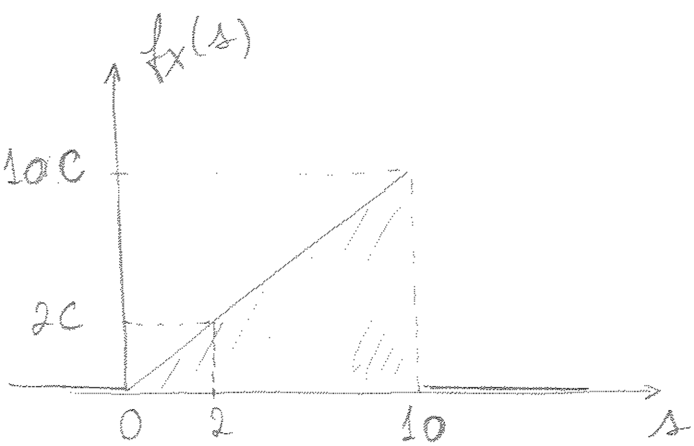
$$= .04 + .32 = \boxed{.36}$$

(Note: only part of the table above is needed to solve this problem)

11) (10 points) When Helen tosses a dart at a target, the distance (in inches) from the center of the target that she hits is well described as a continuous random variable X with probability density function given below, where C is an appropriate number.

$$f_X(s) = \begin{cases} 0 & \text{if } s < 0 \text{ or } s > 10, \\ Cs & \text{if } 0 \leq s \leq 10. \end{cases}$$

a) Find the value of C.
b) If she tosses the dart at the target 5 times, what is the probability that at least once the dart hits the target less than 2 inches from the center? Assume independence of the 5 tosses.
(Provide numerical answers in decimal form.) (Show and explain your work.)



a) $C = ?$ $\quad \int_{-\infty}^{+\infty} f_X(s) = 1 \Rightarrow \frac{1}{2} \times 10 \times 10C = 1 \Rightarrow \boxed{C = \frac{2}{100} = \frac{1}{50} = 0.02}$

b) In each toss, P(less than 2 inches from center)

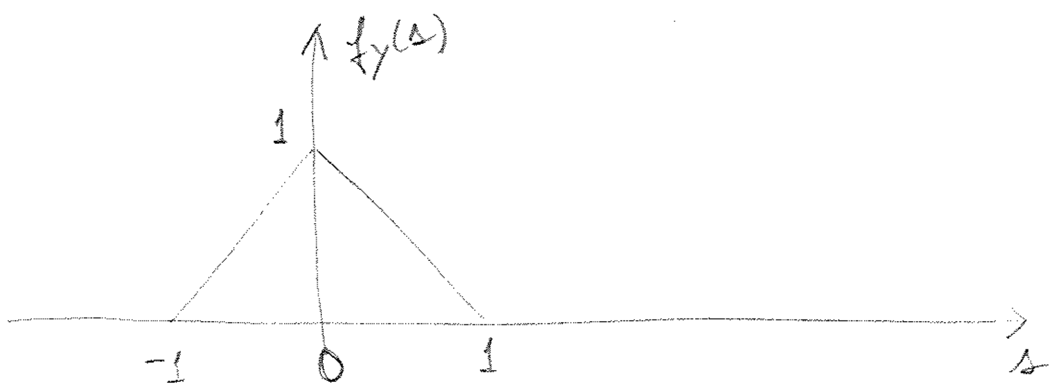
$= P(X \leq 2) = \frac{1}{2} \times 2 \times 2C = \frac{1}{25}$

Five tosses $\Rightarrow$ (Binomial $(5, \frac{1}{25})$) P(at least one hit less than 2 inches from center) $= 1 - \left(1 - \frac{1}{25}\right)^5 = 1 - \left(\frac{24}{25}\right)^5 = \boxed{0.185}$

12) (10 points) A continuous random variable Y has probability density function given below.

$$f_Y(s) = \begin{cases} 0 & \text{if} \quad s < -1 \text{ or } s > 1, \\ 1+s & \text{if} \quad -1 \le s < 0, \\ 1-s & \text{if} \quad 0 \le s \le 1. \end{cases}$$

Compute $\mathrm{Var}(Y)$. (Provide a numerical answer in decimal form.) (Show and explain your work.)



By symmetry (or integration) : $\mu_Y = 0$

$$\int_{-\infty}^{+\infty} s^2 f_Y(s)\, ds = \int_{-1}^{0} s^2(1+s)\, ds + \int_{0}^{1} s^2(1-s)\, ds$$

$$= \int_{-1}^{0} (s^2+s^3)\, ds + \int_{0}^{1} (s^2 - s^3)\, ds = \left[\frac{s^3}{3} + \frac{s^4}{4}\right]_{-1}^{0} + \left[\frac{s^3}{3} - \frac{s^4}{4}\right]_{0}^{1}$$

$$= \left[0 - \left(-\frac{1}{3} + \frac{1}{4}\right)\right] + \left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = 2 \times \frac{4-3}{12} = \frac{1}{6}$$

Therefore $\mathrm{Var}(Y) = \int_{-\infty}^{+\infty} s^2 f_Y(s)\, ds - (\mu_Y)^2 = \frac{1}{6} - 0 = \frac{1}{6} \cong 0.167$

13) (10 points) Suppose that W is a random variable with normal distribution with mean -1 and standard deviation 2. Compute

$$P(-3 < W < -0.5).$$

(Provide a numerical answer in decimal form.) (Show and explain your work.)

