ON THE NIELSEN ROOT THEORY OF
n-VALUED MAPS

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Abstract

Let $\phi\colon X \multimap Y$ be an n-valued map of connected finite polyhedra and let $a \in Y$. Then $x \in X$ is a root of ϕ at a if $a \in \phi(x)$. The Nielsen root number $N(\phi : a)$ is a lower bound for the number of roots at a of any n-valued map homotopic to ϕ. We prove that if X and Y are manifolds without boundary, of the same dimension, then there is an n-valued map ψ homotopic to ϕ such that ψ has finitely many roots at a. We conjecture that if X and Y are q-manifolds without boundary, $q \neq 2$, then there is an n-valued map homotopic to ϕ that has $N(\phi : a)$ roots at a. We verify the conjecture when $X = Y$ is a Lie group by employing a fixed point result of Schirmer. As an application, we calculate the Nielsen root numbers of linear n-valued maps of tori.

Keywords and Phrases: n-valued map, Nielsen number, coincidence, root

Subject Classification: 55M20

1   Introduction

Throughout the paper all spaces are connected finite polyhedra. Let $f\colon X \to Y$ be a map and $a \in Y$, then a *root of f at a* is a point $x \in X$ such that $f(x) = a$. An *n-valued map* is a lower semi-continuous, hence also upper semi-continuous [5], set-valued function $\phi\colon X \multimap Y$ such that $\phi(x)$ is n points of Y for each $x \in X$. A *root of ϕ at $a \in Y$* is a point $x \in X$ such that $a \in \phi(x)$.

Nielsen root theory for single-valued maps was first formalized in [2], but important results concerning roots had been established

much earlier by Hopf in [9]. The Nielsen root theory of n-valued maps was introduced in [6] as a tool for studying the Nielsen coincidence theory of such maps. In the present paper, the focus is on the Nielsen root theory of n-valued maps for its own sake.

Section 2 presents Nielsen root theory for n-valued maps in the context of Nielsen coincidence theory and explains the relationship between the two theories. In Section 3, we prove a root theory analogue of a finiteness theorem for fixed points due to Schirmer in [11]. Section 4 presents a conjecture about the Nielsen root theory of n-valued maps and we partly verify the conjecture in Section 5. In Section 6 we use the techniques of Section 5 to compute the Nielsen root numbers of "linear" n-valued maps of tori that were introduced in [7].

2 Coincidences and Roots

Let $\phi, \psi\colon X \multimap Y$ be n-valued and m-valued maps, respectively, which we will call an (n, m)-*valued pair of maps*. A point $x \in X$ is a *coincidence of ϕ and ψ* if $\phi(x) \cap \psi(x) \neq \emptyset$. If, for $a \in Y$, the map $\psi(x)$ is the constant map at a, then x a coincidence of ϕ and ψ means that $a \in \phi(x)$ and thus x is a root of ϕ at a. An important property of n-valued maps is that they *split* if the domain X is simply-connected, that is, there is the *splitting* as $\phi(x) = \{f_1(x), \ldots, f_n(x)\}$ for all $x \in X$, where each $f_j\colon X \to Y$ is a single-valued map. We use this property to define an equivalence relation on the set of coincidences of an (n, m)-valued pair $\phi, \psi\colon X \multimap Y$ as follows. Coincidences x_0 and x_1 are *equivalent* if there exists a path $c\colon I = [0, 1] \to X$ such that $c(0) = x_0, c(1) = x_1$ and c has the following property: for the splittings $\phi c = \{f_1, \ldots, f_n\}$ and $\psi c = \{g_1, \ldots, g_m\}$ there exist $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$ such that $f_j(0) = g_k(0), f_j(1) = g_k(1)$ and the paths $f_j, g_k\colon I \to Y$ are homotopic relative to the endpoints. The equivalence classes are called the *coincidence classes* of the (n, m)-valued pair ϕ, ψ and they are finite in number because X is compact. If ψ is the constant map at $a \in Y$, roots x_0 and x_1 are equivalent according to this definition if there is a path $c\colon I \to X$ such that $c(0) = x_0, c(1) = x_1$ and for the splitting $\phi c = \{f_1, \ldots, f_n\}$ there is a contractible loop f_j at a for some $j \in \{1, \ldots, n\}$. The equivalence classes are called the *root classes of ϕ at a*.

