# ON THE TOPOLOGY OF $n$-VALUED MAPS


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**Abstract**

This paper presents an exposition of the topological foundations of the theory of $n$-valued maps. By means of proofs that exploit particular features of $n$-valued functions, as distinct from more general classes of multivalued functions, we establish, among other properties, the equivalence of several definitions of continuity. The exposition includes an exploration of the role of configuration spaces in the study of $n$-valued maps. As a consequence of this point of view, we extend the classical Splitting Lemma, that is central to the fixed point theory of $n$-valued maps, to a characterization theorem that leads to a new type of construction of non-split $n$-valued maps.



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## 1 Introduction

An $n$-valued map $\phi\colon X \multimap Y$ is a continuous multivalued function that associates to each $x \in X$ an unordered subset of exactly $n$ points of $Y$. The fixed point theory of $n$-valued maps $\phi\colon X \multimap X$ has been a topic of considerable interest in recent years and there is much research activity at the present time [3] - [12], [15], [16]. The purpose of this paper is to present the topological foundations of this subject.

In Section 2 we assume as few hypotheses on the topological spaces $X$ and $Y$ as necessary in order to obtain the results. For a general multivalued function, continuity is defined as the satisfaction of two independent conditions called upper and lower semi-continuity. The section begins by proving that, in the case of $n$-valued functions, lower semi-continuity implies upper semi-continuity (but not the converse) and thus this single condition is sufficient for the definition of an $n$-valued map. We use this result to prove that, in the setting of $n$-valued functions, the general definition of continuity for multivalued functions is equivalent to a classical definition based on convergence of sequences.

The *graph* $\Gamma(\phi)$ of a multivalued function $\phi\colon X \multimap Y$ is the set

$$\Gamma(\phi) = \{(x, y) \in X \times Y : y \in \phi(x)\},$$

topologized as a subspace of $X \times Y$. The graph plays a significant role in the general theory of multivalued functions, see for instance [17]. If $\phi$ is an $n$-valued map, then its graph has a very useful property: the projection $p_X : \Gamma(\phi) \to X$ defined by $p_X(x, y) = x$ is a finite covering space. This covering space has played a key role in the construction of $n$-valued maps.

In Section 3 we characterize the splitting of an $n$-valued map that extends the classical Splitting Lemma. An $n$-valued map $\phi\colon X \multimap Y$ is *split* if there are single-valued maps $f_1, \ldots, f_n\colon X \to Y$ such that $\phi(x) = \{f_1(x), \ldots, f_n(x)\}$ for all $x \in X$. The Splitting Lemma states that if $X$ is simply connected and locally path connected, then every $n$-valued map $\phi\colon X \multimap Y$ is split. The Splitting Lemma was first proved by Banach and Mazur in [1] to obtain conditions that imply that a local homeomorphism is a global homeomorphism.[1] The Nielsen fixed point theory of an $n$-valued map $\phi\colon X \multimap X$ presented by Schirmer in [23] - [25] depends on a version of the Splitting Lemma which she obtained from a more general result concerning multivalued functions on simply connected compact metric spaces due to O'Neill [21]. Another consequence of the Splitting Lemma, in [6], is that an $n$-valued map $\phi\colon X \multimap Y$ of finite simplicial complexes induces a chain map of their simplicial chain complexes. It follows that the integer Lefschetz number of an $n$-valued self-map of a finite simplicial complex is well-defined and the Lefschetz Fixed Point Theorem can be generalized to such maps.

The configuration space $D_n(Y)$ of a space $Y$ is the set of all unordered sets of $n$ points of $Y$. The set of $n$-valued functions from a space $X$ to a space $Y$ is in one-to-one correspondence with the

[1] The proof in [1] is for $n$-valued maps of metric spaces. However, the authors state that the same result can be generalized to topological spaces.

set of single-valued functions from $X$ to $D_n(X)$ where, to a given $n$-valued function $\phi\colon X \multimap Y$ we associate the single-valued function $\Phi\colon X \to D_n(Y)$ defined by $\Phi(x) = \{\phi(x)\}$. We prove in Section 3 that, under suitable hypotheses, $\phi$ is an $n$-valued map if and only if $\Phi$ is continuous with respect to a natural topology on $D_n(Y)$.[2]

Thus for $\phi\colon X \multimap Y$ an $n$-valued map, the map $\Phi$ induces a homomorphism $\Phi_{\#}\colon \pi_1(X) \to \pi_1(D_n(Y))$. We prove that this homomorphism characterizes the splitting of $n$-valued maps. If $X$ is simply connected, we obtain the Splitting Lemma as a consequence of this result. Moreover, we demonstrate that hypotheses on $X$ more general than simple connectedness can be sufficient to imply that all $n$-valued maps $\phi\colon X \multimap Y$ are split.

