Introduction/Background

My general research interests are in algebraic topology and category theory. I specialize in broadening the techniques of equivariant stable homotopy theory.

Homotopy theory is a branch of algebraic topology that deals with topological spaces up to homotopy and related algebraic invariants, such as homotopy groups, $\pi_n$. Stable homotopy theory uses the observation that a continuous map of spaces $X \xrightarrow{f} Y$ gives rise to a map on their suspensions $\Sigma X \xrightarrow{\Sigma f} \Sigma Y$, notes that this procedure can be repeated to give maps $\Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y$, $\Sigma^3 X \xrightarrow{\Sigma^3 f} \Sigma^3 Y$, etc., and then studies phenomena that survive arbitrary suspension. Equivariant stable homotopy theory does the same procedure, except where $X$ and $Y$ are have group actions and $f$ preserves these actions.

My research is about performing equivariant stable homotopy theory but giving $X$ and $Y$ the action of a category instead of a group. The rest of this document explains in more detail what constructions I generalized, (namely those of “$G$-spectra” and “$G$-Mackey Functors”) and how I made those generalizations, and what I see as the future directions of my research program.

Equivariance, in General

Equivariance, loosely speaking, is the study of symmetries and maps that preserve symmetry. This is often modeled using group actions. For instance, for the 2-element group $C_2$, we can consider any $n$-sphere $S^n$ and have the group act via the antipodal map. In this context, the inclusion of $S^1$ into $S^2$ is equivariant because it preserves the group action. That is, applying the antipodal map of $S^1$ and then including $S^1$ into $S^2$ yields the same result as the reverse order: including $S^1$ into $S^2$ and then applying the antipodal map on $S^2$.

All of this can be neatly encoded via category theory: a group $G$ can be equivalently defined as a category with one object where every morphism happens to be an isomorphism. A $G$-action is then a functor $X : G \to ???$, where ??? is whatever category we’d like. If ??? is the category of sets, topological spaces, or complex vector spaces, then $X$ encodes the structure of a $G$-set, $G$-space, or complex representation of $G$, respectively.

In this framework, viewing group actions as functors from one category to another, there’s no reason we need the first category to be a group. Instead, we can allow the first category to be an arbitrary category $D$. (thinking ‘$D$’ for ‘diagram’) This is the setup of our generalizations.

**Definition.** [2] Given a category $D$, a $D$-set is a functor $X : D \to \text{Set}$, where $\text{Set}$ denotes the category of sets. Similiarly, a $D$-space is a functor $X : D \to \text{Top}$ where $\text{Top}$ is the category of topological spaces, and a $D$-representation is a functor $X : D \to \text{Vect}_k$ where $\text{Vect}_k$ is the category of vector spaces over a fixed field, $k$.

**Example.** Consider the category $J = \xrightarrow{s \to t}$ with two objects, $s$ and $t$, and one morphism between them, $f$. Then, a $J$-set (a functor $X : J \to \text{Set}$) is precisely the data of: a set, $X_s$, another set, $X_t$, ...
and a function $X_s \xrightarrow{X_f} X_t$. If we instead wanted a J-space, $X_s$ and $X_t$ would both be topological spaces, and $X_f$ would be a continuous function.

**Orbits, the Key to Equivariance**

For $G$-sets, we have a structure theorem known as orbit decomposition, whereby a $G$-set can be uniquely decomposed into a disjoint union of orbits, $G$-sets that are isomorphic to $G/H$ for some subgroup $H \leq G$. Orbit decomposition holds for $D$-sets, too, but for this we’ll need a generalized notion of orbit, due to Emmanuel Dror Farjoun:[2]

**Definition.** Given a category $D$, we say that a $D$-set $X : D \to \text{Set}$ is an orbit if the colimit of $X$ is terminal. (that is, a one-point set).

**Example.** If $D$ is a group, then a $D$-orbit (up to isomorphism) is of the form $D/H$ for some subgroup, $H$. However, if $D$ is instead the category $J = s \xrightarrow{f} t$, then a $D$-orbit (a.k.a J-orbit) is a $J$-set $X : J \to \text{Set}$ where $X_s$ can be any set, $X_t$ is a one-point set, and the map $X_s \xrightarrow{X_f} X_t$ is thus uniquely determined. In other words, the category of $J$-orbits is equivalent to the category of sets!