Let $\Phi\colon X \times I \multimap Y$ be an n-valued map, called an *n-valued homotopy*, and let $\Psi\colon X \times I \multimap Y$ be an m-valued homotopy. The restrictions of Φ and Ψ to $X \times \{t\}$ for $0 \leq t \leq 1$ are denoted by ϕ_t and ψ_t respectively. By Lemma 2.1 of [6], a coincidence class of the

(n, m)-valued pair ϕ_t, ψ_t is contained in a unique coincidence class of the (n, m)-valued pair (Φ, Ψ).

A coincidence class $\mathbf{C_0}$ of an (n, m)-valued pair $\phi, \psi\colon X \multimap Y$ is *inessential* if there exists an (n, m)-valued pair of homotopies $\Phi, \Psi\colon X \times I \multimap Y$ such that $\phi_0 = \phi, \psi_0 = \psi$ and the coincidence class $\mathbf{C}$ of the (n, m)-valued pair (Φ, Ψ) containing $\mathbf{C_0}$ has the property $\mathbf{C} \cap (X \times \{1\}) = \emptyset$. Otherwise the coincidence class $\mathbf{C_0}$ is *essential*. The *Nielsen coincidence number* $N(\phi : \psi)$ of the (n, m)-valued pair $\phi, \psi\colon X \multimap Y$ is the number of essential coincidence classes. If ψ is the constant map at $a \in Y$, then the Nielsen number is denoted by $N(\phi : a)$ and called the *Nielsen root number of ϕ at a*.

The Nielsen coincidence number for (n, m)-valued pairs of maps is homotopy invariant by Theorem 2.1 of [6]. That is, let $\Phi, \Psi\colon X \times I \multimap Y$ be an (n, m)-valued pair of homotopies, then $N(\phi_0 : \psi_0) = N(\phi_1 : \psi_1)$. We will make use of the following consequence in which $\Psi(x, t) = a$ for all $(x, t) \in X \times I$:

Proposition 2.1. *If* $\Phi\colon X \times I \multimap Y$ *is an n-valued homotopy, then* $N(\phi_0 : a) = N(\phi_1 : a)$.

Although the concepts of Nielsen root theory have been defined in terms of concepts in Nielsen coincidence theory, root theory is not the special case of the coincidence theory of an (m, n)-valued pair $\phi, \psi\colon X \multimap Y$ in which $\psi(x) = a$ for all $x \in X$. The difference lies in the type of homotopies that are valid for the two theories. In coincidence theory they are (n, m)-valued pairs of maps $\Phi, \Psi\colon X \times I \multimap Y$ whereas in root theory a homotopy is a single map $\Phi\colon X \times I \multimap Y$ because the target $a \in Y$ is constant throughout the homotopy, in other words it requires that $\Psi(x, t) = a$ for all $(x, t) \in X \times I$.

3 Root-finite Approximation

If $\phi\colon X \multimap X$ is an n-valued map, then $x \in X$ is a *fixed point of ϕ* if $x \in \phi(x)$. In [11] Schirmer proved the following result:

Theorem 3.1. *(Schirmer's Fix-finite Approximation Theorem) Let* $\phi\colon X \to X$ *be an n-valued map. Given $\epsilon > 0$ there exists an n-valued map $\psi\colon X \to X$, homotopic to ϕ, with finitely many fixed points, such that the Hausdorff distance $d(\phi, \psi) < \epsilon$.*

An n-valued map $\phi\colon X \multimap Y$ is defined in [11] to be *simplicial* if, for each closed simplex $\bar{\sigma}$ of X, the restriction of ϕ to $\bar{\sigma}$ splits into maps $f_1, f_2, \ldots, f_n$ such that each f_j maps $\bar{\sigma}$ affinely onto a closed simplex of Y. Schirmer's proof of her fix-finite result depended on

the following simplicial approximation theorem for n-valued maps, which is Theorem 4 of [11].

