MATH 131B (Winter 2003, Lecture 1)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											

1) (10 points) Let g be a uniformly continuous function on $\mathbb{R}$ and $\{a_n\}$ be a numerical sequence which satisfies $a_n \to a$, for some $a \in \mathbb{R}$. Define f_n by $f_n(x) = g(x + a_n)$. Does $f_n \to f$ uniformly for some function f? (If your answer is yes, say what the function f is.) Prove your answer.

Yes, $f_n \to f$ uniformly, where

$f(x) = g(x+a)$.

pf: Since g is unif. cont., given $\varepsilon > 0$ $\exists\, \delta > 0$ s.t. $|y-z| \leq \delta \Rightarrow |g(y)-g(z)| \leq \varepsilon$ (I)

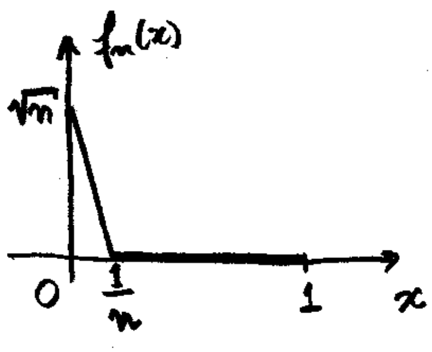
Since $a_n \to a$, given $\delta > 0$ $\exists\, N$ s.t.

$$n \geq N \Rightarrow |a_n - a| \leq \delta \quad \text{(II)}$$

From (I) and (II): $n \geq N \Rightarrow |f_n(x) - f(x)|$

$= |g(x+a_n) - g(x+a)| \leq \varepsilon$ $\square$

2) (10 points) Provide an example of a sequence of functions in C[0,1] (continuous functions from $[0,1]$ to $\mathbb{R}$), which converges according to the $\|\cdot\|_1$ norm but not according to the $\|\cdot\|_\infty$ norm.



$$f_n(x) = \begin{cases} \sqrt{n} - n^{3/2} x & 0 \le x \le \frac{1}{n} \\ 0 & \frac{1}{n} < x \le 1 \end{cases}$$

Set also $f(x) = 0 \quad 0 \le x \le 1$

$f_n \to f$ in $\|\cdot\|_1$, since $\|f_n - f\|_1$

$$= \int_0^1 |f_n(x) - 0|\, dx = \int_0^1 f_n(x)\, dx = \frac{\sqrt{n}}{2n} \to 0$$

But f_n does not converge to any fct in $\|\cdot\|_\infty$ norm, since $\|f_n\|_\infty = \sqrt{n} \to \infty$.

(Recall that if $f_n \to g$ in $\|\cdot\|_\infty$, then $\|f_n\|_\infty \to \|g\|_\infty$.)

3) (10 points) Suppose that $(\mathcal{M}, \rho)$ is a metric space, $x, y \in \mathcal{M}$, and that $\{x_n\}$ is a sequence in this metric space such that $x_n \to x$. Prove that $\rho(x_n, y) \to \rho(x, y)$.

$$\rho(x_n, y) \overset{\circledast}{\leq} \rho(x_n, x) + \rho(x, y)$$

$$\therefore \quad \rho(x_n, y) - \rho(x, y) \leq \rho(x_n, x) \qquad (\mathrm{I})$$

$$\rho(x, y) \overset{\circledast}{\leq} \rho(x, x_n) + \rho(x_n, y)$$

$$\therefore \quad \rho(x_n, y) - \rho(x, y) \geq -\rho(x, x_n) \qquad (\mathrm{II})$$

Since $x_n \to x$, $\rho(x, x_n) \to 0$. So (I), (II) and the squeeze thm. imply

$\rho(x_n, y) - \rho(x, y) \to 0$. Hence

$$\rho(x_n, y) \to \rho(x, y)$$

$\circledast$ triangle inequality.

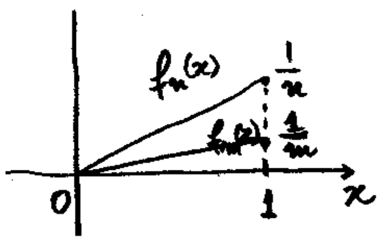
4) (10 points) Is the following metric space complete? Prove your answer. $\{f \in C[0,1] \mid f(0) = 0,\ f(1) > 0\}$, with metric corresponding to the supremum norm $\|\cdot\|_\infty$.

No.

Consider the functions $f_n(x) = x/n$, $0 \leq x \leq 1$.

