LIFT FACTORS FOR THE NIELSEN ROOT THEORY OF n-VALUED MAPS

Robert F. Brown
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
e-mail: rfb@math.ucla.edu
Daciberg Lima Gonçalves
Departamento de Matemática - IME
Universidade de São Paulo
CEP 05508-090 - São Paulo - SP - Brazil
e-mail: dlgoncal@ime.usp.br

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Abstract

A root of an n-valued map $\varphi\colon X \to D_n(Y)$ at $a \in Y$ is a point $x \in X$ such that $a \in \varphi(x)$. We lift the map φ to a split n-valued map of finite covering spaces and its single-valued factors are defined to be the lift factors of φ. We describe the relationship between the root classes at a of the lift factors and those of φ. We define the Reidemeister root number $RR(\varphi)$ in terms of the Reidemeister root numbers of the lift factors. We prove that the Reidemeister root number is a homotopy invariant upper bound for the Nielsen root number $NR(\varphi)$, the number of essential root classes, and we characterize essentiality by means of an equivalence relation called the Φ-relation. A theorem of Brooks states that a single-valued map to a closed connected manifold is root-uniform, that is, its root classes are either all essential or all inessential. It follows that Y if is a closed connected manifold, then the lift factors are root-uniform and we relate this property to the root-uniformity of φ. If X and Y are closed connected oriented manifolds of the same dimension then, by means of the lift factors, we define an integer-valued index of a root class of φ that is invariant under Φ-relation and this implies that if its index is non-zero, then the root class is essential.

Keywords and Phrases: n-valued map, Nielsen number, root, lift factor, Reidemeister number, root class, index, root-uniform

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1 Introduction

Let X and Y be connected, locally path connected, semi-locally simply connected Hausdorff spaces and let $f\colon X \to Y$ be a map. Choosing $a \in Y$, a point $x \in X$ is a *root* of f at a if $f(x) = a$. The Nielsen root number $NR(f)$ of f is a lower bound for the number of roots of $g\colon X \to Y$ for every map g homotopic to f. An extensive survey of the Nielsen root theory of single-valued maps is presented in [2].

Let $D_n(Y)$ be the n^{th} unordered configuration space of Y, that is, the space of unordered subsets of n points of Y, then an *n-valued map* is a continuous function $\varphi\colon X \to D_n(Y)$. A point $x \in X$ is a *root* of φ at a if $a \in \varphi(x)$. The Nielsen root number for n-valued maps, which we will denote by $NR(\varphi)$ and will be define precisely later, was introduced in [8] and the Nielsen root theory of n-valued maps was developed further in [5]. The purpose of the present paper is to extend our understanding of this topic.

An n-valued map $\varphi\colon X \to D_n(Y)$ is *split* if there are maps $f_i\colon X \to Y$ for $i = 1, \ldots, n$ such that $\varphi(x) = \{f_1(x), \ldots, f_n(x)\}$ for all $x \in X$. For such n-valued maps, its Nielsen theory can usually be reduced to the single-valued case. For instance, Theorem 3.2 below states that $NR(\varphi)$ is the sum of the $NR(f_i)$. However, n-valued maps are not usually split. For instance, the n-valued maps of the circle are classified up to n-valued homotopy by an integer d, their degree, and by Corollary 5.1 of [3], such a map is split if and only if d is a multiple of n. Nevertheless, it is possible to obtain information in the non-split case by lifting φ to a split n-valued map of finite covering spaces of X and $D_n(Y)$. The single-valued factors of the split map are called the *lift factors* of φ. This technique, applied to the fixed point theory of such maps, was introduced by Gert-Jan Dugardein, see Section 3 of [6], and it was exploited in that paper.

In Section 2 we define the lift factors for the root theory of an n-valued map φ. The set of roots of a map are partitioned by means of an equivalence relation into subsets called the root classes of the map. The Nielsen root number of the map is the number of such classes that are essential, which can be described informally as the classes that cannot be removed by a homotopy. Thus the root classes are central to Nielsen root theory and, in Section 2, we describe how the root classes of an n-valued map and those of its lift factors relate to each other.

The precise definition of the Nielsen root number $NR(\varphi)$ of an n-valued map appears in Section 3 along with that of the corresponding Reidemeister number, which is defined in terms of the Reidemeister numbers of its lift factors. As in single-valued root theory, the Reidemeister number is a homotopy invariant upper bound for the Nielsen number.

In Section 4, for an n-valued homotopy $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$, we discuss an equivalence relation on the root classes of homotopic n-valued maps called the Φ-relation. We prove that Definition 3.7 of [2] can be generalized to n-valued maps in order to characterize essential root classes in terms of the Φ-relation.

A theorem of Brooks from [1] is the motivation for Sections 5 and 6. He proved that if $f\colon X \to Y$ is a map where Y is a closed connected manifold, then either all its root classes are essential or all are inessential. We call such a map root-uniform and, in Section 5, we study the relationship between an n-valued map and its lift factors with respect to root-uniformity. Then, in Section 6, we apply Brooks' theorem in the setting of a map $\varphi\colon X \to D_n(Y)$ where Y is a closed connected manifold, and thus it applies to the lift factors of φ. In particular, we find conditions under which the Nielsen root number and the Reidemeister root number of φ are equal. This section also contains an example to demonstrate that, in general, n-valued maps to closed connected manifolds are not root-uniform when $n > 1$.

Section 7 concerns maps $\varphi\colon X \to D_n(Y)$ for which X and Y are closed connected oriented manifolds of the same dimension. In the single-valued setting, it is possible to define an integer-valued root index of a root class which, if non-zero, implies that the root class is essential. We show that such an index can be defined for the root classes of φ by means of that of its lift factors. A root class of φ of non-zero index is essential, but if, for some homotopy $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ with $\varphi_0 = \varphi$, a root class is not Φ-related to any root class, then its root index is zero.

2 Lift factors and root classes

An n-valued map $\varphi\colon X \to D_n(Y)$ induces the fundamental group homomorphism $\varphi_\#\colon \pi_1(X) \to \pi_1(D_n(Y))$. The group $\pi_1(D_n(Y))$ is isomorphic to the braid group $B_n(Y)$ (see Chapter I, Section 3 of [9]). A homomorphism $\rho\colon B_n(Y) \to S_n$, the symmetric group, is defined for $\beta \in B_n(Y)$ by $\rho(\beta) = \sigma$ where $\beta(1) = \sigma\beta(0)$. Let $\theta = \rho\varphi_\#\colon \pi_1(X) \to S_n$. Denote by $p\colon \widehat{X} \to X$ the covering space of X corresponding to the kernel $\ker(\theta) \subseteq \pi_1(X)$, which is a finite covering space because the index of $\ker(\theta)$ in $\pi_1(X)$ equals the order of the image of θ which is a subset of S_n. Let $F_n(Y)$ be the configuration space of ordered subsets of n points of Y and let $q\colon F_n(Y) \to D_n(Y)$ be the covering space defined by $q(y_1, \dots, y_n) = \{y_1, \dots, y_n\}$. Choose $\hat{x}^* \in \widehat{X}$ and let $x^* = p(\hat{x}^*) \in X$. Choosing $y^* \in q^{-1}(\varphi(x^*)) \subseteq F_n(Y)$, we claim that there is a lift $\hat{\varphi}^*\colon \widehat{X} \to F_n(Y)$ such that $\hat{\varphi}^*(\hat{x}^*) = y^*$. From the definition of the covering space $p\colon \widehat{X} \to X$ we have

$$\varphi_{\#}(p_{\#}(\pi_1(\widehat{X}))) = \varphi_{\#}(\ker(\theta)).$$

Thus, under the isomorphism of $\pi_1(D_n(Y))$ to $B_n(Y)$, the subgroup $\varphi_{\#}(p_{\#})(\pi_1(\widehat{X}))$ is mapped to $P_n(Y)$, the subgroup of pure braids. On the other hand, the isomorphism takes the subgroup $q_{\#}(\pi_1(F_n(Y))$ of $\pi_1(D_n(Y))$ onto $P_n(Y)$. Therefore, the sufficient condition $\varphi_{\#}(p_{\#}(\pi_1(\widehat{X})) \subseteq q_{\#}(\pi_1(F_n(Y)))$ is satisfied and the lift exists.