Theorem 3.2. *(Schirmer's Simplicial Approximation Theorem) Let $\phi\colon X \multimap Y$ be an n-valued map of finite polyhedra. Given $\epsilon > 0$, there exists an n-valued map $\psi\colon X \multimap Y$, homotopic to ϕ, that is simplicial with respect to some barycentric subdivisions of X and Y, and $d(\phi, \psi) < \epsilon$.*

Let d_Y be the metric of Y. For $x \in X$, let $\phi(x) = \{y_1, y_2, \ldots, y_n\}$ and define $\gamma(x) = \min\{d_Y(y_i, y_j)|i \neq j\}$. As in [11], the *gap* of ϕ, denoted $\gamma(\phi)$, is defined by $\gamma(\phi) = \inf\{\gamma(x)|x \in X\}$. Then $\gamma(\phi) > 0$; see Proposition 4.1 of [5]. On page 79 of [11], Schirmer proved that if the mesh of X is less than $\frac{1}{4}\gamma(\phi)$, then the image of a simplicial n-valued map ϕ on a simplex $\bar{\sigma}$ of X is the union of n disjoint simplices of Y. Therefore, a can be in the image of at most one of the maps f_j of the splitting of the restriction of ϕ to $\bar{\sigma}$.

Theorem 3.3. *(Root-finite Approximation Theorem) Let $\phi\colon X \multimap Y$ be an n-valued map where X and Y are q-manifolds without boundary, and let $a \in Y$. Given $\epsilon > 0$, there exists an n-valued map $\psi\colon X \multimap Y$, homotopic to ϕ, such that $d(\phi, \psi) < \epsilon$ and ψ has finitely many roots at a.*

Proof. Define a point $a \in Y$ in the closure $\bar{\sigma}$ of an open q-simplex σ to be *fully interior* if $a \in \sigma$ and a is in an open q-simplex of every barycentric subdivision of σ. The fully interior points of an open simplex σ are dense in σ by the Baire Category Theorem. First suppose that $a \in \sigma \subset Y$ is fully interior. We may assume that the given ϵ is less than $\frac{1}{4}\gamma(\phi)$. Taking the necessary number of barycentric subdivisions of X and Y for Theorem 3.2, let ψ be a simplicial approximation to ϕ with $d(\phi, \psi) < \epsilon$. Then ψ has no roots at a on the $(q-1)$-skeleton of X since a simplicial map takes a simplex to one that is of the same or lower dimension. The map ψ has no more than one root on each q-simplex of X because the corresponding map f_j is affine on the closure of the simplex and thus two roots would determine a linear set of roots that intersects its boundary.

If $a \in \bar{\sigma}$ is not a fully interior point of σ, there is a homeomorphism $h\colon Y \to Y$ such that $d_Y(y, h(y)) < \frac{\epsilon}{2}$ for all $y \in Y$ and $b = h(a)$ is fully interior. To define h, since the fully interior points are dense in σ, there is a fully interior point $b \in \sigma$ such that $d_Y(a, b) < \frac{\epsilon}{4}$. Thus an $\frac{\epsilon}{2}$ Euclidean neighborhood U of a contains b. Let $h\colon Y \to Y$ be a homeomorphism that is the identity on $Y \setminus U$ and is a Euclidean translation on U such that $h(a) = b$.