**Orbits vs. Subgroups**

When working with group actions, we can “restrict” our group action to that of a subgroup. In our categorical language, this means taking a $G$-set $X : G \to \text{Set}$ and precomposing it with the inclusion $H \hookrightarrow G$ to get an $H$-set, $\text{Res}(X) : H \to \text{Set}$. To generalize restriction to $D$-sets, one might naively try to repeat the procedure above, with $H$ instead becoming a subcategory of $D$. However, this would miss an important relationship between orbits and subgroups of $G$:

**Definition.** Given a $G$-set $Y : G \to \text{Set}$, the slice category of $G$-sets over $Y$, $\text{Set}_{G/Y}^G$, is the category whose objects are $G$-sets $X : G \to \text{Set}$ together with an (equivariant) map $X \xrightarrow{\alpha_X} Y$. The morphisms of $\text{Set}_{G/Y}^G$ are (equivariant) maps $X_1 \xrightarrow{i} X_2$ such that $\alpha_{X_2} \circ i = \alpha_{X_1}$.

The classical relevance of slice categories is:

**Theorem.** Given a subgroup $H \leq G$, the category of $H$-sets, $\text{Set}^H$, is equivalent (in the categorical sense) to $\text{Set}_{G/H}^G$.

I’ve proven a generalization to the $D$-case, which uses the notion of translation category:

**Definition.** Given a $D$-set $T : D \to \text{Set}$, its translation category, $\text{B}_D(T)$, has objects given by elements of $T$ and where homsets are defined by $\text{Hom}_{\text{B}_D(T)}(a, b) = \{ f \in D | T_f(a) = b \}$.

**Theorem.** Given a category $D$ and $D$-set $T$, the slice category of $D$-spaces over $T$, $\text{Top}_{D/T}$ is equivalent to the category of $\text{B}_D(T)$-spaces, $\text{Top}_{\text{B}_D(T)}$.

I’m introducing translation categories and slice categories to do do away with the distinction between $H$ and $G/H$: $H$ is categorically equivalent to $\text{B}_G(G/H)$. Hence, the category of $H$-sets, $\text{Set}^H$, is equivalent to the category of $\text{B}_G(G/H)$-sets, $\text{Set}^{\text{B}_G(G/H)}$, which is exactly what we get from my theorem. **In other words, we can generalize equivariant theory using only the data of orbits, ignoring the data of subgroups.**
Stability and Representation Theory

In stable homotopy theory, one studies the suspension function, $\Sigma$, which sends a pointed topological space $X$ to its reduced suspension $\Sigma X$. Up to homeomorphism, $\Sigma X$ can be viewed as the smash product of $X$ and $S^1$, $X \wedge S^1$. Higher suspensions $\Sigma^k X$ are similarly homeomorphic to $X \wedge S^k$.

When working equivariantly, we consider the possibility that these spheres have a non-trivial group action by being “representation spheres”:

**Definition.** Given a group $G$, an **orthogonal representation** of $G$ is a functor $V : G \to \mathcal{O}$, ($\mathcal{O}$ is the category whose objects are finite dimensional real vector spaces and whose morphisms are orthogonal maps) We denote the category of orthogonal $G$-representations by $\text{Orthog}^G$.

**Definition.** Given an orthogonal $G$-representation $V$, its **representation sphere**, $S^V$ is the (pointed) $G$-space given by one-point compactification, where the new point has trivial $G$-action.

**Definition.** Given a pointed $G$-space $X$ and $G$-representation sphere $S^V$, the $V$**th**-**suspension of $X$ is $\Sigma^V := S^V \wedge X$, where the smash product is computed object-wise.

These readily admit generalizations:

**Definition.** Given a category $D$, an **orthogonal representation** of $D$ is a functor $V : D \to \mathcal{O}$. Like for the group case, we denote the category of orthogonal $D$-representations by $\text{Orthog}^D$.