The group $\pi_1(D_n(Y))$ may be identified with the braid group of $n$ strands on $Y$ [13]. This identification facilitates the construction in Section 3 of non-split $n$-valued self-maps $\phi\colon X \multimap X$ that are of interest in the fixed point theory of such maps, complementing the constructions in [12] and [8].

Finally, in Section 4 we add the hypothesis that the range $Y$ of an $n$-valued function $\phi\colon X \multimap Y$ is a metric space. Therefore, for $x, x' \in X$, the Hausdorff distance between $\phi(x)$ and $\phi(x')$ as subsets of $Y$ is well-defined. We can use the Hausdorff distance to define a form of continuity for $\phi$ and we prove that, in this setting, it is equivalent to the other definitions of continuity that we have presented. As a tool for this proof, we use the concept of the gap of an $n$-valued map that was introduced by Schirmer in [23] and we establish its properties.

Most of the results in this paper are not new and are actually special cases of established facts. The $n$-valued maps belong to a class of multivalued functions called weighted maps, and these have been extensively studied [22]. Our goal is to furnish a person interested in the topic of $n$-valued maps with arguments for their basic topological properties that are as simple and elementary as possible, without any dependence on a more general theory.

However, although the primary purpose of this paper is the exposition of known facts about the topology of $n$-valued maps, in the course of its preparation some interesting features of such maps appeared that, it seems, have not been noted previously. We call the reader's attention to the property of $n$-valued maps presented as Proposition 2.1, that lower semi-continuity implies upper semi-

[2]The configuration space viewpoint was introduced into the fixed point theory of $n$-valued maps in [15] and [16]. However, configuration spaces have been a well-established tool in the fixed point and coincidence theory of single-valued maps, see for instance [14] where the configuration space of ordered pairs is presented as the complement of the diagonal of the Cartesian square of a space.

continuity. This property not only simplifies the proofs of some of the subsequent results, it is quite special to $n$-valued maps, as an example in Section 2 demonstrates. Moreover, while the Splitting Lemma of Corollary 3.1 has been a basic tool since the initiation of the fixed point theory of $n$-valued maps by Schirmer, the Characterization Theorem 3.1 is a considerable extension of this classical result and, as demonstrated in Section 3, it has several significant consequences.

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## 2 Functions of Hausdorff spaces

Throughout this section, we assume only that the spaces are Hausdorff.

A multivalued function $\phi\colon X \multimap Y$ is *lower semi-continuous* if $V$ an open subset of $Y$ implies that the set $\{x \in X\colon \phi(x) \cap V \neq \emptyset\}$ is open in $X$ and *upper semi-continuous* if $V$ open in $Y$ implies that the set $\{x \in X\colon \phi(x) \subseteq V\}$ is open in $X$. Since we will need terminology that distinguishes several definitions of continuity, we will call $\phi$ *multicontinous* if it is both lower and upper semi-continuous.[3]

Let $\phi\colon X \multimap Y$ be a lower semi-continuous function and $x^{(0)} \in X$ with $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$. Let $V_1, \ldots, V_n$ be disjoint open subsets of $Y$ such that $y_j^{(0)} \in V_j$. Since $\phi$ is lower semi-continuous, for each $j$ there is an open subset $U_j$ of $X$ containing $x^{(0)}$ such that if $x \in U_j$, then $\phi(x) \cap V_j \neq \emptyset$ and we define $U(x^{(0)}, \{V_j\}) = \bigcap_{j=1}^n U_j$. If $x \in U(x^{(0)}, \{V_j\})$, then $\phi(x) \subseteq \bigcup_{j=1}^n V_j$ and we can number the points of $\phi(x) = \{y_1, \ldots, y_n\}$ so that $y_j \in V_j$.

**Proposition 2.1.** *Let $\phi\colon X \multimap Y$ be an $n$-valued function. If $\phi$ is lower semi-continuous it is also upper semi-continuous and therefore multicontinuous, that is, an $n$-valued map.*

*Proof.* Let $x^{(0)} \in X$ and let $V$ be an open subset of $Y$ containing $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$. Let $V_1, \ldots, V_n$ be disjoint open subsets of $V$ such that $y_j^{(0)} \in V_j$ for each $j$. If $x \in U(x^{(0)}, \{V_j\})$, then $\phi(x) \subseteq \bigcup_{j=1}^n V_j \subseteq V$ so $\{x \in X\colon \phi(x) \subseteq V\}$ is open in $X$ and we have proved that $\phi$ is upper semi-continuous. $\square$

[3] In Wikipedia, these concepts are called upper and lower *hemi-continuous* and the words upper and lower semi-continuous are reserved for certain generalizations of continuity for single-valued functions which, together, imply continuity in the usual sense. However, in both [2] and [17] the word semi-continuous is used as it is here.