4

Let $\zeta\colon X \multimap Y$ be a simplicial approximation to the n-valued map $h\phi\colon X \multimap Y$ such that $d(\zeta, h\phi) < \frac{\epsilon}{2}$. By the first part of the proof, there are a finite number of points $x \in X$ such that $b \in \zeta(x)$. Let $\psi = h^{-1}\zeta\colon X \multimap Y$. Then $b \in \zeta(x)$ if and only if $a = h^{-1}(b) \in h^{-1}\zeta(x) = \psi(x)$. Therefore ψ has finitely many roots at a and $d(\phi, \psi) < \epsilon$. $\qquad\square$

4   A Root-Minimization Conjecture

In 1930, Heinz Hopf proved [9] that if $f\colon X \to Y$ is a map where X and Y are compact q-manifolds with or without boundary, and $q \neq 2$, then there is a map g homotopic to f with $N(f\colon a)$ roots and thus the minimum is achieved within the homotopy class, that is, $N(f; a)$ is a sharp lower bound. We will refer to this result as "Hopf's root-minimization theorem". Maps of surfaces do not possess the root-minimization property, see [10]. A detailed proof of Hopf's theorem in the case of manifolds without boundary is presented by Brooks in [1].

In [14], Wecken proved that if X is a compact q-manifold without boundary, $q \neq 2$, and $f\colon X \to X$ is a map, then there is a map g homotopic to f such that g has $N(f)$ fixed points, where $N(f)$ denotes the Nielsen fixed point number of f.

In [12], Schirmer defined fixed points x_0, x_1 of an n-valued map $\phi\colon X \multimap X$ to be *equivalent* if there is a path $c\colon I \to X$ such that for the splitting $\phi c = \{f_1, \ldots, f_n\}\colon I \multimap X$ there is some f_j such that $f_j(0) = x_0, f_j(1) = x_1$ and c and f_j are homotopic relative to the endpoints. The equivalence classes are the *fixed point classes* of ϕ. let X be a manifold without boundary. Approximating ϕ by an n-valued map with finitely many fixed points by Theorem 3.1, each fixed point is an isolated fixed point of a single-valued map from a splitting and therefore its fixed point index is defined, see [3]. As in single-valued Nielsen fixed point theory, the *index of a fixed point class* is the sum of the indices of its points. A fixed point class is *inessential* if its index is zero and *essential* otherwise and the Nielsen fixed point number $N(\phi)$ of ϕ is the number of essential fixed point classes. Schirmer proved that this Nielsen number is independent of the fix-finite approximation. She then proved in [13] that Wecken's theorem can be extended to fixed points of n-valued maps, as follows.

Theorem 4.1. *(Schirmer's Fixed Point Minimization Theorem) Given an n-valued map $\phi\colon X \multimap X$ of a compact q-manifold without*

boundary, $q \neq 2$, there is an n-valued map ψ homotopic to ϕ such that ψ has $N(\phi)$ fixed points.

The following conjecture extends Hopf's result in the same way that Schirmer's minimization theorem extends Wecken's.

Conjecture 4.2. Given an n-valued map $\phi\colon X \multimap Y$, where X and Y are compact q-manifolds without boundary, $q \neq 2$, and given $a \in Y$, there is an n-valued map ψ homotopic to ϕ such that ψ has $N(\phi : a)$ roots at a.

There is only one compact 1-manifold without boundary: the circle S^1. Although, for (n, m)-valued pairs of self-maps of the circle S^1, Corollary 5.1 of [6] proves that the Nielsen coincidence number is a sharp lower bound for the number of coincidences of all maps (n, m)-homotopic to them, that does not imply that Conjecture 4.2 is true for the Nielsen root number of an n-valued self-map of the circle. As we pointed out above, homotopies in the setting of (n, m)-valued coincidence theory are (n, m)-valued pairs of maps, so when ψ is the constant map at a the pair of homotopies $\Phi, \Psi\colon S^1 \times I \multimap S^1$ that minimize the number of coincidences of an (n, m)-pair $\phi, \psi\colon S^1 \multimap S^1$ would not necessarily have the property that $\Psi(x, t) = a$ for all $(x, t) \in X \times I$.

However, the conjecture is in fact true for n-valued maps of S^1, as we will next prove.

