MATH 131A (Fall 2002, Lecture 2)
Instructor: Roberto Schonmann
Final Exam

Last Name:

First and Middle Names:

Signature:

UCLA id number (if you are an extension student, say so):

Provide the information asked above and write your name on the top of each page using a pen. You should show your work and explain what you are doing; this is more important than just finding the right answer. You can use the blank pages as scratch paper or if you need space to finish the solution to a question. Please, make clear what your solution and answer to each problem is. When you continue on another page indicate this clearly. You are not allowed to sit next to students with whom you have been studying for this exam or to your friends.

Good Luck !

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											

1) (10 points) Suppose that A is a countable set and B is an uncountable set. Can the intersection $A \cap B$ be an uncountable set? Explain your answer. (You can use without proof anything we proved about countable and uncountable sets.)

No. $A \cap B$ is a subset of A, and A is coutable. Therefore $A \cap B$ is either finite or countable.

2) (10 points) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences that satisfy $a_n \to L$, $b_n \to L$. Define

$$c_n = \begin{cases} a_n & \text{if } n \text{ is odd} \\ b_n & \text{if } n \text{ is even} \end{cases}$$

Show, using only the definition of convergence of sequences, that $c_n \to L$.

Suppose given $\varepsilon > 0$.

Since $a_n \to L$, $\exists N_1$ s.t. $n \geq N_1 \Rightarrow |a_n - L| \leq \varepsilon$ (I)

Since $b_n \to L$, $\exists N_2$ s.t. $n \geq N_2 \Rightarrow |b_n - L| \leq \varepsilon$ (II)

Define $N = \max\{N_1, N_2\}$. Then

$$n \geq N \Rightarrow \begin{cases} \text{for } n \text{ odd} & |c_n - L| = |a_n - L| \overset{(I)}{\leq} \varepsilon \\ \text{for } n \text{ even} & |c_n - L| = |b_n - L| \overset{(II)}{\leq} \varepsilon \end{cases}$$

Therefore

$$n \geq N \Rightarrow |c_n - L| \leq \varepsilon. \quad \square$$

3) (10 points) Prove or give a counterexample to the following statement. If the sequence $\{a_n\}$ is bounded then it converges.

Counterexample

$$a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

Then $|a_n| \leq 1$, so $\{a_n\}$ is bounded.

But clearly $\{a_n\}$ does not converge.

4) (10 points) Define Cauchy sequence.

$\{a_n\}$ is Cauchy if

$\forall \varepsilon > 0 \quad \exists N \quad$ s.t.

$$n, m \geq N \implies |a_n - a_m| \leq \varepsilon$$

5) (10 points) Give an example of a sequence that has exactly one ~~limit~~ point but does not converge.

$$a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ n & \text{if } n \text{ is even} \end{cases}$$

limit point: 0

but the subsequence $\{a_{2k}\}$ has

$a_{2k} = 2k \to \infty$ So a_n does not converge

6) (10 points) Suppose that f and g are two uniformly continuous functions on the same finite closed interval $[a, b]$. Does it follow that the product function fg is uniformly continuous on $[a, b]$? Prove your answer. (You can use any theorem you learned in the course.)

Yes.

f, g uniformly continuous

$\Rightarrow$ f, g continuous $\overset{①}{\Rightarrow}$ $f \cdot g$ continuous

$\overset{②}{\Rightarrow}$ $f \cdot g$ uniformly continuous.

Explanations: ① product of continuous fcts is continuous

② $f \cdot g$ is continuous on $[a, b]$, hence uniformly continuous.

7) (10 points) What is the negation of the following statement? For any $\epsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$.

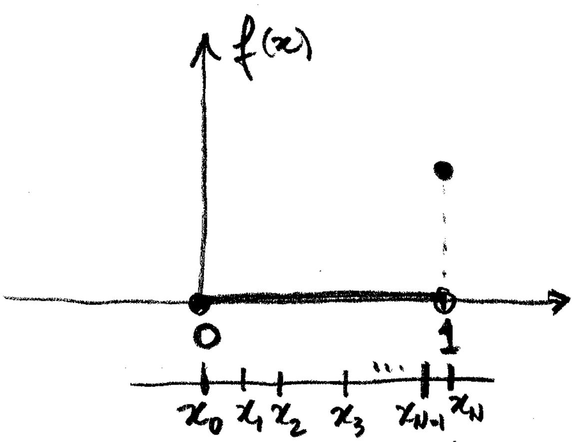
For some $\varepsilon > 0$ there is no $\delta > 0$ such that if $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

Alternative answer:

For some $\varepsilon > 0$ for all $\delta > 0$ there are x, y s.t. $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$.

8) (10 points) Give an example of a discontinuous function defined on $[0, 1]$ that is Riemann integrable, and explain how one can prove that it is Riemann integrable.

Example: $f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$



For any partition P: $x_0 < x_1 < x_2 < \dots < x_N$

$x_0 = 0, \quad x_N = 1$

have $m_i = 0 \qquad M_i = \begin{cases} 0 & \text{if } i = 1, \dots, N-1 \\ 1 & \text{if } i = N \end{cases}$

So $L_P(f) = 0 \qquad U_P(f) = x_N - x_{N-1}$

Since $x_N - x_{N-1}$ can be arbitrarily small,

$\inf_P U_P(f) = 0$. Clearly $\sup_P L_P(f) = 0$

So the condition $\inf_P U_P(f) = \sup_P L_P(f)$ holds.

9) (10 points) Suppose that the function g has domain $\mathbb{R}$ and satisfies $-1 \leq g(x) \leq 1$ for all $x \in \mathbb{R}$. Define

$$f(x) = \begin{cases} x^2\, g(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Using only the definition of derivative, compute $f'(0)$, or show that f is not differentiable at 0.

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 g(1/h)}{h}$$

$$= \lim_{h \to 0} h\, g(1/h)$$

When $h > 0$, $-h \leq h\, g(1/h) \leq h$

When $h < 0$, $h \leq h\, g(1/h) \leq -h$

In either case $-|h| \leq h\, g(1/h) \leq |h|$

By the squeeze theorem, $\lim_{h \to 0} h\, g(1/h) = 0$

So $f'(0) = 0$.

10) (10 points) State the Mean Value Theorem. (You do not have to prove it.)

Suppose f continuous on $[a,b]$ and differentiable on (a,b). Then there exists $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$