MATH 3C (Spring 2007, Lecture 1)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Circle the discussion section in which you are enrolled:

1A (T 9, Matthew Keegan)

1B (R 9, Matthew Keegan)

1C (T 9, Yan Wang)

1D (R 9, Yan Wang)

1E (T 9, Christopher McKinlay)

1F (R 9, Christopher McKinlay)

When the instructions to a question ask you to explain your answer, you should show your work and explain what you are doing carefully; this is then more important than just finding the right answer. Please, write clearly and make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. To cancel anything from your solution, erase it or cross it out. You are not allowed to sit close to students with whom you have studied for this exam, or to your friends.

Enjoy the exam, and Good Luck !

Question	1	2	3	4	5	6	7	8
Score								

Question	9	10	11	12	13	14	15	Total
Score								

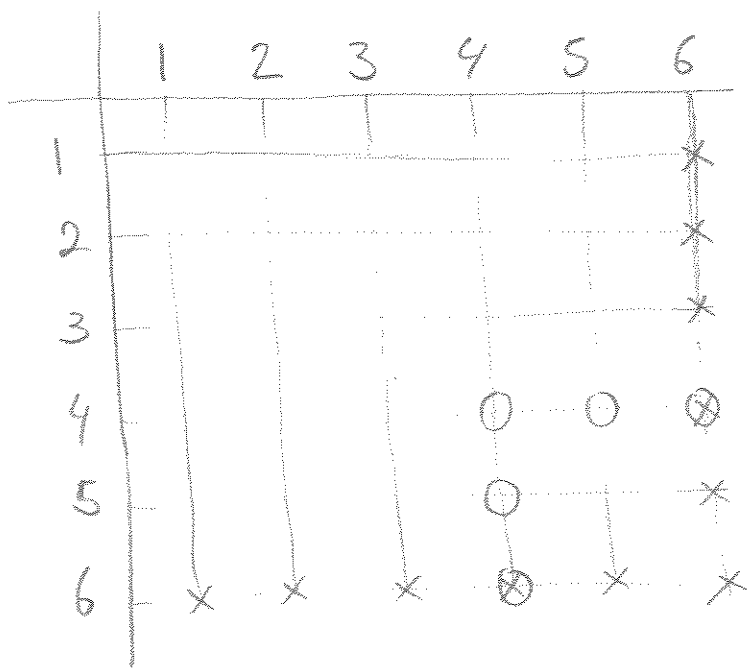
1) (10 points) In a group of students, 8 are freshmen and 8 are sophomores. In how many ways can the students in this group form a waiting line, if freshmen and sophomores should alternate in the line? (No need to compute factorials, powers, permutations and combinations.) (No explanation needed, just the answer is enough.)

f s f s ... f s   or   s f s f ... s f

$8! \times 8!$ $+$ $8! \times 8!$

Answer: $2 \times 8! \times 8!$

2) (10 points) Roll a die 2 times. What is the conditional probability that the maximum of the faces shown is 6, given that the minimum of the faces shown is 4? (Give answer as a fraction or in decimal form.) (Show and explain your work.)



O minimum is 4 : A

x maximum is 6 : B

$$\boxed{P(B|A) = \frac{P(BA)}{P(A)} = \frac{2/36}{5/36} = \frac{2}{5} = .4}$$

3) (10 points) You are dealt 8 cards from a standard deck of 52 cards. Compute the probability that among those cards you have: 2 cards of one denomination, 2 cards of another denomination, 3 cards of a third denomination, and one single card of a fourth denomination. (No need to compute factorials, powers, permutations and combinations.) (No explanation needed, just the answer is enough.)

denomination of pairs

suits of 2 pairs

denomination of 3 cards of same denomination

suit of 3 cards of same denomination

denomination of single card

$$\frac{\binom{13}{2} \times \binom{4}{2}^2 \times \binom{11}{1} \times \binom{4}{3} \times \binom{10}{1} \times \binom{4}{1}}{\binom{52}{8}}$$

suit of single card

4) (10 points) Roll a fair die. Consider the events $A = \{1,3\}$, $B = \{3,4\}$. Are A and B independent? (Explain your answer carefully.)

$$P(A) = \frac{2}{6} \qquad P(B) = \frac{2}{6}$$

$$A \cap B = \{3\} \Rightarrow P(A \cap B) = \frac{1}{6}$$

$$P(A) \times P(B) = \frac{2}{6} \times \frac{2}{6} = \frac{1}{9} \neq \frac{1}{6} = P(A \cap B)$$

So A and B are not independent.