If $\hat{\varphi}^*(\hat{x}^*) = y^* = (y_1^*, \dots, y_n^*) \in F_n(Y)$, then for $i = 1, \dots, n$, define $\hat{g}_i\colon \widehat{X} \to Y$ such that $\hat{\varphi}^* = (\hat{g}_1, \dots, \hat{g}_n)$ is the split n-valued map where $\hat{g}_i(\hat{x}^*) = y_i^*$. We call the maps $\hat{g}_j\colon \widehat{X} \to Y$ the *lift factors* of the n-valued map $\varphi\colon X \to D_n(Y)$. (Compare Section 2 of [6].)

Theorem 2.1. *The lift factors $\hat{g}_1, \dots, \hat{g}_n\colon \widehat{X} \to Y$ of $\varphi\colon X \to D_n(Y)$ are independent of the choice of the lift $\hat{\varphi}\colon \widehat{X} \to F_n(Y)$ of φ.*

Proof. Let $\hat{\varphi}^*\colon \widehat{X} \to F_n(Y)$ be the lift of φ such that $\hat{\varphi}^*(\hat{x}^*) = \hat{y}^*$. Let $\hat{\varphi}\colon \widehat{X} \to F_n(Y)$ be another lift of φ such that $\hat{\varphi}(\hat{x}^*) = \bar{y} = (\bar{y}_1, \dots, \bar{y}_n)$. There is a deck transformation $\sigma \in S_n$ of $q\colon F_n(Y) \to D_n(Y)$ such that $\sigma(\hat{y}^*) = \bar{y}$. Then $\sigma\hat{\varphi}^*(\hat{x}^*) = \bar{y} = \hat{\varphi}(\hat{x}^*)$ and thus, by Proposition 1.34 of [10], $\hat{\varphi}^* = \sigma\hat{\varphi}$. In terms of lift factors, $\hat{\varphi}^* = (\hat{g}_1, \dots, \hat{g}_n)$ and therefore

$$\hat{\varphi} = \sigma\hat{\varphi}^* = \sigma(\hat{g}_1, \dots, \hat{g}_n) = (\hat{g}_{\sigma(1)}, \dots, \hat{g}_{\sigma(n)}).$$

We conclude that the lift factors $\hat{g}_i\colon \widehat{X} \to Y$ are the same maps for all lifts $\hat{\varphi}$ of φ. □

Theorem 2.1 implies that the set of lifts is in one-to-one correspondence with the set of permutations of the n-tuple $(\hat{g}_1, \dots, \hat{g}_n)$ of the lift factors and therefore there are $n!$ lifts of φ.

If $\varphi = \{f_1, \dots, f_n\}\colon X \to D_n(Y)$ is a split n-valued map, then the image of $\varphi_{\#}\colon \pi_1(X) \to \pi_1(D_n(Y))$ is isomorphic to a subgroup of the group $P_n(Y)$ of pure braids. Therefore θ is the constant homomorphism and consequently $\widehat{X} = X$. Then, if we impose an order on the f_i, we can define $\hat{\varphi} = (f_1, \ldots, f_n)\colon X \to F_n(Y)$ and the f_i may be viewed as the lift factors $\hat{g}_i = f_i$ of φ.

Let A and B be spaces such that a group G acts on A and a group H acts on B and let $\psi\colon G \to H$ be a homomorphism. A map $f\colon A \to B$ is a *homomorphism of group actions* if $f(ga) = \psi(g)f(a)$ for all $g \in G$ and $a \in A$.

Lemma 2.1. *Let $p\colon \widetilde{X} \to X$ be a regular covering space with deck transformation group $\mathbf{D}(\widetilde{X})$ and let $q\colon \widetilde{Y} \to Y$ be a regular covering space with deck transformation group $\mathbf{D}(\widetilde{Y})$. If $\tilde{f}\colon \widetilde{X} \to \widetilde{Y}$ is a lift of a map $f\colon X \to Y$, then $\tilde{f}$ is a homomorphism of the action of $\mathbf{D}(\widetilde{X})$ on $\widetilde{X}$ to the action of $\mathbf{D}(\widetilde{Y})$ on $\widetilde{Y}$.*

Proof. Choose $\tilde{x}_0 \in \widetilde{X}$ and let $\tilde{y}_0 = \tilde{f}(\tilde{x}_0)$ and $x_0 = p(\tilde{x}_0)$ so that $y_0 = q(\tilde{y}_0) = f(x_0)$. Let $\tilde{f}_{x_0}\colon p^{-1}(x_0) \to q^{-1}(y_0)$ be the restriction of $\tilde{f}$. It is sufficient to prove that $\tilde{f}_{x_0}$ is a homomorphism of the actions of the groups of deck transformations. Let $\tilde{x}_i, \tilde{x}_j \in p^{-1}(x_0)$. By Chapter 5, Theorem 7.2 of [12], we can represent the deck transformation that takes $\tilde{x}_i$ to $\tilde{x}_j$ by some $\alpha \in \pi_1(X, x_0)$, that is, $\alpha\tilde{x}_i = \tilde{x}_j$. Let $w\colon I \to X$ be a loop at x_0 such that $[w] = \alpha$. Let $\tilde{w}\colon I \to \widetilde{X}$ be the lift of w such that $\tilde{w}(0) = x_i$, then $\tilde{w}(1) = x_j$. Let $\tilde{y}_i = \tilde{f}(x_i)$ and $\tilde{y}_j = \tilde{f}(x_j)$, then $\tilde{f}\tilde{w}(0) = \tilde{y}_i$ and $\tilde{f}\tilde{w}(1) = \tilde{y}_j$. Since $\tilde{f}$ is a lift of f, then

$$[q\tilde{f}\tilde{w}] = [fp\tilde{w}] = [fw] = f_{\#}[w] = f_{\#}(\alpha).$$

So $\tilde{y}_j = f_{\#}(\alpha)\tilde{y}_i$ and we conclude that $\tilde{f}(\alpha\tilde{x}) = f_{\#}(\alpha)\tilde{f}(\tilde{x})$ and therefore $\tilde{f}$ is a homomorphism of the group actions. □

Roots x_0, x_1 of $\varphi\colon X \to D_n(Y)$ at $a \in Y$ are *equivalent*, that is, in the same *root class*, if there is a path $v\colon I \to X$ from x_0 to x_1 such that for the splitting $\varphi v = \{f_1, \ldots, f_n\}\colon I \multimap Y$, there is some $f_j\colon I \to Y$ that is a contractible loop at a. Roots $\hat{x}_0, \hat{x}_1$ of $\hat{g}_i\colon \widehat{X} \to Y$ at a are *equivalent* and hence in the same *root class* if there is a path $\hat{v}$ from $\hat{x}_0$ to $\hat{x}_1$ such that $\hat{g}_i\hat{v}\colon I \to Y$ is a contractible loop at a.