Let $p\colon \mathbb{R} \to S^1$ be the universal covering space and represent the points of S^1 by $p(t)$ for $0 \leq t < 1$. Let $\phi\colon S^1 \multimap S^1$ be an n-valued map and let $\phi p = \{f_0, f_1, \ldots, f_{n-1}\}\colon I \multimap Y$ where $f_j(0) = p(t_j)$ for $0 \leq t_0 < t_1 < \cdots < t_{n-1} < 1$. Let $\tilde{f}_j\colon I \to \mathbb{R}$ be the lift of f_j such that $\tilde{f}_j(0) = t_j$ then, following [4], $\tilde{f}_0(1) = v + t_J$ for some integers v and J, where $0 \leq J \leq n-1$ and the *degree $Deg(\phi)$* of ϕ is defined by $Deg(\phi) = nv + J$. Theorem 2.3 of [4] states that if $\phi, \psi\colon S^1 \multimap S^1$ are homotopic n-valued maps, then $Deg(\phi) = Deg(\psi)$.

For integers d and $n \geq 1$, the *n-valued power map $\phi_{n,d}\colon S^1 \multimap S^1$* is defined by

$$\phi_{n,d}(p(t)) = \{p(\frac{d}{n}t), p(\frac{d}{n}t + \frac{1}{n}), \ldots, p(\frac{d}{n}t + \frac{n-1}{n})\}.$$

Theorem 4.3. *Let $\phi\colon S^1 \multimap S^1$ be an n-valued map of degree $Deg(\phi) = d$. Then there is an n-valued map $\psi\colon S^1 \multimap S^1$ homotopic to ϕ such that ψ has $N(\phi : 1) = |d|$ roots.*

Proof. By Theorem 3.1 of [4], since ϕ is n-valued and of degree d, then it is homotopic to $\phi_{n,d}$. By Proposition 5.1 of [6], $N(\phi_{n,d} : 1) = |d|$ and therefore $N(\phi : 1) = |d|$ by Proposition 2.1 above. Since $\phi_{n,d}$ has $|d|$ roots at 1, then $\psi = \phi_{n,d}$ is the required n-valued map. $\square$

5 Root Minimization on Lie Groups

Let X be a compact Lie group with identity element e and let $\phi \colon X \multimap X$ be an n-valued map. For $\{p_1, \ldots, p_n\}$ an unordered set of points of X and $x \in X$, let $x\{p_1, \ldots, p_n\} = \{xp_1, \ldots, xp_n\}$. We define an n-valued map $\theta \colon X \multimap X$ as follows: if $\phi(x) = \{p_1, \ldots, p_n\}$, let $\theta(x) = x\{p_1, \ldots, p_n\}$. The fixed points of θ are the roots of ϕ at e. It will be convenient to abbreviate the definition of θ by letting $\mathbf{1}$ be the identity map of X and writing $\theta = \mathbf{1}\phi$. We note that, with this notation, $\phi = \mathbf{1}^{-1}\theta$.

Lemma 5.1. *Let $\phi \colon X \multimap X$ be an n-valued map of a compact Lie group. Roots x_0, x_1 of ϕ at e are equivalent if and only if they are equivalent as fixed points of $\theta = \mathbf{1}\phi$. Therefore, the fixed point classes of θ are the root classes of ϕ.*

Proof. Since x_0, x_1 are equivalent roots of ϕ at e, there is a path $c \colon I \to X$ such that $c(0) = x_0, c(1) = x_1$ and, for the splitting $\phi c = \{f_1, \ldots, f_n\} \colon I \multimap X$, there exists $f_j \colon I \to X$ that is a contractible loop at e. Therefore, there is a map $H \colon I \times I \to X$ such that $H(s,0) = H(s,1) = H(0,t) = e$ for all $s, t \in I$ and $H(1,t) = f_j(t)$ for all $t \in I$. From the splitting $\phi(c(t)) = \{f_1(t), \ldots, f_n(t)\}$ we have

$$
\begin{aligned}
\theta(c(t)) = c(t)\phi(c(t)) &= c(t)\{f_1(t), \ldots, f_n(t)\} \\
&= \{c(t)f_1(t), \ldots, c(t)f_n(t)\}.
\end{aligned}
$$