5) (10 points) A group of people has 10 adults and 20 children. One of the adults is called Ann and another one Bob. We want to select 2 adults and 8 children to go on a trip. They will go in a vehicle with room for 10, and one of the selected adults has to be the driver. (All the adults are assumed to be potential drivers.) Suppose that every possible choice of driver and passengers is equally likely. What is the probability that Ann will be the driver or Bob will be a passenger (not the driver)? (Provide a numerical answer as a fraction or in decimal form.) (Show and explain your work.)

A: Ann is driver

B: Bob passenger (not driver)

Want $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A) = \frac{1}{10}$, $P(B) = \frac{1}{10}$

since out of 10 adults one is driver and one is passenger

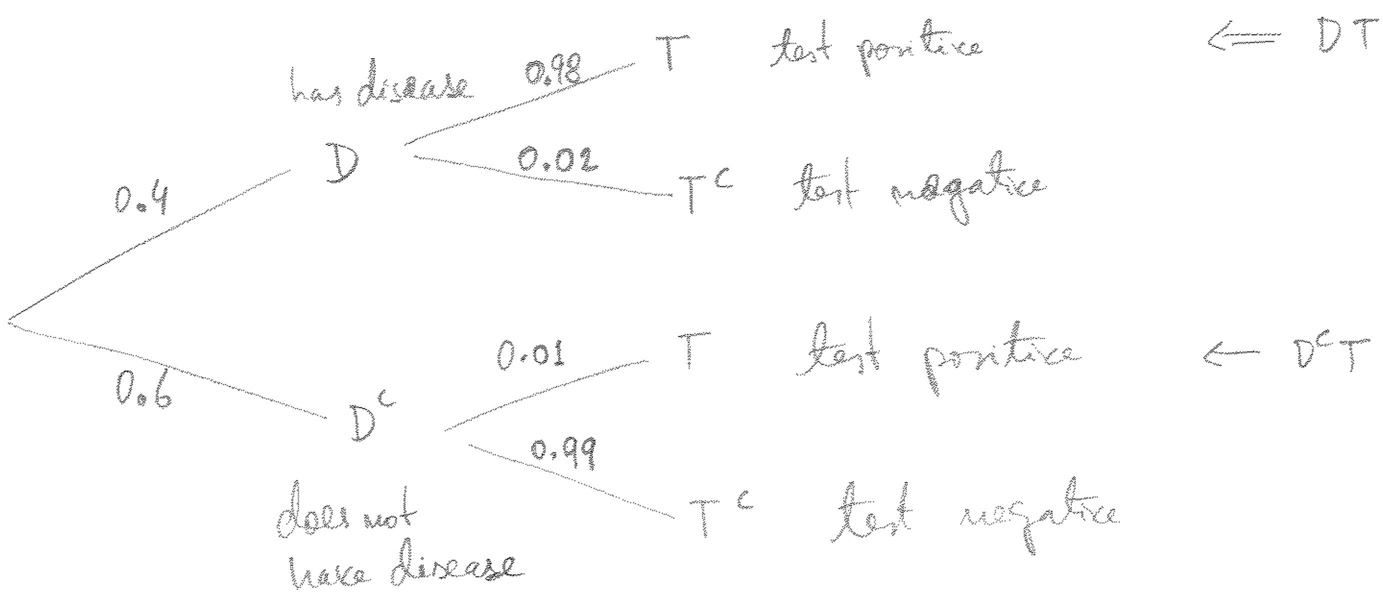
Ann driver, Bob passenger

$$P(A \cap B) = \frac{1 \times 1}{10 \times 9} = \frac{1}{90}$$

driver's choice, passanger's choice

So $P(A \cup B) = \frac{1}{10} + \frac{1}{10} - \frac{1}{90} = \frac{9+9-1}{90} = \boxed{\frac{17}{90}}$

6) (10 points) A screening test for a disease shows a false positive with probability 1% and a false negative with probability 2%. In the population 40% of people have that disease. Given that someone tested positive for the disease, what is the probability that he/she has the disease? (Provide a numerical answer in decimal form, or as a percentage.) (Show and explain your work.)