Lift factors $\hat{g}_i, \hat{g}_j\colon \widehat{X} \to Y$ are *equivalent* if there exists $\sigma \in \mathrm{im}(\theta)$ such that $\sigma(i) = j$. Denote the set of lift factors of φ by LF_φ. Define an action $\mu\colon \mathrm{im}(\theta) \times LF_\varphi \to LF_\varphi$ by sending each lift factor $\hat{g}_i$ to the lift factor $\hat{g}_{\sigma(i)}$. The equivalence classes of lift factors are the orbits of this action.

Denote the set of roots of φ by $\mathrm{root}(\varphi)$ with the corresponding notation for single-valued maps.

Theorem 2.2. *Lift factors $\hat{g}_i$ and $\hat{g}_j$ are equivalent so that there exists $\sigma = \theta(\alpha)$, for some $\alpha \in \pi_1(X)$, such that $\sigma(i) = j$ if and only if $\hat{g}_j = \hat{g}_i h_\alpha$ where h_α is the deck transformation corresponding to α. The deck transformation h_α defines a one-to-one correspondence between the root classes of $\hat{g}_i$ and the root classes of $\hat{g}_j$.*

Proof. Suppose $\hat{g}_i$ and $\hat{g}_j$ are equivalent, that is $j = \sigma(i)$ where $\sigma = \theta(\alpha) \in S_n$ for some $\alpha \in \pi_1(X)$. By Lemma 2.1, the map $\hat{\varphi}\colon \widehat{X} \to F_n(Y)$ is a homomorphism of group actions with respect to the actions of $\mathrm{im}(\theta)$ on $\widehat{X}$ by deck transformations and of the symmetric group S_n on $F_n(Y)$. Therefore for $\hat{x} \in \widehat{X}$ we have

$$\begin{aligned} \hat{\varphi}(h_\alpha \hat{x}) &= (\hat{g}_1(h_\alpha \hat{x}), \ldots, \hat{g}_n(h_\alpha \hat{x})) \\ &= \sigma(\hat{g}_1(\hat{x}), \ldots, \hat{g}_n(\hat{x})) \\ &= (\hat{g}_{\sigma(1)}(\hat{x}), \ldots, \hat{g}_{\sigma(n)}(\hat{x})) \end{aligned}$$

so $\sigma(i) = j$ implies $\hat{g}_j = \hat{g}_i h_\alpha$.

Conversely, suppose $\hat{g}_j = \hat{g}_i h_\alpha$ and let $\sigma = \theta(\alpha)$, then $\hat{g}_i h_\alpha = \hat{g}_{\sigma(i)}$. Therefore $\hat{g}_j = \hat{g}_{\sigma(i)}$ so $j = \sigma(i)$ and thus $\hat{g}_i$ and $\hat{g}_j$ are equivalent.

To prove that h_α defines a one-to-one correspondence between the root classes of $\hat{g}_i$ and the root classes of $\hat{g}_j = \hat{g}_i h_\alpha$, let $\hat{x} \in \mathrm{root}(\hat{g}_i)$, then

$$a = \hat{g}_i(\hat{x}) = \hat{g}_i(h_\alpha h_\alpha^{-1} \hat{x}) = \hat{g}_j(h_\alpha^{-1} \hat{x})$$

so $h_\alpha^{-1}(\mathrm{root}(\hat{g}_i)) \subseteq \mathrm{root}(\hat{g}_j)$. Since $\hat{g}_j(\hat{x}) = \hat{g}_i(h_\alpha(\hat{x}))$, we conclude that the restriction $h_\alpha^{-1}\colon \mathrm{root}(\hat{g}_i) \to \mathrm{root}(\hat{g}_j)$ of h_α^{-1} is a homeomorphism. Suppose $\hat{x}_0, \hat{x}_1 \in \widehat{R}_i$, a root class of $\hat{g}_i$, so there is a path $\hat{v}\colon I \to \widehat{X}$ from $\hat{x}_0$ to $\hat{x}_1$ such that $\hat{g}_i \hat{v}$ is a contractible loop at a. Then $h_\alpha^{-1} \hat{v}$ is a path from $h_\alpha^{-1}(\hat{x}_0)$ to $h_\alpha^{-1}(\hat{x}_1)$ such that $\hat{g}_j(h_\alpha^{-1} \hat{v}) = \hat{g}_i \hat{v}$ so it is a contractible loop at a and therefore $h_\alpha^{-1}(\hat{x}_0)$ and $h_\alpha^{-1}(\hat{x}_1)$ are in a root class $\widehat{R}_j = h_\alpha^{-1}(\widehat{R}_i)$ of $\hat{g}_j$ and thus h_α^{-1} maps root classes of $\hat{g}_i$ to root classes of $\hat{g}_j$. □

Theorem 2.3. *Let $\varphi\colon X \to D_n(Y)$ be an n-valued map with lift factors $\hat{g}_i\colon \widehat{X} \to Y$ for $i = 1, \ldots, n$. The roots and root classes of φ and of its lift factors are related in the following ways.*

(a) $p^{-1}(\mathrm{root}(\varphi)) = \bigcup_{i=1}^n \mathrm{root}(\hat{g}_i)$.

(b) *Let R be a root class of φ then, for each $i = 1, \ldots, n$, either $p^{-1}(R) \cap \mathrm{root}(\hat{g}_i) = \emptyset$ or $p^{-1}(R) \cap \mathrm{root}(\hat{g}_i) = \widehat{R}_i$, which is a root class of $\hat{g}_i$.*

(c) *If R is a root class of φ and $p^{-1}(R) \cap \mathrm{root}(\hat{g}_i) = \widehat{R}_i$, then $p(\widehat{R}_i) = R$.*

Proof. (a) If $x_0 \in \operatorname{root}(\varphi)$ and $\hat{x}_0 \in p^{-1}(x_0)$, then $\hat{g}_i(\hat{x}_0) = a$ for some $1 \leq i \leq n$, and therefore $p^{-1}(\operatorname{root}(\varphi)) \subseteq \operatorname{root}(\hat{\varphi})$ which is the disjoint union of the sets $\operatorname{root}(\hat{g}_i)$. Conversely, if $p(\hat{x}_0) = x_0$ and x_0 is a root of φ at a, then since $q\hat{\varphi} = \varphi p$ it follows that $\hat{x}_0$ is a root of $\hat{\varphi}$ at a. Therefore, $p^{-1}(\operatorname{root}(\varphi)) = \operatorname{root}(\hat{\varphi})$ for any lift $\hat{\varphi} \colon \widehat{X} \to F_n(Y)$ of φ.