To demonstrate that an upper semi-continuous $n$-valued function need not be continuous, define a 2-valued function $\phi\colon \mathbb{R} \multimap \mathbb{R}$ by $\phi(x) = \{x, x/2\}$ if $x \neq 0$ and $\phi(0) = \{0, 1\}$. If $V \subseteq \mathbb{R}$ is an open set containing $\phi(0)$, then there exists $\epsilon > 0$ such that $(-\epsilon, \epsilon) \subseteq V$ and if $|x| < \epsilon$, then $\phi(x) \subset (-\epsilon, \epsilon) \subseteq V$. Since $\phi$ is continuous at $x \neq 0$ and thus upper semi-continuous, we conclude that $\phi$ is an upper semi-continuous function. However,

$$\{x \in \mathbb{R}\colon \phi(x) \cap (1/2, 3/2) \neq \emptyset\} = (1/2, 3) \cup \{0\}$$

which is not open. Therefore $\phi$ is not lower semi-continuous and thus not continuous.

The property of $n$-valued maps presented in Proposition 2.1 fails immediately beyond the class of $n$-valued functions. For instance, define an "almost 2-valued function" $\phi\colon [0,1] \multimap [0,1]$ by $\phi(x) = \{0, 1\}$ if $x \neq 0$ and $\phi(0) = \{0\}$. Then $\phi$ is lower semi-continuous because if $U_0$ and $U_1$ are neighborhoods of 0 and 1 respectively, then $\{x\colon \phi(x) \cap U_0 \neq \emptyset\} = [0, 1]$ and $\{x\colon \phi(x) \cap U_1 \neq \emptyset\} = (0, 1]$. But $\{x\colon \phi(x) \subseteq U_0\} = \{0\}$ so $\phi$ is not upper semi-continuous.

The function $\phi\colon X \multimap Y$ is *locally split* if, given $x^{(0)} \in X$ there is a neighborhood $U$ of $x^{(0)}$ and maps $g_1, \ldots, g_n\colon U \to Y$ such that $\phi(x) = \{g_1(x) \ldots, g_n(x)\}$ for all $x \in U$.

**Proposition 2.2.** *An $n$-valued map $\phi\colon X \multimap Y$ is locally split.*

*Proof.* Let $x^{(0)} \in X$ with $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$ and $V_1, \ldots, V_n$ disjoint open subsets of $Y$ with $y_j^{(0)} \in V_j$. For $x \in U = U(x^{(0)}, \{V_j\})$ with $\phi(x) = \{y_1, \ldots, y_n\}$, we number the $y_j$ such that $y_j \in V_j$. For $j = 1, \ldots, n$, define $g_j\colon U \to Y$ by $g_j(x) = y_j$, then $\phi(x) = \{g_1(x) \ldots, g_n(x)\}$. To prove that the $g_j$ are continuous, let $x_* \in U$ and let $W$ be an open subset of $Y$ that contains $g_j(x_*)$. Let $Z = \{x \in X\colon \phi(x) \cap (W \cap V_j) \neq \emptyset\}$ which contains $x_*$ and is open because $\phi$ is lower semi-continuous. Then $U \cap Z$ is an open subset of $X$ containing $x_*$ such that if $x \in U \cap Z$, then $g_j(x) = \phi(x) \cap (W \cap V_j) \subseteq W$. □

It follows easily from Proposition 2.2 that

**Proposition 2.3.** *If $\phi\colon X \multimap Y$ is an $n$-valued map, then the map $p_X\colon \Gamma(\phi) \to X$ defined by $p_X(x, y) = x$ is a covering space.*

Thus an $n$-valued map determines a finite covering space. Conversely, a finite covering space determines an $n$-valued map, as follows.

**Proposition 2.4.** *If $p\colon \widetilde{X} \to X$ is a covering space of degree $n$, then $p^{-1}\colon X \multimap \widetilde{X}$ in an $n$-valued map.*

*Proof.* Let $U$ be an open subset of $\widetilde{X}$ and let $x_0 \in X$ such that $p^{-1}(x_0) \cap U \neq \emptyset$. Because $p\colon \widetilde{X} \to X$ is a covering space, then $p$ is an open map so $p(U)$ is an open subset of $X$ containing $x_0$. Since $\phi(x) \cap U \neq \emptyset$ for all $x \in p(U)$, we conclude that $p^{-1}$ is a lower semi-continuous function and therefore, by Proposition 2.1, an $n$-valued map. □

The $n$-valued map $p^{-1}$ is used in [8] to construct $n$-valued self-maps of $\widetilde{X}$ and in [12] to construct $n$-valued self-maps of $X$.

