TWO IMPORTANT THEOREMS THAT ARE REALLY ONE

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Abstract: An elementary proof of the equivalence of the Intermediate Value Theorem and a fixed point theorem on the real line illustrates the equivalence of the Brouwer Fixed Point Theorem and the Poincaré-Miranda Theorem.

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Section 2.8 of the widely used textbook *Single Variable Calculus* by John Rogawski and Colin Adams [4] is devoted to a theorem that the book states this way:

Theorem 1. *(Intermediate Value Theorem) If the function $f(x)$ is continuous on $[a, b]$ with $f(a) \neq f(b)$ and y is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = y$.*

The point y is "between" $f(a)$ and $f(b)$ means either that $f(a) < y < f(b)$ or that $f(a) > y > f(b)$, depending on how $f(a)$ and $f(b)$ are related to each other.

In order to avoid "between", with its two cases, the theorem is sometimes stated in this neater way:

Theorem 2. *(Intermediate Value Theorem) If the function $g(x)$ is continuous on $[a, b]$ and $g(a)g(b) < 0$, then $g(c) = 0$ for some $c \in (a, b)$.*

What do I mean by claiming that Theorem 2 is also the Intermediate Value Theorem? Theorems 1 and 2 are the same theorem in the sense that they are *equivalent*, which means that each can be proved directly from the other. Certainly Theorem 2 is a special case of Theorem 1 since one of $g(a)$ and $g(b)$ is positive and the other is negative. But if you assume that Theorem 2 is true, then so is Theorem 1 because if $f(x)$ satisfies the hypotheses of Theorem 1, then $g(x) = f(x) - y$ satisfies the hypotheses of Theorem 2 and $0 = g(c) = f(c) - y$ means that $f(c) = y$.

In that textbook, Problem 32 of Section 2.8 is to prove

Theorem 3. *Assume that $f(x)$ is continuous and that $a \leq f(x) \leq b$ for $a \leq x \leq b$. Then $f(c) = c$ for some $c \in [a, b]$.*[1]

Proof. If $f(a) = a$ or $f(b) = b$ there is nothing to prove so we can assume $f(a) > a$ and $f(b) < b$, then $g(x) = f(x) - x$ satisfies the hypotheses of Theorem 2 and $g(c) = 0$ implies that $f(c) = c$. □

[1] In [4] the problem is stated for $a = 0$ and $b = 1$, but this more general notation is consistent with the way that we are stating the other results and the solution is really the same either way.

The point c such that $f(c) = c$ is called a *fixed point* of the function f. Thus Theorem 3 is an example of a *fixed point theorem.*

Now we'll go a bit beyond what is in [4]. The following fixed point theorem has somewhat more general hypotheses because, rather than functions that take $[a, b]$ to itself, it concerns a class of function from $[a, b]$ to the real line.

Theorem 4. *If a function $f(x)$ is continuous on $[a, b]$ with $f(a) \geq a$ and $f(b) \leq b$, then there exists $c \in [a, b]$ such that $f(c) = c$.*

Proof. Again there is nothing to prove unless $f(a) > a$ and $f(b) < b$. Let $g(x) = x - f(x)$, then $g(a)g(b) < 0$ so by Theorem 2 there exists $c \in [a, b]$ such that g(c) = c - f(c) = 0 and thus $f(c) = c$. $\square$

So Theorem 4 is also a consequence of the Intermediate Value Theorem. But the authors of [4] could have gone a step further, because we will show by a similar sort of argument that this fixed point theorem in turn *implies* the Intermediate Value Theorem and thus those theorems are equivalent.

Theorem 5. *Given that for a function $g(x)$ continuous on $[a, b]$ with $g(a) \geq a$ and $g(b) \leq b$ there exists $c \in [a, b]$ such that $g(c) = c$, then $g(a)g(b) < 0$ implies $g(c) = 0$ for some $c \in (a, b)$.*

Proof. If $g(a) > 0$ and $g(b) < 0$, let $f(x) = g(x) + x$ which satisfies the hypotheses of Theorem 4 so $f(c) = g(c) + c = c$ and thus $g(c) = 0$. The other possibility is that $g(a) < 0$ and $g(b) > 0$ in which case let $f(x) = x - g(x)$ and then $f(c) = c - g(c) = c$ means $g(c) = 0$. Therefore, the fixed point theorem implies the Intermediate Value Theorem and consequently these theorems are equivalent. $\square$

Theorem 3 is a special case of our first "important theorem", as promised in the title: the Brouwer Fixed Point Theorem. Instead of the function $f(x)$ that takes the real line to itself, the theorem discovered by L. E. J. Brouwer early in the 20th century concerns a function that takes Euclidean n-space $\mathbb{R}^n$ to itself.

Theorem 6. *(Brouwer Fixed Point Theorem) Let*

$$B^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \colon a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}.$$

Suppose $f(x)$ is continuous and $f(x) \in B^n$ for all $x \in B^n$, then there exists $c \in B^n$ such that $f(c) = c$.

Here is some evidence that the Brouwer Fixed Point Theorem is important: it has been mentioned in more than 500 research papers. Since that list includes more than 50 papers published since 2015, we can see that this theorem is related to research topics of current interest and that is why it is important.

Theorem 3, the solution to Problem 32, is the $n = 1$ case of the Brouwer Fixed Point Theorem. The Intermediate Value Theorem is also the $n = 1$ case of a more general theorem, one that was stated by Henri Poincaré in 1883. It also concerns a function $g(x)$ that takes $\mathbb{R}^n$ to itself. We can think of such a function as a vector-valued function $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$.

Theorem 7. *Suppose $g(x) = (g_1(x), g_2(x), \ldots, g_n(x))$ is continuous on B^n such that*

$$g_i(x_1, \ldots x_{i-1}, a_i, x_{i+1}, \ldots, x_n) \leq 0$$

and

$$g_i(x_1, \ldots x_{i-1}, b_i, x_{i+1}, \ldots, x_n) \geq 0$$

where $a_i < b_i$ for $i = 1, 2, \ldots, n$, then $g(c) = g(c_1, c_2, \ldots, c_n) = (0, 0, \ldots, 0) = \mathbf{0}$ for some $c \in \mathbb{R}^n$.

In 1940, Carlo Miranda [3] proved that Poincaré's theorem is equivalent to the Brouwer fixed point theorem. As a result, Theorem 6 became known as the Poincaré-Miranda Theorem[2], and that is our other "important theorem". The paper [3] has appeared in the list of references of over 120 research papers, more than half of them published since 2015, so it is very much a part of contemporary mathematical research.

[2] To find out more about the Poincaré-Miranda Theorem, in particular how it is proved, see [1]

An elegant demonstration of the equivalence of the Brouwer Fixed Point Theorem and the Poincaré-Miranda Theorem has recently been published [2].

Thus, by proving that the fixed point theorem of Problem 32 of [4] is equivalent to the Intermediate Value Theorem, we have not only an interesting exercise that is related to that familiar theorem, but we have actually proved the $n = 1$ case of Miranda's surprising discovery that the Brouwer Fixed Point Theorem and an n-dimensional version of the Intermediate Value Theorem are equivalent. Those two important theorems state the same mathematical fact in different forms and so, as our title states, they really are one.[3]

References

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