(b) Let $x_0, x_1 \in \operatorname{root}(\varphi)$ be in the same root class so there is a path $v \colon I \to X$ with $v(0) = x_0, v(1) = x_1$ and for $\phi v = \{f_1, \ldots, f_n\} \colon I \to D_n(Y)$ there is $1 \leq i \leq n$ such that $f_i v$ is a contractible loop at a. For $\hat{x}_0 \in p^{-1}(x_0)$, let $\hat{v}$ be the lift of v such that $\hat{v}(0) = \hat{x}_0$ and set $\hat{x}_1 = \hat{v}(1)$. Write $\hat{\varphi}\hat{v} = (\hat{g}_1, \ldots, \hat{g}_n)\hat{v} = (\hat{g}_1\hat{v}, \ldots, \hat{g}_n\hat{v}) \colon I \to F_n(Y)$. By the uniqueness of splittings (Proposition 2.1 of [8]), there exists $\hat{g}_j$ such that $\hat{g}_j\hat{v} = f_i$. Therefore $\hat{g}_j(\hat{v})$ is a contractible loop at a and thus $\hat{x}_0$ and $\hat{x}_1$ are in the same root class of $\hat{\varphi}$. Conversely, suppose $\hat{x}_0, \hat{x}_1 \in \operatorname{root}(\hat{\varphi})$ are in the same root class so there is a path $\hat{v} \colon I \to \hat{X}$ with $\hat{v}(0) = \hat{x}_0, \hat{v}(1) = \hat{x}_1$ and $\hat{g}_i(\hat{v})$ is a contractible loop at a. Then $x_0 = p(\hat{x}_0)$ and $x_1 = p(\hat{x}_1)$ are in the same root class of φ by means of the path $v = p(\hat{v})$. Thus the inverse image under p of a root class of φ is the union of root classes of lift factors.

(c) Since $p^{-1}(R) \cap \operatorname{root}(\hat{g}_i) \neq \emptyset$, there exists $x_0 \in R$ and $\hat{x}_0 \in p^{-1}(x_0)$ such that $\hat{g}_i(\hat{x}_0) = a$ so $\hat{x}_0 \in \widehat{R}_i$. Given $x_1 \in R$, as above let v be a path from x_0 to x_1 such that some $f_i v$ is a contractible loop at a and lift v to $\hat{v}$ at $\hat{x}_0$. Since $p(\widehat{R}_i) \subseteq R$ and we have shown that $\hat{x}_1 = \hat{v}(1) \in \widehat{R}_i$, we conclude that $p(\widehat{R}_i) = R$. □

Theorem 2.4. *Let R be a root class of an n-valued map $\varphi \colon X \to D_n(Y)$.* (a) *If* $\operatorname{root}(\hat{g}_i) \cap p^{-1}(R) = \widehat{R}_i$, *then* $\operatorname{root}(\hat{g}_j) \cap p^{-1}(R) = \widehat{R}_j$ *if and only if $\hat{g}_j$ is equivalent to $\hat{g}_i$.* (b) *If $\hat{g}_j$ is equivalent to $\hat{g}_i$ so that $\hat{g}_j = \hat{g}_i h_\alpha$ for some deck transformation h_α, then $h_\alpha^{-1}(\widehat{R}_i) = \widehat{R}_j$.*

Proof. (a) If $\hat{g}_j$ is equivalent to $\hat{g}_i$, then, by Theorem 2.2, $\hat{g}_j = \hat{g}_i h_\alpha$ where h_α is a deck transformation. Let $\hat{x}_0 \in \widehat{R}_i$, then $\hat{g}_j(h_\alpha^{-1}(\hat{x}_0)) = \hat{g}_i(\hat{x}_0) = a$ so $\operatorname{root}(\hat{g}_j) \cap p^{-1}(R) \neq \emptyset$ and thus by Theorem 2.3(b) it is a root class $\widehat{R}_j$ of the lift factor $\hat{g}_j$.

Conversely, suppose $\operatorname{root}(\hat{g}_i) \cap p^{-1}(R) = \widehat{R}_i$ and $\operatorname{root}(\hat{g}_j) \cap p^{-1}(R) = \widehat{R}_j$. Let $x_0 \in R$, then since $p(\widehat{R}_i) = p(\widehat{R}_j) = R$ by Theorem 2.3(c), there exists $\hat{x}_i \in \widehat{R}_i$ and $\hat{x}_j \in \widehat{R}_j$ such that $p(\hat{x}_i) = p(\hat{x}_j) = x_0$. Therefore there is a deck transformation h_η such that $h_\eta(\hat{x}_i) = \hat{x}_j$. The cardinality of $p^{-1}(x_0)$ equals the cardinality of $\operatorname{im}(\theta)$. For each $\alpha \in \operatorname{im}(\theta)$ there is a point of $p^{-1}(x_0)$ that is a root of some lift factor. The points are distinct for distinct elements of $\mathrm{im}(\theta)$ because $\mathrm{im}(\theta)$ acts transitively on $p^{-1}(x_0)$. Therefore we conclude that the lift factors $\hat{g}_i$ and $\hat{g}_j$ are equivalent.

(b) Let $\hat{x}_0 \in \widehat{R}_i$ which means that $p(\hat{x}_0) \in R$ and $\hat{g}_i(\hat{x}_0) = a$. Since h_α is a deck transformation, $ph_\alpha^{-1}(\hat{x}_0) = p(\hat{x}_0) \in R$. Furthermore, $\hat{g}_j h_\alpha^{-1}(\hat{x}_0) = \hat{g}_i h_\alpha h_\alpha^{-1}(\hat{x}_0) = \hat{g}_i(\hat{x}_0) = a$ so $h_\alpha^{-1}(\hat{x}_0) \in \widehat{R}_j$. □

Example 2.1. *Let S^1 be the complex numbers of norm one. Define $\varphi\colon X = S^1 \to D_2(S^1)$ by $\varphi(z) = \{\sqrt{z}, -\sqrt{z}\}$ where if $z = e^{it}$ for some $0 \leq t < 2\pi$ then $\sqrt{z} = e^{it/2}$. The homomorphism $\theta\colon \mathbb{Z} = \pi_1(S^1) \to S_2$ is an epimorphism so $\ker(\theta) = 2\mathbb{Z}$. Therefore, the covering space $p\colon \widehat{X} \to X$ is the double cover $p\colon S^1 \to S^1$ defined by $p(z) = z^2$. The map φ lifts to $\hat{\varphi} = (\hat{g}_1, \hat{g}_2)\colon S^1 \to F_2(S^1)$ where the lift factors are defined by $\hat{g}_1(z) = z$ and $\hat{g}_2(z) = -z$. The root class of φ is $R = \{1\}$, the root class of $\hat{g}_1$ is $\widehat{R}_1 = \{1\}$ and the root class of $\hat{g}_2$ is $\widehat{R}_2 = \{-1\}$. Since θ is an epimorphism, the lift factors $\hat{g}_i$ and $\hat{g}_2$ are equivalent.*

3 Reidemeister and Nielsen root numbers

Let $\varphi\colon X \to D_n(Y)$ be an n-valued map and denote its lift factors by $\hat{g}_j\colon \widehat{X} \to Y$ for $j = 1, \ldots, n$.

Following [2], a left coset of $\pi_1(Y)$ by $\hat{g}_i(\pi_1(\widehat{X}))$ is a *Reidemeister class* of the lift factor $\hat{g}_i$ and the number of such classes is called the *root Reidemeister number* $RR(\hat{g}_i)$ of $\hat{g}_i$. Suppose lift factors $\hat{g}_i$ and $\hat{g}_j$ are equivalent so there exists $\sigma = \theta(\alpha) \in S_n$ for some $\alpha \in \pi_1(X)$ such that $\sigma(i) = j$ and therefore, by Theorem 2.2, $\hat{g}_j = \hat{g}_i h_\alpha$ where h_α is the deck transformation that corresponds to α. Then since h_α is a homeomorphism which implies that it induces an isomorphism of fundamental groups, the set of left cosets of $\pi_1(Y)$ by $\hat{g}_{i\#}(\pi_1(\widehat{X}))$ is in one-to-one correspondence with the set of left cosets of $\pi_1(Y)$ by $\hat{g}_{j\#}(\pi_1(\widehat{X}))$, which implies that $RR(\hat{g}_i) = RR(\hat{g}_j)$. Denote the equivalence class of the lift factor $\hat{g}_i$ by $[\hat{g}_i]$ and the common value of the root Reidemeister number of the root factors in the equivalence class by $RR[\hat{g}_i]$. The *root Reidemeister number* $RR(\varphi)$ of the n-valued map $\varphi\colon X \to D_n(Y)$ is the sum of all the $RR[\hat{g}_i]$. If $n = 1$, then this is the definition of [2]. If $\varphi = \{f_1, \ldots, f_n\}\colon X \to D_n(Y)$ is split, then since the f_i are its lift factors, $RR(\varphi)$ is the sum of the $RR(f_i)$ by definition.