In [1], Banach and Mazur called an $n$-valued function $\phi\colon X \multimap Y$ with $\phi(x) = \{y_1, \ldots, y_n\}$ continuous if, given a sequence $x^{(k)}$ in $X$ such that $\lim_{k\to\infty} x^{(k)} = x^{(0)}$ with $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$, the sets $\phi(x^{(k)})$ can be ordered as $y_1^{(k)}, \ldots, y_n^{(k)}$ so that $lim_{k\to\infty} y_i^{(k)} = y_i^{(0)}$ for all $i = 1, \ldots, n$. We will call an $n$-valued function that satisfies this definition *sequentially continuous*.

As another application of Proposition 2.1 we have

**Proposition 2.5.** *If an $n$-valued function $\phi\colon X \multimap Y$ is sequentially continuous and $X$ is a first-countable space, then $\phi$ is multicontinuous.*

*Proof.* It follows from Proposition 2.1 that $\phi$ is multicontinuous if and only if it is lower semi-continuous. Suppose $\phi$ is not lower semi-continuous, then there exists $x^{(0)} \in X$ and an open subset $V$ of $Y$ such that $\phi(x^{(0)}) \cap V \neq \emptyset$ but, for any open subset $U$ of $X$ containing $x^{(0)}$, there a point $x \in U$ such that $\phi(x) \cap V = \emptyset$. Since $X$ is first countable, there is a basis $\mathcal{U} = \{U_k\}_{k=1}^{\infty}$ for the topology of $X$ at $x^{(0)}$. For each $k$, there is a point $x^{(k)} \in U_k$ such that $\phi(x^{(k)}) \cap V = \emptyset$. Let $y_j^{(0)} \in \phi(x^{(0)})$ such that $y_j^{(0)} \in V$. Since $\mathcal{U}$ is a basis for the topology of $X$ at $x^{(0)}$, then $\lim_{k\to\infty} x^{(k)} = x^{(0)}$, but $\phi(x^{(k)}) = \{y_1^{k)}, \ldots, y_n^{(k)}\}$ cannot be ordered so that $\lim_{k\to\infty} y_j^{(k)} = y_j^{(0)}$, and therefore $\phi$ is not sequentially continuous. □

The converse of the previous result holds for all Hausdorff spaces:

**Proposition 2.6.** *If an $n$-valued function $\phi\colon X \multimap Y$ is multicontinuous, then it is sequentially continuous.*

*Proof.* Let $x^{(0)} \in X$ such that $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$. Let $(x^{(k)})$ be a sequence in $X$ such that $\lim_{k\to\infty} x^{(k)} = x^{(0)}$. Suppose $V_j \subseteq Y$ are open sets such that $y_j^{(0)} \in V_j$. We may assume, without loss of generality, that the subsets $V_j$ are disjoint. Since $\phi$ is lower semi-continuous, we observed at the beginning of this section that therefore there is an open set $U(x^{(0)}, \{V_j\})$ in $X$ containing $x^{(0)}$ such that if $x \in U(x^{(0)}, \{V_j\})$, then $\phi(x) = \{y_1, \ldots, y_j\}$ can be ordered so that $y_j \in V_j$. Since $\lim_{k\to\infty} x^{(k)} = x^{(0)}$, there exists $N$ such that if $k > N$, then $x^{(k)} \in U(x^{(0)}, \{V_j\})$. Therefore, if $k > N$ then $y_j^{(k)} \in V_j$, so $\lim_{k\to\infty} y_j^{(k)} = y_j^{(0)}$ and we conclude that $\phi$ is sequentially continuous. □

# 3 Configuration spaces and the characterization of splitting

As in the previous section, we assume only that all spaces are Hausdorff unless other conditions are stated.

Given a space $Y$ and a positive integer $n$, we define $F_n(Y)$, the *ordered configuration space of* $Y$, by

$$F_n(Y) = \{(y_1, \ldots, y_n) \in Y^n = Y \times \cdots \times Y : y_i \neq y_j \ for \ i \neq j\}.$$

The symmetric group $S_n$ acts freely on $F_n(Y)$ and the quotient space $D_n(Y) = F_n(Y)/S_n$ is the *unordered configuration space of* $Y$. Topologize $F_n(Y)$ as a subset of the product space $Y^n$ and then give $D_n(Y)$ the quotient topology so that, for the quotient map $q \colon F_n(Y) \to D_n(Y)$, a subset $W$ of $D_n(Y)$ is open if and only if $q^{-1}(W)$ is open in $F_n(Y)$.