Thus $\theta c = \{g_1, \ldots, g_n\}$ where $g_j(t) = c(t)f_j(t)$ so $g_j(0) = x_0, g_j(1) = x_1$ and the path $g_j \colon I \to X$ is homotopic relative to the endpoints to the path c by means of the homotopy $K \colon I \times I \to X$ defined by $K(s,t) = c(t)H(s,t)$. Conversely, suppose x_0 and x_1 are equivalent as fixed points of θ by means of a path c that is homotopic relative to the endpoints to a path g_j defined by $g_j(t) = c(t)f_j(t)$ in the splitting of θc, by means of a homotopy $K \colon I \times I \to X$. The homotopy H defined by $H(s,t) = c(t)^{-1}K(s,t)$ is a contraction of the loop f_j at e, so x_0 and x_1 are equivalent as roots of ϕ at e. $\qquad\square$

Theorem 5.2. *Let $\phi \colon X \multimap X$ be an n-valued map of a compact Lie group and let $\theta = \mathbf{1}\phi$, then $N(\phi \colon e) = N(\theta)$.*

Proof. By Lemma 5.1, a root class $\mathbf{R_0}$ of ϕ is a fixed point class of θ. If $\mathbf{R_0}$ is an inessential root class, there is an n-valued homotopy $\Phi \colon X \times I \multimap X$ with $\phi_0 = \phi$ such that the root class $\mathbf{R}$ of Φ containing $\mathbf{R_0}$ has the property $\mathbf{R} \cap (X \times \{1\}) = \emptyset$. Therefore, by the homotopy and additivity properties of the fixed point index, the value of the index on $\mathbf{R_0}$ is zero and thus $\mathbf{R_0}$ is an inessential fixed

point class of θ. Conversely, if $\mathbf{R_0}$ is an inessential fixed point class of θ, then Theorem 4.1 implies that there is a homotopy $\Theta\colon X \times I \multimap X$ such that $\theta_0 = \theta$ and the fixed point class $\mathbf{R}$ of Θ containing $\mathbf{R_0}$ has the property $\mathbf{R} \cap (X \times \{1\}) = \emptyset$ and therefore $\mathbf{R_0}$ is an inessential root class. Since the correspondence between the root classes of ϕ and the fixed point classes of θ therefore preserves essentiality, it follows that $N(\phi : e) = N(\theta)$. □

The following result verifies the root-minimization conjecture for n-valued self-maps of compact Lie groups.

Theorem 5.3. *Let* $\phi\colon X \multimap X$ *be an n-valued map of a compact Lie group. There exists an n-valued map $\psi\colon X \multimap X$ homotopic to ϕ such that ψ has $N(\phi : e)$ roots at e.*

Proof. Let $\theta = \mathbf{1}\phi$. By Theorem 4.1 there is an n-valued map $\zeta\colon X \multimap X$ homotopic to θ that has $N(\theta)$ fixed points. Let $\psi = \mathbf{1}^{-1}\zeta$. The n-valued map ψ has $N(\theta) = N(\phi : e)$ roots by Theorem 5.2. □

6 Linear n-Valued Maps of Tori

Let T^q denote the q-torus with identity element e and let $p_q\colon \mathbb{R}^q \to T^q$ be its universal covering space. Let A be a $q \times q$ integer matrix. Define an n-valued map $\tilde{\phi}_{n,A}\colon I^q \multimap T^q$ by

$$\tilde{\phi}_{n,A}(t) = \{\tilde{\phi}_{n,A}^{(0)}(t), \ldots, \tilde{\phi}_{n,A}^{(n-1)}(t)\}$$

where

$$\tilde{\phi}_{n,A}^{(k)}(t) = p_q\left(\frac{1}{n}(At + k)\right)$$

for $k = 0, \ldots, n-1$.