$$P(D|T) = \frac{P(DT)}{P(T)} = \frac{0.4 \times 0.98}{0.4 \times 0.98 + 0.6 \times 0.01}$$

$$= \frac{0.392}{0.392 + 0.006} = \boxed{0.985 = 98.5\%}$$

7) (10 points) A box contains 2 fair coins and 1 coin with two tails. One coin is selected at random from this box and flipped 5 times. Let X be the random variable that gives the number of heads in the 5 flips of the coin. Compute $P(X \geq 4)$. (Provide a numerical answer as a fraction or in decimal form.) (No explanation needed, just the answer is enough.)

$$\underbrace{\frac{2}{3}}_{\substack{\text{fair}\\\text{coin}\\\text{selected}}} \times \Bigg(\underbrace{\binom{5}{1}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 + \binom{5}{0}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0}_{\substack{\text{4 or 5}\\\text{heads, given}\\\text{fair coin}\\\text{is flipped}}} \Bigg) = \frac{2}{3} \times \left(5 \times \frac{1}{2^5} + \frac{1}{2^5}\right)$$

$$= \frac{2}{3} \times \frac{6}{2^5}$$

$$= \boxed{\frac{1}{8} = 0.125}$$

8) (10 points) Suppose that the independent events A and B have probabilities $P(A) = .8$ and $P(B) = .5$. Compute $P(A \cup B)$. (Provide a numerical answer in decimal form.) (Show and explain your work.)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= P(A) + P(B) - P(A) \times P(B) \quad \text{(by independence)}$$

$$= .8 + .5 - .8 \times .5$$

$$= .8 + .5 - .4$$

$$= \boxed{.9}$$

9) (10 points) A friend of yours wanted to write a computer code to compute the variance of a discrete random variable X. She knew the correct formula: $\mathrm{Var}(X) = \sum_n (n - \mu_X)^2 p_X(n)$. But, by mistake she left the square out of the formula, making the computer compute instead $\sum_n (n - \mu_X) p_X(n)$. What numerical answer did the computer give her? (Explain your answer carefully.)

$$\sum_n (n-\mu_X)\, p_X(n) \overset{(1)}{=} \sum_n n\, p_X(n) - \mu_X \sum_n p_X(n)$$

$$\overset{(2)}{=} \mu_X - \mu_X \cdot 1 = \mu_X - \mu_X = 0$$

(1) properties of summation

(2) $\sum_n n\, p_X(n) = \mu_X$, $\sum_n p_X(n) = 1$

$\boxed{\text{Answer : } 0}$

10) (10 points) A fair coin is flipped 3 times. Let X be the random variable that gives the number of heads shown in the first two flips of the coin, and let Y be the random variable that gives the number of heads shown in the last two flips of the coin. (For instance, for the outcome hht we have $X = 2$, $Y = 1$, and for the outcome tth we have $X = 0$, $Y = 1$.) Provide a table that gives the joint distribution of X and Y, i.e., a table that gives the numbers $P(X = k, Y = l)$, $k = 0, 1, 2$, $l = 0, 1, 2$. (The numbers in the table should be given as fractions, or in decimal form.) (Show and explain your work.)

Answer:

$l \backslash k$	0	1	2
0	1/8	1/8	0/8
1	1/8	2/8	1/8
2	0/8	1/8	1/8

Table of $P(X=k, Y=l)$

Diagram used to obtain answer

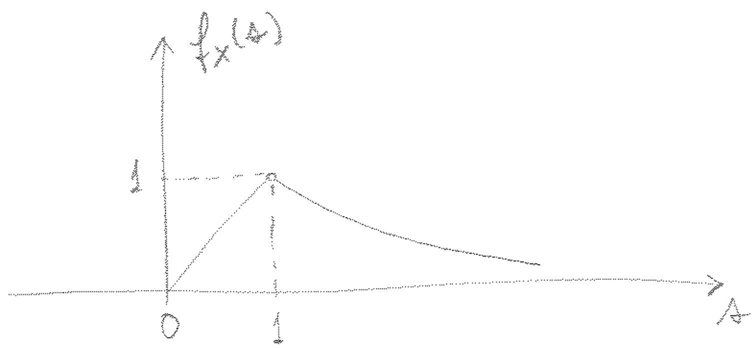
Ω	X	Y
hhh	2	2
hht	2	1
hth	1	1
htt	1	0
thh	1	2
tht	1	1
tth	0	1
ttt	0	0

$|\Omega| = 8$

11) (10 points) A continuous random variable X has probability density function given below, where C is an appropriate number.

$$f_X(s) = \begin{cases} 0 & \text{if } s < 0, \\ Cs & \text{if } 0 \le s < 1, \\ Cs^{-3} & \text{if } s \ge 1. \end{cases}$$

Find the value of C and compute the mean μ_X of X. (Provide numerical answers in decimal form.) (Show and explain your work.)