**Definition.** Given an orthogonal $D$-representation $V$, its representation sphere, $S^V$ is the (pointed) $D$-space given by object-wise one-point compactification. (The action of $D$ sends the new point of $S^V(d)$ to the new point of $S^V(e)$ for all morphisms $d \to e$ in $D$.)

**Definition.** Given a pointed $D$-space $X$ and $D$-representation sphere $S^V$, the $V$**th**-**suspension of $X$ is $\Sigma^V := S^V \wedge X$, where the smash product is computed object-wise.

$D$-Spectra

When working stably (equivariantly or not) we’d like a concise way of keeping track our spaces up to all possible suspensions. In the equivariant case, this is done by $G$-spectra, and by spectra in the non-equivariant case. There are multiple distinct definitions of $G$-spectra in the literature,[4][6][7] but they all yield equivalent categories. I now give my generalization, which is most similar to an approach by Lewis and May:

**Definition.** Given a locally finite category $D$, a $D$-**spectrum** is a functor $Y : \text{Orthog}^D \to \text{Top}^D$, together with compatible maps $Y(V) \wedge S^W \to Y(V \oplus W)$.

$D$-Mackey Functors

As in the non-equivariant case, we have an equivariant notion of homotopy groups $\pi_n$. Naively, we could define our $n$**th**-homotopy groups of a $G$-spectrum $X$ as the homotopy classes of $G$-equivariant maps from $S^n$ to $X$, $[S^n, X]^G$. However, this would miss a lot of structure: since $S^n$ has trivial $G$-action, maps out of it can only detect points in $X$ that have trivial orbit type, $G/G$. To detect the
points of non-trivial orbit type \( G/H \), we could use \([S^n \wedge G/H_+, X]^G\). But there are also structure maps between these “homotopy groups,” encoding the fact that an equivariant map \( G/H \to G/K \) allows points of orbit type \( G/K \) to be detected by maps out of \( G/H \). All of this can be assembled as a \( G\text{-Mackey Functor} \), originally due to Dress[1] and with refinement by Lindner[5]. For conciseness, I’ll only give my definition of a \( D\text{-Mackey Functor} \), which is equivalent to Lindner’s description when \( D \) is a group:

**Definition.** The \( D\text{-Burnside Category} \( \mathcal{B}_D \), is the category whose objects are finite \( D \)-sets and where \( \text{Hom}_{\mathcal{B}_D}(T_1, T_2) \) is the group completion of \( \{ T_1 \leftarrow A \to T_2 \mid A \text{ is a finite } D\text{-set} \} \)

**Definition.** A \( D\text{-Mackey Functor} \) is an additive functor \( M : \mathcal{B}_D \to \text{Ab} \). (Note: Since a \( D\text{-Mackey Functor} \) is additive, it is determined by its values on the full subcategory of \( D \)-orbits. For this reason, some authors only define \( G\text{-Mackey Functors} \) on orbits \( G/H \).)

Finally, the reason for all this structure is:

**Theorem.** If \( G \) is a finite group, then for any \( n \in \mathbb{N} \) and \( G \)-spectrum \( X \), there is a \( G\text{-Mackey Functor} \) \( \pi_n \) given by \( \pi_n(T) = [S^n \wedge T_+, X]^G \). (The structure maps are given as compositions of certain “restriction” and “transfer” maps, which would require more exposition than I have room.)

I believe that the above theorem admits a generalization:

**Conjecture 1.** If \( D \) is a locally-finite category, then for any \( n \in \mathbb{N} \) and \( D \)-spectrum \( X \), there is a \( D\text{-Mackey Functor} \) \( \pi_n \) defined on objects by \( \pi_n(T) = [S^n \wedge T_+, X]^D \).