Theorem 3.1. *If $\varphi, \varphi'\colon X \to D_n(Y)$ are homotopic, then $RR(\varphi) = RR(\varphi')$.*

Proof. Let $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ be a homotopy such that $\varphi_0 = \varphi$ and $\varphi_1 = \varphi'$. By the covering homotopy property, we can lift Φ to $\widehat{\Phi} = \{\hat{\varphi}_t\}\colon \widehat{X} \times I \to F_n(Y)$ such that $\hat{\varphi}_0 = \hat{\varphi}$. The homotopy $\widehat{\Phi}$ splits as $\widehat{\Phi} = \{\hat{\varphi}_t\} = \{(\hat{g}_{1t}, \ldots, \hat{g}_{nt})\}$ so each lift factor $\hat{g}_{i0}$ of $\hat{\varphi}$ is homotopic to the lift factor $\hat{g}_{i1}$ of $\hat{\varphi}'$ and thus they induce the same fundamental group homomorphisms, so there is a one-to-one correspondence between $\pi_1(Y)/\hat{g}_{i0\#}(\pi_1(\widehat{X}))$ and $\pi_1(Y)/\hat{g}_{i1\#}(\pi_1(\widehat{X}))$. Moreover, since φ and φ' induce the same homomorphism of fundamental groups, they define the same equivalence relation on the lift factors and therefore $RR(\varphi) = RR(\varphi')$. □

If R is a root class of φ and $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ is a homotopy with $\varphi_0 = \varphi$, by [8] there is a unique root class $\mathbf{R}$ of Φ such that $R = \mathbf{R} \cap (X \times \{0\})$. The root class R is *essential* if $\mathbf{R} \cap (X \times \{1\}) \neq \emptyset$ for all homotopies such that $R = \mathbf{R} \cap (X \times \{0\})$ and it is *inessential* otherwise. The *Nielsen root number* $NR(\varphi)$ of φ is the number of essential root classes.

Theorem 3.2. *Let $\varphi = \{f_1, \ldots, f_n\}\colon X \to D_n(X)$ be a split n-valued map, then $NR(\varphi) = \sum_{i=1}^{n} NR(f_i)$.*

Proof. The roots, and hence the root classes, of the f_i are disjoint. Moreover, by Theorem 2.1 of [4], if $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ is an n-valued homotopy such that $\varphi_0 = \varphi$, which is split, then Φ splits as $\Phi = \{\Phi_1, \ldots, \Phi_n\}$ such that $\Phi_i(x, 0) = f_i(x)$ for all $x \in X$. Therefore, a root class of φ is a root class of some f_i and it is essential as a root class of φ if and only if it is an essential root class of f_i. □

Theorem 3.3. *Let $\varphi\colon X \to D_n(Y)$ be an n-valued map with Nielsen root number $NR(\varphi)$ and Reidemeister root number $RR(\varphi)$, then $NR(\varphi) \le RR(\varphi)$.*

Proof. Let R be an essential root class of φ. By Theorem 2.3 there is a root class $\widehat{R}_i$ of a lift factor $\hat{g}_i$ such that $p(\widehat{R}_i) = R$. By Theorem 2.4(a), if $\hat{g}_j$ is equivalent to $\hat{g}_i$, then there is a root class $\widehat{R}_j$ such that $p(\widehat{R}_j) = R$. Thus, for each essential root class of φ there is a contribution to the sum of the $RR[\hat{g}_i]$ and therefore $NR(\varphi) \le RR(\varphi)$. □

In Example 2.1, since the lift factors $\hat{g}_1$ and $\hat{g}_2$ induce isomorphisms of $\pi_1(S^1)$, then $RR(\hat{g}_1) = RR(\hat{g}_2) = 1$ and since the lift factors are equivalent, $RR(\varphi) = RR[\hat{g}_1] = 1$.

The possible relationships between the Nielsen number $NR(\varphi)$ and the Reidemeister number $RR(\varphi)$ are illustrated by the following split 2-valued maps of surfaces. Let $T = S^1 \times S^1$ be the torus. In all the examples the range of φ is $F_2(T)$ and $a = (1, 1)$.

Example 3.1. *Define* $\varphi = (f_1, f_2)\colon T \to F_2(T)$ *by setting* $\varphi(e^{it_1}, e^{it_2}) = \{(e^{it_1}, 1), (e^{i(t_1+\epsilon}, 1)\}$ *for a small* $\epsilon > 0$. *Then* $NR(f_1) = NR(f_2) = 0$ *because both* f_1 *and* f_2 *are homotopic to maps without roots at* a *so* $NR(\varphi) = 0$ *by Theorem 3.2. On the other hand,* $\pi_1(T)/f_{1\#}(\pi_1(T)) = \mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z} \oplus 0 = \mathbb{Z}$ *so* $RR(\varphi) = \infty$.

Example 3.2. *For an example like 3.1, but with* $NR(\varphi) > 0$, *we let* $\varphi(e^{it_1}, e^{it_2}) = \{(e^{it_1}, e^{it_2}), (e^{i(t_1+\epsilon}), 1)\}$ *now* $NR(\varphi) = 1$ *since* $NR(f_1) = 1$, *but still* $RR(\varphi) = \infty$.

Example 3.3. *Define maps* $f_1, f_2\colon T\#T \to T$ *by the matrices determined by the* $f_{i\#}\colon \pi_1(T\#T) \to \pi_1(T)$ *as follows:*

$$f_{1\#} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

and

$$f_{2\#} = \begin{bmatrix} 0 & 0 & 3 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

Since $\pi_1(T)$ *is abelian, a map* $k\colon T\#T \to T$ *factors through the one-point union* $T \vee T$. *That is,* $k = (k_1 \vee k_2)q$ *where* $q\colon T\#T \to T \vee T$ *is a quotient map. If*

$$k_\# = [k_{1\#} \quad k_{2\#}] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \end{bmatrix}$$

then the degree of k *is* $\deg(k_1) + \deg(k_2) = \det(k_{1\#}) + \det(k_{2\#})$. *If we define* $k\colon T\#T \to T$ *for* $x \in T\#T$ *by* $k(x) = f_1(x)(f_2(x))^{-1}$, *then*

$$k_\# = f_{1\#} - f_{2\#} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

A theorem attributed to Kneser states that if $f\colon M \to N$ *is a map of closed oriented surfaces and* $a \in N$, *then* f *is homotopic to a map without roots at* a *if and only if the degree of* f *is zero. Since the degree of the map* k *is zero, it follows that* f_1 *and* f_2 *are homotopic to maps* f_1' *and* f_2', *respectively, such that* $\varphi = (f_1', f_2')\colon T\#T \to F_2(T)$. *The degree of* f_1' *is also zero and thus* f_1' *is homotopic to a map without roots at* a *so* $NR(f_1') = 0$. *The image of* $f_{1\#}'$ *is the subgroup of* $\mathbb{Z} \oplus \mathbb{Z}$ *generated by the columns of its matrix so* $f_{1\#}'$ *is an epimorphism and* $RR(f_1') = 1$. *On the other hand, the columns of the matrix of* $f_{2\#}'$ *generate the subgroup* $0 \oplus -5\mathbb{Z}$ *which implies that* $RR(f_2') = 5$. *The degree of* f_2' *is nonzero so one of its* *root classes is essential. Therefore, by a theorem of Brooks [1] (see Theorem 6.1 below) all the root classes of f_2' are essential and thus $NR(f_2') = RR(f_2') = 5$ so $NR(\varphi) = 5$ by Theorem 3.2 whereas, since φ is split, then $RR(\varphi) = RR(f_1') + RR(f_2') = 1 + 5 = 6$.*