$C = ?$: $\int_{-\infty}^{\infty} f_X(s)\,ds = 1$

$$\int_{-\infty}^{+\infty} f_X(s)\,ds = \int_0^1 Cs\,ds + \int_1^\infty Cs^{-3}\,ds = \left[C\frac{s^2}{2}\right]_0^1 + \left[C\frac{s^{-2}}{-2}\right]_1^\infty$$

$$= C\left[\frac{1}{2} - 0\right] + C\left[0 - \left(-\frac{1}{2}\right)\right] = C$$

$$\Rightarrow \boxed{C = 1}$$

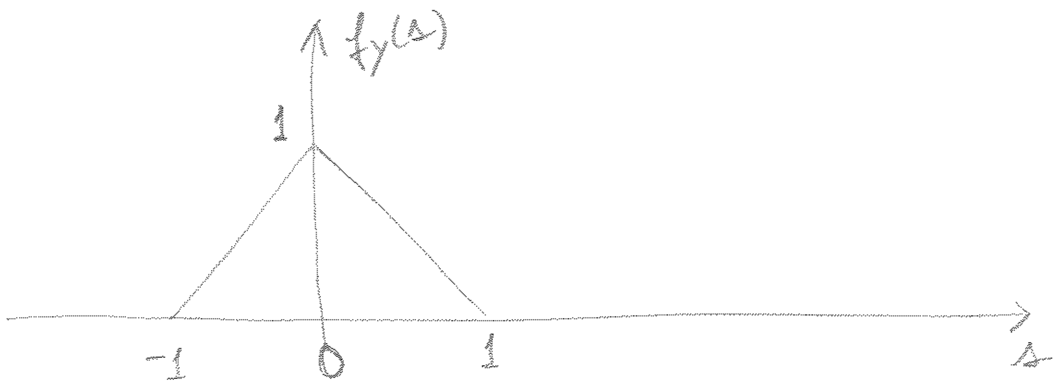
$$\boxed{\mu_X} = \int_{-\infty}^{+\infty} s f_X(s)\,ds = \int_0^1 s^2\,ds + \int_1^\infty s^{-2}\,ds = \left[\frac{s^3}{3}\right]_0^1 + \left[\frac{s^{-1}}{-1}\right]_1^\infty$$

$$= \left[\frac{1}{3} - 0\right] + \left[0 - (-1)\right] = \boxed{\frac{4}{3}}$$

12) (10 points) A continuous random variable Y has probability density function given below.

$$f_Y(s) = \begin{cases} 0 & \text{if} \quad s < -1 \text{ or } s > 1, \\ 1+s & \text{if} \quad -1 \le s < 0, \\ 1-s & \text{if} \quad 0 \le s \le 1. \end{cases}$$

Compute $\mathrm{Var}(Y)$. (Provide a numerical answer in decimal form.) (Show and explain your work.)



By symmetry (or integration): $\mu_Y = 0$

$$\int_{-\infty}^{+\infty} s^2 f_Y(s)\, ds = \int_{-1}^{0} s^2 (1+s)\, ds + \int_{0}^{1} s^2 (1-s)\, ds$$

$$= \int_{-1}^{0} (s^2 + s^3)\, ds + \int_{0}^{1} (s^2 - s^3)\, ds = \left[\frac{s^3}{3} + \frac{s^4}{4}\right]_{-1}^{0} + \left[\frac{s^3}{3} - \frac{s^4}{4}\right]_{0}^{1}$$

$$= \left[0 - \left(-\frac{1}{3} + \frac{1}{4}\right)\right] + \left[\left(\frac{1}{3} - \frac{1}{4}\right) - 0\right] = 2 \times \frac{4-3}{12} = \frac{1}{6}$$

Therefore $\mathrm{Var}(Y) = \int_{-\infty}^{+\infty} s^2 f_Y(s)\, ds - (\mu_Y)^2 = \frac{1}{6} - 0 = \boxed{\frac{1}{6} \approx 0.167}$

13) (10 points) Suppose that W is a random variable with normal distribution with mean -1 and standard deviation 2. Compute

$$P(-3 < W < 0).$$

(Provide a numerical answer in decimal form.) (Show and explain your work.)