It is observed in [15] that the set of $n$-valued functions from a space $X$ to a space $Y$ is in one-to-one correspondence with the set of single-valued functions from $X$ to $D_n(Y)$. Specifically, if $\phi \colon X \multimap Y$ is an $n$-valued function, then the corresponding function $\Phi \colon X \to D_n(Y)$ is defined by $\Phi(x) = \{\phi(x)\}$.

**Proposition 3.1.** *If $\phi \colon X \multimap Y$ is an $n$-valued map, then the corresponding function $\Phi \colon X \to D_n(Y)$ is continuous.*

*Proof.* Let $x^{(0)} \in X$ then, by Proposition 2.2, there is a neighborhood $U$ of $x^{(0)}$ and single-valued maps $g_1, \ldots, g_n \colon U \to Y$ such that $\phi(x) = \{g_1(x), \ldots, g_n(x)\}$ for all $x \in U$. Therefore, the restriction of $\Phi \colon X \to D_n(Y)$ to $U$ is the composition of the function $\hat{\phi} \colon U \to F_n(x)$ defined by $\hat{\phi}(x) = (g_1(x), \ldots, g_n(x))$ and the projection $q \colon F_n(Y) \to D_n(Y)$, both of which are continuous. Therefore, the function $\Phi$ is continuous. □

Proposition 3.1 allows us to replace an $n$-valued map of spaces by a single-valued map, the range of which is an unordered configuration space.

We will illustrate the configuration space concept by describing the ordered and unordered configuration spaces of the circle $S^1$. In addition to the relative simplicity of this setting, it is natural to focus on it because the $n$-valued self-maps of the circle may be viewed as the motivating example for this branch of fixed point theory. The reciprocal of the square-root function on the complex numbers of norm one was introduced in [24]: it is a non-split 2-valued self-map of the circle such that every 2-valued map of the circle homotopic to it has at least three fixed points.[4]

Our presentation is based on a paper of Westerland [27] and comments of Dylan Thurston [26]. We begin with the ordered configuration space $F_n(S^1)$. This space is the disjoint union of $(n-1)!$ sets determined by the $n!/n = (n-1)!$ orderings of $n$ points $z_1, \ldots, z_n$ of $S^1$. For a given ordering, choose the value of $z_1$ as the $S^1$ coordinate, then the successive differences of the polar coordinates between the adjacent points in the cyclic ordering of the points determines $n-1$ positive real numbers whose sum is less than one and so they define an open $n-1$-simplex. Thus each component of $F_n(S^1)$ is homeomorphic to the product of the circle and an open $n-1$-simplex and consequently it is the homotopy type of the circle.

The quotient map $q\colon F_n(S^1) \to D_n(S^1)$ is a covering space of order $n!$ and the restriction of $q$ to each component of $F_n(S^1)$ is of order $n$. Thus $D_n(S^1)$ is a $K(\pi, 1)$ because $F_n(S^1)$ is. To determine the fundamental group $\pi$ of $D_n(S^1)$, we first note that that group must be torsion free since $D_n(S^1)$ is a finite-dimensional manifold and it admits the cyclic group $\mathbb{Z}$ as a subgroup of finite index. Since $\pi$ is a virtually cyclic group that is torsion free, the classification of virtually cyclic groups implies that $\pi = \mathbb{Z}$. The homomorphism of cyclic groups $q_\#\colon \pi_1(F_n(S^1)) \to \pi_1(D_n(S^1))$ induced by $q$ is multiplication by $n$.

The converse of Proposition 3.1 also holds, as follows.

**Proposition 3.2.** *Let $\phi\colon X \multimap Y$ be an $n$-valued function. If $X$ is locally path-connected and semilocally simply connected and $\Phi\colon X \to D_n(Y)$ is continuous, then $\phi$ is multicontinuous, that is, an $n$-valued map.*[5]

*Proof.* Let $x^{(0)} \in X$ be an arbitrary point then, since $X$ is locally path-connected and semilocally simply connected, there is a path-connected neighborhood $U$ of $x^{(0)}$ such that $i_\#\colon \pi_1(U) \to \pi_1(X)$, the homomorphism induced by the inclusion, is trivial. Therefore the homomorphism $(\Phi|U)_\#\colon \pi_1(U) \to \pi_1(D_n(Y))$ induced by the

[4] The fixed point theory of $n$-valued maps of the circle was subsequently developed in [5].