Example 3.4. *For an example in which $NR(\varphi) = 0$ but $RR(\varphi) > 0$, define $\varphi = (f_1, f_2)\colon T\#T \to F_2(T)$ as follows. The map f_1 is as in the previous example. Define $f_2\colon T\#T \to T$ by $f_2 = rf_1$ where $r\colon T \to T$ is $r(e^{it_1}, e^{it_2}) = (e^{i(t_1+\epsilon)}, e^{it_2})$. By calculations like those in the previous example, $NR(\varphi) = 0$ but $RR(f_1) = RR(f_2) = 1$ so $RR(\varphi) = RR(f_1) + RR(f_2) = 2$.*

4 Φ-related root classes

If $C\colon I \to X$, then $C^{-1}\colon I \to X$ is defined by $C^{-1}(t) = C(1-t)$. Let $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ be an n-valued homotopy, then $\Phi^{-1}\colon X \times I \to D_n(Y)$ is defined by $\Phi^{-1}(x,t) = \varphi_{1-t}(x)$.

Lemma 4.1. *Let $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ be an n-valued homotopy and $C\colon I \to X$ a path. Define $< \Phi, C >\colon I \to D_n(Y)$ by $< \Phi, C > (t) = \varphi_t(C(t))$, then $< \Phi^{-1}, C^{-1} > (t) = < \Phi, C >^{-1} (t)$.*

Proof. From the definitions we have

$$< \Phi^{-1}, C^{-1} > (t) = \varphi_{1-t}(C(1-t)) = < \Phi, C >^{-1} (t).$$

□

Let $\Phi = \{\varphi_t\}, \Phi' = \{\varphi_t'\}\colon X \times I \to D_n(Y)$ be homotopies such that $\varphi_0 = \varphi_0'$ and $\varphi_1 = \varphi_1'$. Define $[\Phi] = [\Phi']$ if there exists $\Psi\colon X \times I \times I \to D_n(Y)$ such that $\Psi(x,0,t) = \Phi(x,t), \Psi(x,1,t) = \Phi'(x,t), \Psi(x,s,0) = \varphi_0(x) = \varphi_0'(x)$, and $\Psi(x,s,1) = \varphi_1(x) = \varphi_1'(x)$. Let $C, C'\colon I \to X$ such that $C(0) = C'(0)$ and $C(1) = C'(1)$. Define $[C] = [C']$ if there exists $K\colon I \times I \to X$ such that $K(0,t) = C(t), K(1,t) = C'(t), K(s,0) = C(0) = C'(0)$ and $K(s,1) = C(1) = C'(1)$.

Lemma 4.2. *If $[\Phi] = [\Phi']$ and $[C] = [C']$, then*

$$[< \Phi, C >] = [< \Phi', C' >].$$

Proof. From the hypotheses we have the maps $\Psi\colon X \times I \times I \to D_n(Y)$ and $K\colon I \times I \to X$. The homotopy $< \Psi, K >\colon I \times I \to D_n(Y)$ that is defined by $< \Psi, K > (s,t) = \Psi(K(s,t), s, t)$, has the required properties because

$$\begin{aligned} < \Psi, K > (0,t) &= \Psi(K(0,t), 0, t) = \Psi(C(t)), 0, t) \\ &= \Phi(C(t), t) = < \Phi, C > (t) \end{aligned}$$

and $< \Psi, K > (1,t) = \Phi'(C'(t),t) = < \Phi', C' > (t)$. □

Let $\varphi, \psi \colon X \to D_n(Y)$ be maps and choose $a \in Y$. Roots $x_0, x_1 \in X$ at a of φ and ψ respectively are Φ-*related*, for $\Phi = \{\varphi_t\} \colon X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$, if there exists $C \colon I \to X$ such that $C(0) = x_0, C(1) = x_1$ and for $< \Phi, C >= \{f_1, \dots, f_n\} \colon I \to D_n(Y)$ some $f_i \colon I \to Y$ is a contractible loop at a. If x_0 and x_1 are Φ-related, we write $x_0 \Phi x_1$.

Lemma 4.3. *If $x_0 \Phi x_1$, then $x_1 \Phi^{-1} x_0$.*

Proof. Since $x_0 \Phi x_1$, there exists $C \colon I \to X$ such that $C(0) = x_0, C(1) = x_1$ and for $< \Phi, C > = \{f_1, \dots, f_n\}$ there is a contractible loop f_i at a. Then $C^{-1}(0) = x_1, C^{-1}(1) = x_0$ and, by Lemma 4.1, $< \Phi^{-1}, C^{-1} > = < \Phi, C >^{-1} = \{f_1^{-1}, \dots, f_n^{-1}\}$ and f_i^{-1} is a contractible loop at a so $x_1 \Phi^{-1} x_0$. □

Lemma 4.4. *If $\Phi = \{\varphi_t\}, \Phi' = \{\varphi'_t\} \colon X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi, \varphi_1 = \varphi'_0 = \psi$ and $\varphi'_1 = \zeta$ and define $\Phi\Phi' = \Phi'' = \{\varphi''_t\} \colon X \times I \to D_n(Y)$ by $\varphi''_t = \varphi_{2t}$ for $0 \le t \le 1/2$ and $\varphi''_t = \varphi'_{2t-1}$ for $1/2 \le t \le 1$. Let x_0, x_1 and x_2 be roots of φ, ψ and ζ, respectively, at a. If $x_0 \Phi x_1$ and $x_1 \Phi' x_2$, then $x_0 \Phi\Phi' x_2$.*

Proof. Since $x_0 \Phi x_1$ and $x_1 \Phi' x_2$, there are paths $C, C' \colon I \to X$ such that $C(0) = x_0, C(1) = C'(0) = x_1, C'(1) = x_2$ and for $< \Phi, C > = \{f_1, \dots, f_n\}$ and $< \Phi', C' > = \{f'_1, \dots, f'_n\}$, there are contractible loops f_i and f'_j at a, respectively. Now CC' is a path from x_0 to x_2 defined by $CC'(t) = C(t)$ for $0 \le t \le 1/2$ and by $CC'(t) = C'(2t-1)$ for $1/2 \le t \le 1$. The map $< \Phi\Phi', CC' > \colon I \to D_n(Y)$ can be split as $< \Phi\Phi', CC' > = \{f''_1, \dots, f''_n\}$ where $f''_k(t) = f_i(2t)$ for $0 \le t \le 1/2$ and $f''_k(t) = f'_j(2t-1)$ for f_i and f'_j such that $f_i(1) = f'_j(0)$. Writing $f''_k = f_i f'_j$, in particular, the map $f''_k = f_i f'_j$ is a contractible loop at a and therefore $x_0 \Phi\Phi' x_2$. □