I plan to prove this conjecture by the end of November 2021. The issue has been with the structure maps. I’ve come up with suitable candidate maps that generalize what happens in the finite group case, but I haven’t yet been able to show they’re compatible in general. But I have shown the conjecture holds in at least one non-group case:

**Theorem.** Conjecture 1 holds when \( D = J = s \xrightarrow{f} t \).

**J-Mackey Functors**

My success in working with \( J\text{-Mackey Functors} \) has been largely due to the following computation:

**Theorem.** For \( J = s \xrightarrow{f} t \), a \( J\text{-Mackey Functor} \) is determined by its values on a (non-full) subcategory of \( \mathcal{B}_J \) equivalent to \( \text{Mat}(\mathbb{N}) \), the category whose objects are the elements of \( \mathbb{N} \) and where morphisms between \( m \) and \( n \) are \( \mathbb{N} \)-valued \( n \times m \) matrices. (Composition is matrix multiplication.) This concrete description sets me up for:

**Objective 2.** Classify all \( J\text{-Mackey Functors} \).

So far, my main approach has been in trying to find a nice “generators and relations” description of \( \text{Mat}(\mathbb{N}) \). This is a combinatorial problem with many parts that can be stated elementarily. (i.e. “how many ways are there to write the \( \mathbb{N} \)-valued matrix \( A \) as a composition of \( k \) matrices of the following type...?”) For this reason, I’m interested in pursuing this as a research project with undergraduate and/or graduate students.
**Other Future Research Goals**

**Motivating Goal 3.** Generalize the tools used in the classification of $J$-Mackey Functors to enable potential classification of $D$-Mackey Functors for other categories, $D$. For this task, classifying $J$-Mackey Functors was a natural start because $J$ encodes as little data as possible: just one non-trivial morphism. Hence, the patterns that appear in $J$-Mackey Functors should be pervasive in general $D$-Mackey Functors. In particular:

**Question 4.** How can one do equivariant stable homotopy for infinite groups, such as $\mathbb{Z}$? The case of $\mathbb{Z}$ seems especially ripe to examine using $J$-spectra. This is because $\mathbb{Z}$ is generated by a single element and the functor from $J = s \xrightarrow{f} t$ to $\mathbb{Z}$ that sends $f$ to 1 picks out that element. Thus, in theory, understanding the stable homotopy theory for $J$ gives a lens to understand it for $\mathbb{Z}$. If the techniques can be pushed far enough, being able to work with infinite discrete groups would dramatically affect the field of equivariant stable homotopy theory.

Regardless of the success of $D$-spectra in dealing with infinite discrete groups, it would still be helpful if $D$ were allowed to encode more data in other ways:

**Objective 5.** Expand definitions of $D$-Spectra and $D$-Mackey Functors to include $D$ that have some form of topological structure, generalizing Lie groups.

This isn’t the only chance for possible topological enrichment:

**Objective 6.** Explore what it would mean to have a $D$-Spectral Mackey Functor. In the $G$-equivariant case, a Spectral Mackey Functor is a Mackey Functor that takes values in spectra, rather than in abelian groups. By a theorem of Guillou and May,[3] the category of $G$-Spectral Mackey Functors is equivalent to the category of orthogonal $G$-spectra, leading to:

**Goal Theorem 7.** Show that the category of $D$-Spectral Mackey Functors is equivalent to the category of $D$-spectra.

On the other hand, I’d like to expand the possible $D$ with more categorical data:

**Motivating Goal 8.** Generalize $D$-spectra and $D$-Mackey Functors to include $D$ that have higher categorical data, such as that of a 2-category or $\infty$-category. This would enable equivariant structures to handle even more information.

Lastly, for a different enhancement of $G$-Mackey Functors, there is the related notion of $G$-Tambara Functors, which are functors $\mathcal{M} : \mathcal{U}_G \to \text{Ab}$, where $\mathcal{U}_G$ is the category whose objects are finite $G$-sets and where $\text{Hom}_{\mathcal{U}_G}(T_1, T_2)$ is the group completion of $\{T_1 \leftarrow A \rightarrow B \rightarrow T_2 \mid A, B$ are finite $G$-sets}. By replacing the category of finite $G$-sets by the category of finite $D$-sets, we get a notion of $D$-Tambara Functor, leading to the goal:

**Objective 9.** Investigate the structure of $D$-Tambara Functors.
References Cited