[5] By means of a more elaborate proof, based on Corollary 9.3 of [20], it can be demonstrated that this proposition does not require the hypotheses that the domain is locally path-connected and semilocally simply connected. However, such a proof is beyond the scope of the present paper.

restriction of $\Phi$ to $U$ is trivial because it factors through $i_{\#}$. By [18], Proposition 1.33, the map $\Phi|U$ admits a lifting to $F_n(Y)$ and we can view it as consisting of maps $g_1, \ldots, g_n \colon U \to Y$ such that $g_i(x) \neq g_j(x)$ for $i \neq j$. Therefore the restriction to $U$ of the corresponding $n$-valued function $\phi \colon X \multimap Y$ is split by the maps $g_j$. Let $x^{(0)} \in U$ and let $V$ be an open subset of $Y$ that contains some $g_j(x^{(0)})$. Since $g_j$ is continuous, there exists an open neighborhood $W$ of $x^{(0)}$ in $U$ such that $g_j(W) \subseteq V$. Therefore $\phi(x) \cap V \neq \emptyset$ for all $x \in W$ and we have proved that $\phi$ is lower semi-continuous which, by Proposition 2.1, implies that $\phi$ is an $n$-valued map. □

By Propositions 3.1 and 3.2, if the domain satisfies appropriate hypotheses, we obtain another characterization of continuity for $n$-valued functions, as follows:

**Corollary 3.1.** *If $X$ is locally path-connected and semilocally simply connected, an $n$-valued function $\phi \colon X \multimap Y$ is multicontinous if and only if the corresponding function $\Phi \colon X \to D_n(Y)$ is continuous.*

By [13] the fundamental group of the configuration space $D_n(Y)$ is $B_n(Y)$, the braid group of $n$ strands on $Y$, and the fundamental group of $F_n(Y)$ is $P_n(Y)$, the subgroup of $B_n(Y)$ of pure braids. Since if $\phi \colon X \multimap Y$ is an $n$-valued map, then the corresponding function $\Phi \colon X \to D_n(Y)$ is continuous, so it induces a homomorphism $\Phi_{\#} \colon \pi_1(X) \to B_n(Y)$ (compare [15]). We use that homomorphism to obtain the following generalization of the Splitting Lemma.

**Theorem 3.1.** *(Splitting Characterization Theorem) Let $\phi \colon X \multimap Y$ be an $n$-valued map, where $X$ is connected and locally path-connected. Then $\phi$ is split if and only if the image of $\Phi_{\#} \colon \pi_1(X) \to B_n(Y)$ is contained in the image of the homomorphism $q_{\#} \colon P_n(Y) \to B_n(Y)$ induced by the projection $q \colon F_n(Y) \to D_n(Y)$. In particular, if $X$ is simply connected, then all $n$-valued maps $\phi \colon X \multimap Y$ are split.*[6]

*Proof.* Suppose that $\phi$ is split. This implies that there exist maps $g_1, \ldots, g_n \colon X \to Y$ such that $\phi(x) = \{g_1(x), \ldots, g_n(x)\}$ and we may define a map $\hat{\phi} \colon X \to F_n(Y)$ by $\hat{\phi}(x) = (g_1(x), \ldots, g_n(x))$. Then the corresponding map $\Phi \colon X \to F_n(Y)$ is the composition of $\hat{\phi}$ with the projection $q \colon F_n(Y) \to D_n(Y)$ and it follows that the image of the induced homomorphism $\Phi_{\#} \colon \pi_1(X) \to B_n(Y)$ is contained in the image of $q_{\#} \colon P_n(Y) \to B_n(Y)$.

[6] This form of the Splitting Lemma corresponds to that of Banach and Mazur [1] for Hausdorff spaces. The proof of Schirmer [23], based on a result of O'Neill [21], does not require that the domain be locally path-connected, but it only applies when both domain and range are compact spaces.

Suppose that the image of the homomorphism $\Phi_\#\colon \pi_1(X) \to B_n(Y)$ is contained in the image of $q_\#\colon P_n(Y) \to B_n(Y)$, the homomorphism induced by the projection $q\colon F_n(Y) \to D_n(Y)$. By [18], Proposition 1.33 there is a lifting $\hat{\phi}\colon X \to F_n(Y)$ of $\Phi$. The coordinate maps of $\hat{\phi}$ split $\phi$ and the result follows. $\square$

Not only must $n$-valued maps with domain a simply connected space be split but, more generally, if $X$ and $Y$ are spaces such that there are no nontrivial homomorphism from $\pi_1(X)$ to $B_n(Y)$, then Theorem 3.1 implies that every $n$-valued map from $X$ to $Y$ must be split. For example, let $G$ be a finite group, let $X_{\scriptscriptstyle G}$ be a 2-complex such that $\pi_1(X_{\scriptscriptstyle G}) = G$ and let $Y$ be a surface that is not the projective plane then, since $B_n(Y)$ is a torsion free group, all $n$-valued maps $\phi\colon X_{\scriptscriptstyle G} \multimap Y$ must be split.