Lemma 4.5. *Let $\Phi = \{\varphi_t\}, \Phi' = \{\varphi'_t\} \colon X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi'_0 = \varphi, \varphi_1 = \varphi'_1 = \psi$ and suppose x_0 and x_1 are roots of φ and ψ at a, respectively. If $[\Phi] = [\Phi']$ and $x_0 \Phi x_1$, then $x_0 \Phi' x_1$.*

Proof. Since $x_0 \Phi x_1$ then there is a path C from x_0 to x_1 such that for $< \Phi, C > = \{f_1, \dots, f_n\}$, some f_i is a contractible loop at a. By Lemma 4.2, $[\Phi] = [\Phi']$ implies that $[< \Phi, C >] = [< \Phi', C >]$ and therefore $x_0 \Phi' x_1$. □

Theorem 4.1. *If $x_0, x'_0 \in X$ are roots of $\varphi \colon X \to D_n(Y)$ at a that are in the same root class and $x_1, x'_1 \in X$ are roots of $\psi \colon X \to D_n(Y)$ at a that are in the same root class, then $x_0 \Phi x_1$ implies $x'_0 \Phi x'_1$. Therefore, the Φ-related condition is defined for root classes.*

Proof. Define $\bar{\varphi} = \{\bar{\varphi}_t\}, \bar{\psi} = \{\bar{\psi}_t\}\colon X \times I \to D_n(Y)$ by $\bar{\varphi}_t = \varphi$ and $\bar{\psi}_t = \psi$ for all $t \in I$. The points $x_0, x'_0 \in X$ are in the same root class of φ, so there is a path C from x_0 to x'_0 such that for $\varphi C = \{f_1, \ldots, f_n\}\colon I \to D_n(Y)$ some f_i is a contractible loop at a. Since $< \bar{\varphi}, C > (t) = \bar{\varphi}_t(C(t)) = \varphi(C(t)) = \{f_1, \ldots, f_n\}$ such that some f_i is a contractible loop at a, then $x_0\bar{\varphi}x'_0$ and, similarly, $x_1\bar{\psi}x'_1$. Since $\bar{\varphi}^{-1} = \bar{\varphi}$, then by Lemma 4.3, $x_0\bar{\varphi}x'_0 = x'_0\bar{\varphi}^{-1}x_0 = x'_0\bar{\varphi}x_0$. Given that $x_0\Phi x_1$, then Lemma 4.4 implies $x_0\Phi\bar{\varphi}x'_1$ and then also that $x'_0\bar{\varphi}(\Phi\bar{\psi})x'_1$. Define $\Delta\colon X \times I \times I \to D_n(Y)$ by $\Delta(x, s, t) = \varphi(x)$ for $0 \leq t \leq (1-s)/2, \Delta(x, s, t) = \Phi(x, t)$ for $(1-s)/2 \leq t \leq (s+3)/4$ and $\Delta(x, s, t) = \psi(x)$ for $(s+3)/4 \leq t \leq 1$. The homotopy Δ proves that $[\bar{\varphi}(\Phi\bar{\psi})] = [\Phi]$ and therefore, by Lemma 4.5, $x'_0\Phi x'_1$. □

If $\Phi = \{\varphi_t\}\colon X \to D_n(X)$ is an n-valued homotopy, R_0 is a root class of φ_0 and R_1 a root class of φ_1 that is Φ-related to R_0, then we write $R_0\Phi R_1$.

Lemma 4.6. *Let $\varphi\colon X \to D_n(y)$ be an n-valued map and $\bar{\varphi} = \{\bar{\varphi}_t\}\colon X \times I \to D_n(Y)$ the n-valued homotopy such that $\bar{\varphi}_t = \varphi$ for all $t \in I$. Let R, R' be root classes of φ. If $R\bar{\varphi}R'$, then $R = R'$.*

Proof. Let $x \in R$ and $x' \in R'$. Since $R\bar{\varphi}R'$, then there exists $C\colon I \to X$ such that $C(0) = x, C(1) = x'$ and for $< \bar{\varphi}, C >= \{f_1, \ldots, f_n\}\colon I \to D_n(Y)$ defined by $< \bar{\varphi}, C > (t) = \bar{\varphi}_t(C(t))$ there is some f_i that is a contractible loop at a. Since $\bar{\varphi}_t(C(t)) = \varphi C(t)$, then f_i proves that x and x' are equivalent roots and, since the root classes are disjoint, then $R = R'$. □

Lemma 4.7. *Let $\varphi, \psi\colon X \to D_n(Y)$ be n-valued maps and $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ an n-valued homotopy such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. Let R_φ be a root class of φ and R_ψ be a root class of ψ.*

(a) *If $R_\varphi\Phi R_\psi$, then $R_\psi\Phi^{-1}R_\varphi$ where $\Phi^{-1}(x, t) = \Phi(x, 1-t)$.*

(b) *Let $\Phi = \{\varphi_t\}, \Phi' = \{\varphi'_t\}\colon X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi, \varphi_1 = \varphi'_0 = \psi$ and $\varphi'_1 = \zeta\colon X \to D_n(Y)$. Let $R_\varphi, R_\psi, R_\zeta$ be root classes of φ, ψ and ζ, respectively. If $R_\varphi\Phi R_\psi$ and if $R_\psi\Phi' R_\zeta$, then $R_\varphi\Phi\Phi' R_\zeta$.*

(c) *Let $\Phi = \{\varphi_t\}, \Phi' = \{\varphi'_t\}\colon X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi'_0 = \varphi, \varphi_1 = \varphi'_1 = \psi$ and let R_φ and R_ψ be root classes of φ and ψ, respectively. If $[\Phi] = [\Phi']$, then $R_\varphi\Phi R_\psi$ implies that $R_\varphi\Phi' R_\psi$.*

Proof. By Theorem 4.1, parts (a), (b) and (c) follow from Lemmas 4.3, 4.4 and 4.5, respectively. □

Theorem 4.2. *Let $\varphi, \psi\colon X \to D_n(y)$ be n-valued maps and $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ an n-valued homotopy such that $\varphi_0 = \varphi$ and*

$\varphi_1 = \psi$. Let R_φ be a root class of φ and let R_ψ and R'_ψ be root classes of ψ. If $R_\varphi \Phi R_\psi$ and $R_\varphi \Phi R'_\psi$ then $R_\psi = R'_\psi$. Therefore, the homotopy Φ determines a one-to-one correspondence between a subset of the set of the root classes of φ and a subset of the set of root classes of ψ.