$\{f_n\}$ is a Cauchy sequence in $\|\cdot\|_\infty$ since given $\varepsilon > 0$, take $N \geq 1/\varepsilon$ and then

$$m, n \geq N \Rightarrow \|f_n - f_m\|_\infty = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{N} \leq \varepsilon .$$



But if we had $f_n \to g$ in $\|\cdot\|_\infty$ for some $g \in C[0,1]$ with $g(1) > 0$, then

$$|f_n(1) - g(1)| \leq \|f_n - g\|_\infty \to 0 .$$

Hence $f_n(1) \to g(1)$. But actually $f_n(1) = \frac{1}{n} \to 0$.

5) (10 points) When V is a linear space with norm $||\cdot||$, one defines a metric on V by $\rho(v,u) = ||v-u||$. Prove that the triangle inequality for $\rho(\cdot,\cdot)$ follows from that for $||\cdot||$.

$\forall\ v, u, w \in V,$

$$\rho(v,u) = \|v-u\| = \|(v-w) + (w-u)\|$$

$$\leq \|v-w\| + \|w-v\| = \rho(v,w) + \rho(w,v)$$

triangle ineq. for $\|\cdot\|$

6) (10 points) Define $\limsup a_n$.

$$\limsup a_n = \lim_{j\to\infty} \sup\{a_m \mid n \geq j\}$$

7) (10 points) Suppose that a_j is a sequence of real numbers, such that the series $\sum_{j=1}^{\infty} a_j$ converges. Define $b_{n,j} = a_{j+n}$, $n, j = 1, 2, \ldots$. Show that for each n,

$$c_n = \sum_{j=1}^{\infty} b_{n,j}$$

is well defined (i.e., this series converges), and that $c_n \to 0$ as $n \to \infty$.

Define $S_m = \sum_{j=1}^{m} a_j$, $S_m \to S$ (since $\sum_{j=1}^{\infty} a_j$ converges)

Now define $S_{n,m} = \sum_{j=1}^{m} b_{n,j} = \sum_{j=1}^{m} a_{n+j}$

$$= \sum_{k=n+1}^{m+n} a_k = S_{m+n} - S_n$$

Hence $\lim_{m\to\infty} S_{n,m} = S - S_n$

This means that $c_n = S - S_n$ is well defined as $\sum_{j=1}^{\infty} b_{n,j}$.

Also $\lim_{n\to\infty} c_n = \lim_{n\to\infty} (S - S_n) = S - S = 0.$

8) (10 points) Prove that

$$\zeta(x) = \sum_{j=1}^{\infty} \frac{1}{j^x}$$

defines a continuous function for $x > 1$. (This is called the Riemman zeta function.)

Suppose $a > 1$. Consider first $x \in [a, \infty)$.

Then $j^x \geq j^a \therefore j^{-x} \leq j^{-a} = M_j$.

$$\sum_{j=1}^{\infty} M_j \leq \sum_{j=1}^{\infty} j^{-a} < \infty \quad (\text{since } a > 1)$$

Hence, by Weierstrass' M-test

$\sum_{j=1}^{\infty} \frac{1}{j^x}$ converges uniformly on $[a, \infty)$.

Since for each j, j^x is a continuous function of x, the limit (call it $\zeta(x)$) is continuous in $x \in [a, \infty)$. Since $a > 1$ is arbitrary these claims hold on $(1, \infty)$.

9) (10 points) Is there a power series which converges to the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

for all $x \in (-1, 1)$? Explain your answer (no partial credit for right answer for wrong reason).

No. Suppose that such a series existed, call it $\sum_{j=0}^{\infty} a_j (x-x_0)^j$. Let R be its radius of convergence. Then would need to have $(-1,1) \subset [x_0 - R, x_0 + R]$. But the limit of the series would be a fct which is C^∞ on (x_0-R, x_0+R) and hence cannot be discontinuous (as f is) at 0.

10) (10 points) Suppose that $f : \mathbb{N} \to \mathbb{Q}$ is a one-to-one onto function. Can the series $\sum_{j=1}^{\infty} f(j)$ converge? Prove your answer. (No credit for right answer for wrong reason.)

The series cannot converge.

Let's prove it by contradiction.
If it converged, then $f(j) \to 0$ as $j \to \infty$.

This would imply, given $\varepsilon = 1$ $\exists J$ s.t.

$$j \geq J \implies |f(j)| \leq 1.$$

But this implies that all the rational numbers in $(-\infty,-1) \cup (1,\infty)$ are contained in $\{f(1), f(2), \dots, f(J-1)\}$.

This is absurd since an infinite set cannot be contained in a finite set.