Standardize: $Z = \frac{W-(-1)}{2} = \frac{W+1}{2} \sim N(0,1)$

$$P(-3 < W < 0) = P\left(\frac{-3+1}{2} < Z < \frac{0+1}{2}\right)$$

$$= P(-1 < Z < 0.5) = \phi(0.5) - \phi(-1)$$

$$= \phi(0.5) - (1 - \phi(+1)) = \phi(0.5) + \phi(1) - 1$$

$$= .6915 + .8413 - 1 = \boxed{.5328}$$

(Table)

14) (10 points) You have an appointment with a friend at 4:00 pm. You know from experience that your friend will arrive at a time that differs from 4:00pm by a random amount T with normal distribution with mean 0 and standard deviation 5 minutes. (A negative value of T means that your friend arrives early, a positive value of T means that your friend arrives late.) Given that your friend has not arrived yet at 4:05, what is the conditional probability that he will not yet have arrived at 4:10? (Provide a numerical answer in decimal form.) (Show and explain your work.)

$$T \sim N(0, 5^2)$$

$$P(T>10 \mid T>5) = \frac{P(T>10 \cap T>5)}{P(T>5)}$$

$$= \frac{P(T>10)}{P(T>5)} = \frac{P(Z>2)}{P(Z>1)} = \frac{1-\Phi(2)}{1-\Phi(1)}$$

standardize: $\frac{T}{5} = Z \sim N(0,1)$

$$= \frac{1-.9772}{1-.8413}$$

$$= \frac{.0228}{.1587}$$

$$= \boxed{0.1437}$$

15) (10 points) Toss a fair coin 100 times. Use the central limit theorem to find an approximation for the probability that the number of heads is at least 60. (Provide a numerical answer in decimal form.) (Show and explain your work.)

$S_n = X_1 + \dots + X_n$ = # of heads $\qquad n = 100$

$X_i = \begin{cases} 1 & \text{i-th toss is heads} \\ 0 & \text{i-th toss is tails} \end{cases}$ $\qquad \mu = EX_i = 1/2$, $\sigma^2 = \operatorname{Var} X_i = \frac{1}{2}(1 - \frac{1}{2}) = \frac{1}{4}$

CLT: $\bar{X}_n = \frac{S_n}{n} \overset{\text{appr.}}{\sim} N(\mu, \sigma^2/n) = N(\frac{1}{2}, \frac{1}{400}) = N(0.5, (0.05)^2)$

Note $\mu_{\bar{X}_n} = \mu = 0.5$, $\sigma_{\bar{X}_n} = \frac{\sigma}{\sqrt{n}} = \frac{1}{2 \times 10} = \frac{1}{20} = 0.05$

Want $P(S_n \geq 60) = P(\bar{X}_n \geq \frac{60}{100}) = P(\bar{X}_n \geq 0.6)$

Using CLT: $P(\bar{X}_n \geq 0.6) \cong P(Z \geq \frac{0.6 - 0.5}{0.05})$

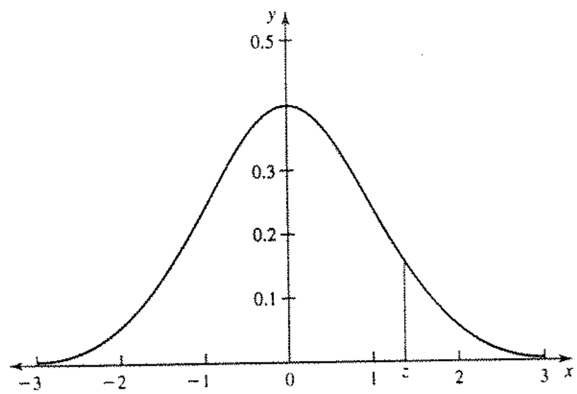
$\left(Z = \frac{\bar{X}_n - 0.5}{0.05} \overset{\text{appr.}}{\sim} N(0,1)\right)$

$= P(Z \geq 2) = 1 - \underset{\text{(Table)}}{\phi(2)} = 1 - .9772$

$= \boxed{0.0228}$

TABLE OF THE STANDARD NORMAL DISTRIBUTION

Areas under the Standard Normal Curve from $-\infty$ to z (see Figure B.1).



◀ **Figure B.1**

z	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5754
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7258	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7518	.7549
0.7	.7580	.7612	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7996	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986