Viewing a braid $\beta$ in $B_n(Y)$ as an $n$-valued map of the unit interval to $Y$ where $\beta(1)$ is a permutation of $\beta(0)$, define $\rho\colon B_n(Y) \to S_n$ by sending $\beta$ to the corresponding permutation. Since the braid group sequence

$$1 \to P_n(Y) \xrightarrow{q_\#} B_n(Y) \xrightarrow{\rho} S_n \to 1$$

is exact ([19], page 16), then the Splitting Characterization Theorem implies that if $\phi\colon X \multimap Y$ induces $\Phi_\#\colon \pi_1(X) \to B_n(Y)$ such that the composition $\rho\Phi_\#\colon \pi_1(X) \to S_n$ is nontrivial, then $\phi$ does not split.

From the point of view of fixed point theory, the study of split $n$-valued self-maps of a finite polyhedron can be reduced to the single-valued setting because Schirmer proved in [24] that if $\phi = \{f_1, \ldots, f_n\}\colon X \multimap X$, then the Nielsen number of $\phi$ is calculated from that of the $f_i$ by the formula $N(\phi) = \sum_{i=1}^n N(f_i)$. On the other hand, examples from [8] demonstrate that $N(\phi)$ can exhibit a variety of behaviors in the non-split case. This motivates the use of our Splitting Characterization Theorem to construct non-split $n$-valued self-maps $\phi\colon X \multimap X$ which we will do by defining $\phi$ so that the image of $\Phi_\#$ does not lie in the subgroup $q_\#(P_n(Y)) \subseteq B_n(Y)$ of pure braids on $Y$. In order to make the construction, the only requirement is that the first Betti number of $X$ be non-zero. Thus the integer cohomology group $H^1(X)$ is of rank at least one. Identifying $H^1(X)$ with the Buschinsky group of homotopy classes of maps $[X, S^1]$, that group is nontrivial so we may choose a map $a\colon X \to S^1$ which induces an epimorphism $a_\#\colon \pi_1(X) \to \pi_1(S^1)$. Let $\beta \in B_n(X)$ be a braid on $X$ such that $\rho(\beta) \in S_n$ is not the identity permutation. Since $B_n(Y) = \pi_1(D_n(Y))$, the homotopy class $[b]$ corresponding to the braid $\beta$ is represented by a map $b\colon S^1 \to D_n(X)$ which may be viewed as an $n$-valued map $b\colon S^1 \multimap X$. Then $\phi = b \circ a\colon X \multimap X$ is a non-split $n$-valued map.

## 4 Maps to a metric range

Let $\phi\colon X \multimap Y$ be an $n$-valued function and write $\phi(x) = \{y_1, \dots, y_n\}$ for $x \in X$. For $Y$ a metric space with metric $d_Y$, define $\gamma_\phi\colon X \to \mathbf{R}$ by

$$\gamma_\phi(x) = \min_{1 \le i \ne j \le n} d_Y(y_i, y_j).$$

In [23], the *gap* $\gamma(\phi)$ of $\phi$ is defined as

$$\gamma(\phi) = \inf\{\gamma_\phi(x)\colon x \in X\}.$$

The following useful fact is stated in [23] without a proof. We take this opportunity to present its brief proof.

**Proposition 4.1.** *Let $Y$ be a metric space with metric $d_Y$ and let $\phi\colon X \multimap Y$ be a multicontinuous $n$-valued function, then the function $\gamma_\phi\colon X \to \mathbf{R}$ is continuous. Therefore, if $X$ is compact, then the gap $\gamma(\phi) > 0$ for all $\phi$.*

*Proof.* Let $x_0 \in X$, then by Proposition 2.2 there is an neighborhood $U$ of $x_0$ and maps $g_1, \dots g_n\colon U \to Y$ such that $\phi(x) = \{g_1(x), \dots, g_n(x)\}$ for all $x \in U$. Then for $x \in U$, the gap is defined by

$$\gamma_\phi(x) = \min_{1 \le i \ne j \le n} \{d_Y(g_i(x), g_j(x))\}.$$

Since the $g_i$ are continuous, $\gamma_\phi$ is continuous on $U$ and therefore it is a continuous function. $\square$

The gap, and in particular the fact that it is positive on compact spaces, is used in [23] to obtain a simplicial approximation theorem for $n$-valued maps and, subsequently, in [6] to define a simplicial chain map induced by the $n$-valued map. It appears also in the proof of the uniqueness of splitting in [9] and in the reduction of the computation of the Nielsen number of an $n$-valued map to a coincidence number for single-valued maps in [10].

