

MATH 131B (Winter 2003, Lecture 1)
Instructor: Roberto Schonmann
Midterm Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	Total
Score						

1) (10 points) Let $\{(x_n, y_n)\}$ be a sequence of points in a rectangle $S = [a, b] \times [c, d]$. Prove that $\{(x_n, y_n)\}$ has a subsequence that converges to a point of S . [Hint: use the corollary to the Bolzano-Weirstrass theorem which states that any sequence $\{z_n\}$ of real numbers in a finite closed interval $[r, s]$ has a subsequence that converges to a point of $[r, s]$.]

By that corollary to BW $\exists \{x_{n_k}\}$ s.t.

$$x_{n_k} \rightarrow x, \text{ some } x \in [a, b]. \quad (\text{I})$$

By that same corollary to BW, the sequence $\{y_{n_k}\}$ has a subsequence

$$\{y_{n_{k_j}}\} \text{ s.t. } y_{n_{k_j}} \rightarrow y, \text{ some } y \in [c, d]. \quad (\text{II})$$

Now $x_{n_{k_j}} \rightarrow x$ also, from (I).

Using this and (II), we get

$$(x_{n_{k_j}}, y_{n_{k_j}}) \rightarrow (x, y) \in [a, b] \times [c, d].$$

2) (10 points) Suppose that f and g are discontinuous functions from \mathbb{R} to \mathbb{R} , and that $\{f_n\}$ is a sequence of functions from \mathbb{R} to \mathbb{R} such that $f_n \rightarrow f$ pointwise. Can you conclude that $gf_n \rightarrow gf$ pointwise? Prove your answer. [No credit will be given for a right answer given for the wrong reason.]

Yes, $gf_n \rightarrow gf$ pointwise.

Pf: $\forall x \in \mathbb{R} \quad f_n(x) \rightarrow f(x)$
(since $f_n \rightarrow f$ pointwise)

Hence $g(x) f_n(x) \rightarrow g(x) f(x) \quad \forall x \in \mathbb{R}$,
(by a property of sequences of real numbers: $a_n \rightarrow a \Rightarrow c \cdot a_n \rightarrow c \cdot a$).

But this means $gf_n \rightarrow gf$ pointwise.

3) (10 points) Prove the theorem that states that if $\{f_n\}$ is a sequence of continuous functions on $[a, b]$ and $f_n \rightarrow f$ uniformly, then

$$\int_a^b f_n(x) dx \rightarrow \int_a^b f(x) dx.$$

[Recall that this is a theorem proved in Section 5.2.]

Given $\varepsilon > 0 \quad \exists N$ s.t.

$$n \geq N \quad \Rightarrow \quad |f_n(x) - f(x)| \leq \frac{\varepsilon}{b-a} \quad \forall x \in [a, b]$$

$$\Rightarrow \left| \int_a^b f_n(x) dx - \int_a^b f(x) dx \right| = \left| \int_a^b (f_n(x) - f(x)) dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)| dx \leq \int_a^b \frac{\varepsilon}{b-a} dx$$

$$= \frac{\varepsilon}{b-a} (b-a) = \varepsilon \quad \square$$

4) (10 points) For $f \in C[a, b]$, define $\|f\|_1 = \int_a^b |f(x)| dx$. Show that $\|\cdot\|_1$ satisfies the triangle inequality

$$\|f + g\|_1 \leq \|f\|_1 + \|g\|_1.$$

$$\|f + g\|_1 = \int_a^b |f(x) + g(x)| dx$$

$$\stackrel{\textcircled{1}}{\leq} \int_a^b (|f(x)| + |g(x)|) dx$$

$$\stackrel{\textcircled{2}}{=} \int_a^b |f(x)| dx + \int_a^b |g(x)| dx$$

$$= \|f\|_1 + \|g\|_1.$$

① triangle ineq. for $|\cdot|$; property of \int .

② linearity of \int .

5) (10 points) Suppose that (\mathcal{M}, ρ) is a metric space with the property that for all $x, y \in \mathcal{M}$, $\rho(x, y) \in \{0, 1, 2, 3, \dots\}$. Show that if $\{x_n\}$ is a sequence of points in \mathcal{M} which converges to a point $x \in \mathcal{M}$, then there exists N such that

$$n \geq N \Rightarrow x_n = x.$$

Take $\varepsilon = 1/2$. Since $x_n \rightarrow x$, $\exists N$ s.t.

$$n \geq N \Rightarrow \rho(x_n, x) \leq \varepsilon = 1/2$$

$$\stackrel{\textcircled{1}}{\Rightarrow} \rho(x_n, x) = 0$$

$$\stackrel{\textcircled{2}}{\Rightarrow} x_n = x$$

① since $\rho(x_n, x) \in \{0, 1, 2, \dots\}$.

② property of metric.