Standardize : $Z = \frac{W-(-1)}{2} = \frac{W+1}{2} \sim N(0,1)$

$$P(-3 < W < -0.5) = P\left(\frac{-3+1}{2} < Z < \frac{-0.5+1}{2}\right)$$

$$= P(-1 < Z < +0.25) = \phi(+0.25) - \phi(-1)$$

$$= \phi(0.25) - (1-\phi(1))$$

$$= \phi(1) + \phi(0.25) - 1 = .8413 - .5987 - 1$$

$$= \boxed{0.4400}$$

14) (10 points) A certain substance in the blood is distributed as a normal random variable with mean 200 and standard deviation 10 (when measured in mg/dl). People who have an amount larger than 225 mg/dl are diagnosed as having a certain medical condition. When the amount is larger than 215 mg/dl, doctors call the patient. (If the amount is between 215 mg/dl and 225 mg/dl they recommend life style changes, to prevent the condition from developing; if the amount is larger than 225 mg/dl they start treating the condition with medications; but if the amount is less than 215 mg/dl they do not even call the patient.) If after having a sample of your blood taken to check on that substance you receive a phone call from your doctor, what is the conditional probability that you have the medical condition? (Provide a numerical answer in decimal form.) (Show and explain your work.)

$X \sim N(200, (10)^2)$ substance in blood

$X > 215 \Rightarrow$ Dr. calls

$X > 225 \Rightarrow$ Medical condition

Want $P(X > 225 \mid X > 215)$

$$= \frac{P(X>225 \cap X>215)}{P(X>215)} = \frac{P(X>225)}{P(X>215)}$$

$$\boxed{Z = \frac{X-200}{10} \qquad Z \sim N(0,1)}$$

$$= \frac{P\left(Z > \frac{225-200}{10}\right)}{P\left(Z > \frac{215-200}{10}\right)} = \frac{P(Z > 2.5)}{P(Z > 1.5)}$$

$$= \frac{1-\Phi(2.5)}{1-\Phi(1.5)} = \frac{1-.9938}{1-.9332} = \frac{.0062}{.0668} = \boxed{0.0928}$$

15) (10 points) Toss a fair coin 100 times. Use the central limit theorem to find an approximation for the probability that the number of heads is at most 55. (Provide a numerical answer in decimal form.) (Show and explain your work.)

$S_n = X_1 + \dots + X_n = \#$ of heads $\qquad n = 100$

$X_i = \begin{cases} 1 & i\text{-th toss is heads} \\ 0 & i\text{-th toss is tails} \end{cases} \qquad \mu = EX_i = 1/2$

$\sigma^2 = \operatorname{Var} X_i = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

CLT: $\bar{X}_n = \frac{S_n}{n} \overset{\text{appr.}}{\sim} N(\mu, \sigma^2/n) = N(1/2, 1/400)$

$= N(0.5, (0.05)^2)$

Note $\mu_{\bar{X}_n} = \mu = 0.5 \qquad \sigma_{\bar{X}_n} = \sigma/\sqrt{n} = \frac{1}{20} = 0.05$

Want $P(S_n \le 55) = P(\bar{X}_n \le \frac{55}{100}) = P(\bar{X}_n \le .55)$

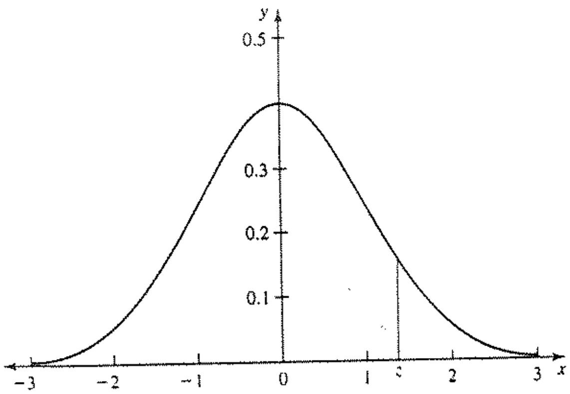
Using CLT: $P(\bar{X}_n \le .55) = P(Z \le \frac{.55 - .5}{.05}) = P(Z \le 1)$

$\left(Z = \frac{\bar{X}_n - .5}{.05} \overset{\text{appr.}}{\sim} N(0,1) \right)$

$= \Phi(1) = \boxed{.8413}$

TABLE OF THE STANDARD NORMAL DISTRIBUTION

Areas under the Standard Normal Curve from $-\infty$ to z (see Figure B.1).



◀ Figure B.1

z	0	1	2	3	4	5	6	7	8	9
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5754
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7258	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7518	.7549
0.7	.7580	.7612	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7996	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986