Let $\phi\colon X \multimap Y$ be an $n$-valued function where $Y$ is a metric space with metric $d_Y$. For $x, x' \in X$, the *Hausdorff distance*, when specialized to that between $\phi(x) = \{y_1, \dots, y_n\}$ and $\phi(x') = \{y'_1, \dots, y'_n\}$, denoted $d_H(\phi(x), \phi(x'))$, is the greater of

$$\max_{1 \le j \le n} \min_{1 \le i \le n} d_Y(y_j, y'_i)$$

and

$$\max_{1\leq i\leq n}\ \min_{1\leq j\leq n}\ d_Y(y_j, y'_i).$$

We define $\phi\colon X \multimap Y$ to be *Hausdorff continuous at* $x_0 \in X$ if, given $\epsilon > 0$, there exists an open subset $U$ of $X$ containing $x_0$ such that if $x \in U$, then $d_H(\phi(x_0), \phi(x)) < \epsilon$. It is *Hausdorff continuous on* $X$ if it is Hausdorff continuous at every point of $X$.

**Proposition 4.2.** *An $n$-valued function $\phi\colon X \multimap Y$, where $Y$ is a metric space, is multicontinuous if and and only if it is Hausdorff continuous on $X$.*

*Proof.* We first prove that if $\phi\colon X \multimap Y$ is Hausdorff continuous, then it is multicontinuous. By Proposition 2.1, we need only prove that $\phi$ is lower semi-continuous. Let $x^{(0)} \in X$ such that $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$ and let $V \subseteq Y$ be an open set such that $\phi(x^{(0)}) \cap V \neq \emptyset$; we assume $y_1^{(0)} \in V$. Choose $\epsilon > 0$ so that $B(y_1^{(0)}, \epsilon) \subseteq V$ then, since $\phi$ is Hausdorff continuous, there is an open subset $U$ of $X$ containing $x^{(0)}$ such that $x \in U$ implies that $d_H(\phi(x), \phi(x^{(0)})) < \epsilon$. Therefore, for $\phi(x) = \{y_1, \ldots, y_j\}$ we know that

$$\max_{1\leq i\leq n}\ \min_{1\leq j\leq n}\ d_Y(y_j, y_i^{(0)}) < \epsilon.$$

and, in particular, that

$$\min_{1\leq j\leq n}\ d_Y(y_j, y_1^{(0)}) < \epsilon$$

so some $y_j \in B(y_1^{(0)}, \epsilon)$. Therefore, $x \in U$ implies that $\phi(x) \cap V \neq \emptyset$ and we have demonstrated that the set of $x \in X$ with this property is open.

Now suppose that $\phi\colon X \multimap Y$ is multicontinuous, that $x^{(0)} \in X$, and we are given $\epsilon > 0$. We may assume that $\epsilon < \gamma_\phi(x^{(0)})/2$. Let $\phi(x^{(0)}) = \{y_1^{(0)}, \ldots, y_n^{(0)}\}$ and define disjoint open sets $V_j = B(y_j^{(0)}, \epsilon)$. We will show that if $x$ is in the open subset $U(x^{(0)}, \{V_j\})$ of $X$ defined in Section 2, then $d_H(\phi(x), \phi(x^{(0)})) < \epsilon$ so $\phi$ is Hausdorff continuous at $x_0$. Let $\phi(x) = \{y_1, \ldots, y_n\}$. If $i \neq j$, then

$$\begin{aligned}
\gamma_\phi(x^{(0)}) &\leq d_Y(y_i^{(0)}, y_j^{(0)}) \\
&\leq d_Y(y_i^{(0)}, y_j) + d_Y(y_j, y_j^{(0)}) \\
&< d_Y(y_i^{(0)}, y_j) + \gamma_\phi(x^{(0)})/2
\end{aligned}$$

so

$$d_Y(y_i^{(0)}, y_j) > \gamma_\phi(x^{(0)})/2.$$

Therefore, since $d_Y(y_i^{(0)}, y_i) < \gamma_\phi(x^{(0)})/2$ for all $i$, we conclude that

$$d_H(\phi(x), \phi(x^{(0)})) = \max_{1 \le i \le n} d_Y(y_i^{(0)}, y_i) < \epsilon.$$

$\square$

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