Define integer vectors $x = (x_0, \ldots, x_{n-1})$ and $y = (y_0, \ldots, y_{n-1})$ to be *congruent mod n*, written $x = y \mod n$, if x_j and y_j are congruent mod n for all $j = 0, \ldots, n-1$. By Theorem 3.1 of [7], the n-valued map $\tilde{\phi}_{n,A}$ induces an n-valued map $\phi_{n,A}\colon T^q \multimap T^q$ if and only if all the rows of A are congruent to each other mod n. The map $\phi_{n,A}$ is called a *linear n-valued map* of the q-torus.[1]

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[1]For the linear n-valued maps of tori introduced in [7], there is an additional parameter that somewhat extends the variety of such maps. However, since the Nielsen number of linear n-valued maps of tori is independent of the value of that parameter, we limit ourselves to one value of the parameter for the purpose of this exposition.

Theorem 6.1. *The Nielsen root number of a linear n-valued map $\phi_{n,A}\colon T^q \multimap T^q$ is*

$$N(\phi_{n,A} : e) = n \left| det(\frac{1}{n}A) \right|.$$

Proof. Define an n-valued map $\tilde{\theta}_{n,A}\colon I^q \multimap T^q$ by

$$\tilde{\theta}_{n,A}(t) = \{\tilde{\theta}_{n,A}^{(0)}(t), \ldots, \tilde{\theta}_{n,A}^{(n-1)}(t)\}$$

where, for $k = 0, \ldots, n-1$,

$$
\begin{aligned}
\tilde{\theta}_{n,A}^{(k)}(t) &= p_q(t)p_q\left(\frac{1}{n}(At+k)\right) \\
&= p_q\left(t + \frac{1}{n}(At+k)\right) \\
&= p_q\left(\frac{1}{n}(nt + \frac{1}{n}(At+k))\right) \\
&= p_q\left(\frac{1}{n}((nE+A)t+k)\right) \\
&= p_q\left(\frac{1}{n}(Bt+k)\right)
\end{aligned}
$$

and $B = nE + A$ for E the $q \times q$ identity matrix. Since the rows of nE are congruent mod n to the zero vector, $\tilde{\theta}_{n,A}$ induces an n-value map $\theta_{n,A}\colon T^q \multimap T^q$ if and only if the rows of A are all congruent mod n. In that case, $\theta_{n,A}$ is a linear n-valued map: $\theta_{n,A} = \phi_{n,B}$. By [8], the Nielsen fixed point number of $\phi_{n,B}$ is

$$N(\phi_{n,B}) = n \left| det(E - \frac{1}{n}B) \right|.$$

Since $\theta_{n,A} = \mathbf{1}\phi_{n,A}$, Theorem 5.2 implies that the Nielsen root number of $\phi_{n,A}$ at e is

$$
\begin{aligned}
N(\phi_{n,A} : e) = N(\phi_{n,B}) &= n \left| det(E - \frac{1}{n}B) \right| \\
&= n \left| det(E - \frac{1}{n}(nE+A)) \right| \\
&= n \left| det(\frac{1}{n}A) \right|
\end{aligned}
$$

which completes the proof. $\qquad\qquad\square$

Example 6.2. Let $\phi_{2,A}\colon T^3 \multimap T^3$ where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ -5 & -1 & -3 \end{bmatrix}$$

then

$$N(\phi_{2,A} : (1,1,1)) = 2|\det(\tfrac{1}{2}A)| = 2|-4| = 8$$

so if $\psi\colon T^3 \multimap T^3$ is a 2-valued map homotopic to $\phi_{2,A}$, then there are at least 8 solutions to the equation $\psi(z_1, z_2, z_3) = (1,1,1)$ for $z_i \in S^1, i = 1, 2, 3$.

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