Proof. Since $R_\varphi \Phi R_\psi$, then $R_\psi \Phi^{-1} R_\varphi$ by Lemma 4.7(a). Therefore $R_\psi \Phi^{-1} \Phi R'_\psi$ by Lemma 4.7(b). Let $\bar{\psi} = \{\bar{\psi}_t\} \colon X \times I \to D_n(Y)$ such that $\bar{\psi}_t = \psi$ for all t. We claim that $[\Phi^{-1}\Phi] = [\bar{\psi}]$ so that $R_\psi \bar{\psi} R'_\psi$ by Lemma 4.7(c) and therefore $R_\psi = R'_\psi$ by Lemma 4.6. To prove that $[\Phi^{-1}\Phi] = [\bar{\psi}]$, define $\Delta \colon I \times I \times X \to D_n(Y)$ as follows. Set $\Delta(s,t,x) = \psi(x)$ for $0 \leq t \leq s/2$ and $1 - s/2 \leq t \leq 1$ and define $\Delta(s,t,x) = \varphi_{\alpha(t)}$ where

$$\alpha(t) = 2(1+s)t + s^2 - s - 1$$

for $1/2 \leq t \leq 1 - s/2$ and $\Delta(s,t,x) = \varphi_{\beta(t)}$ where

$$\beta(t) = -2(1+s)t + s^2 + s + 1$$

for $s/2 \leq t \leq 1/2$. □

Theorem 4.3. *Let $\varphi \colon X \to D_n(Y)$ be a map and $\Phi = \{\varphi_t\} \colon X \times I \to D_n(Y)$ a homotopy such that $\varphi_0 = \varphi$. Let R_0 be a root class of φ and $\mathbf{R}$ the root class of Φ such that $\mathbf{R} \cap (X \times \{0\}) = R_0$, then $\mathbf{R} \cap (X \times \{1\}) \neq \emptyset$ if and only if there is a root class R_1 of φ_1 such that $R_0 \Phi R_1$.*

Proof. Suppose there is a root class R_1 of φ_1 such that $R_0 \Phi R_1$. Therefore, there exists $C \colon I \to X$ such that $x_0 = C(0) \in R_0, x_1 = C(1) \in R_1$ and for the splitting $< \Phi, C >= \{f_1, \ldots, f_n\} \colon I \to D_n(Y)$ some $f_i \colon I \to Y$ is a contractible loop at a. Let $\mathbf{R} \subseteq X \times I$ be the root class of Φ such that $\mathbf{R} \cap (X \times \{0\}) = R_0$. Define $C' \colon I \to X \times I$ by $C'(t) = (C(t), t)$, then $\Phi C'(t) = \varphi_t C(t) =< \Phi, C > (t)$ and therefore $(x_1, 1) \in \mathbf{R}$ so $\mathbf{R} \cap (X \times \{1\}) \neq \emptyset$.

Conversely, suppose $\mathbf{R} \cap (X \times \{1\}) \neq \emptyset$ and choose $(x_1, 1) \in \mathbf{R}$ so there is a path $D \colon I \to X \times I$ such that $D(0) = (x_0, 0)$ and $D(1) = (x_1, 1)$ and $\Phi D = \{f_1, \ldots, f_n\} \colon I \to Y$ such that some $f_i \colon I \to Y$ is a contractible loop at a. Define $\pi_X \colon X \times I \to X$ and $\pi_I \colon X \times I \to I$ to be the projections, that is, $\pi_X(x,t) = x$ and $\pi_I(x,t) = t$. Then $\pi_I D \colon I \to I$ such that $\pi_I D(0) = 0$ and $\pi_I D(1) = 1$. Let $\Theta \colon I \times I \to I$ such that $\Theta(s,0) = 0$ and $\Theta(s,1) = 1$ for all $s \in I$, $\Theta(0,t) = \pi_I D(t)$ and $\Theta(1,t) = t$ for all $t \in I$. Let $D' \colon I \to X \times I$ be defined by $D'(t) = (\pi_X D(t), t)$, then $\Gamma \colon I \times I \to X \times I$ defined by $\Gamma(s,t) = (\pi_X D(t), \Theta(s,t))$ is a homotopy from D to

D' relative to the endpoints. Since $\Phi D = \{f_1, \ldots, f_n\}\colon I \to D_n(Y)$ such that some $f_i\colon I \to Y$ is a contractible loop at a, then $\Phi D' = \{f'_1, \ldots, f'_n\}\colon I \to D_n(Y)$ such that $f'_i\colon I \to Y$ is a contractible loop at a. But

$$\Phi D'(t) = \Phi(\pi_X D(t), t) = \varphi_t(\pi_X D(t) =< \Phi, \pi_X D > (t).$$

Therefore, letting R_1 be the root class of φ_1 that contains x_1, we conclude that $R_0 \Phi R_1$. □

We thus have the following equivalent characterization of essential root classes.

Corollary 4.1. *A root class R_0 of an n-valued map $\varphi\colon X \to D_n(Y)$ is essential if and only if for every homotopy $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ with $\varphi_0 = \varphi$ there exists a root class R_1 of φ_1 such that $R_0 \Phi R_1$.*

Although the homotopy invariance of the Nielsen root number for n-valued maps can be deduced from the corresponding property for the coincidence number as in [8], it is also a consequence of properties of the Φ-related concept.

Corollary 4.2. *Let $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ be an n-valued homotopy. If R_0 is an essential root class of φ_0 and R_1 is a root class of φ_1 such that $R_0 \Phi R_1$, then R_1 is also an essential root class. Therefore $NR(h_0) = NR(h_1)$.*

Proof. Let $\Gamma = \{\gamma_t\}\colon X \times I \to D_n(Y)$ be a homotopy such that $\gamma_0 = \varphi_1$, then $\Phi \circ \Gamma = \{(\varphi\gamma)_t\}\colon X \times I \to D_n(Y)$ defined by $(\varphi\gamma)_t = \varphi_{2t}$ for $0 \le t \le 1/2$ and $(\varphi\gamma)_t = \gamma_{1-2t}$ for $1/2 \le t \le 1$ is a homotopy such that $(\varphi\gamma)_0 = \varphi_0$. Since R_0 is an essential root class of φ_0, there is a root class R_2 of $(\varphi\gamma)_1 = \gamma_1$ such that $R_0(\Phi \circ \Gamma)R_2$. By Lemma 4.7(a), $R_0 \Phi R_1$ implies that $R_1 \Phi^{-1} R_0$. Therefore $R_1(\Phi^{-1} \circ \Phi \circ \Gamma)R_2$ by Lemma 4.7(b) and, since $[\Phi^{-1} \circ \Phi \circ \Gamma] = [\Gamma]$, then Lemma 4.7(c) implies that $R_1 \Gamma R_2$. Theorem 4.3 then implies that $\mathbf{R} \cap (X \times \{1\}) = R_2 \neq \emptyset$ so R_1 is essential. By Corollary 4.1, the homotopy Φ induces a one-to-one correspondence between the root classes of φ_0 and those of φ_1 and since the Φ-relation preserves essentiality, we conclude that $NR(h_0) = NR(h_1)$. □

5 Root-uniform maps

An n-valued map $\varphi\colon X \to D_n(Y)$ is *root-essential* (resp. *root-inessential*) if, for every n-valued map $\psi\colon X \to D_n(Y)$ homotopic to φ, every root class of ψ is essential (resp. inessential). If a map is either root-essential or root-inessential, it is said to be *root-uniform.*

In [1], Brooks proved that every single-valued map $f\colon X \to Y$, where Y is a closed manifold, is root-uniform. We will discuss consequences of Brooks' theorem for n-valued maps in the next section. In this section, we we will explore the root-uniform concept for $\varphi\colon X \to D_n(Y)$ in the more general setting where X and Y are as in the previous sections.

Let $\varphi = \{f_1, \dots, f_n\}\colon X \multimap D_n(Y)$ be a split n-valued map. Then since the root classes of the f_i are disjoint and a homotopy of a split n-valued map is split by Theorem 2.1 of [3], then φ is root-essential (resp. root-inessential) if and only if every f_i is root-essential (resp. root-inessential).

If φ is not necessarily split, we still have

Theorem 5.1. *Let R be a root class of $\varphi\colon X \to D_n(Y)$ and let $\widehat{R}_i$ be a root class of a lift factor $\hat{g}_i\colon \widehat{X} \to Y$ such that $p(\widehat{R}_i) = R$. If $\widehat{R}_i$ is an essential root class of $\hat{g}_i$, then R is an essential root class of φ.*

Proof. Let $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ be a homotopy such that $\varphi_0 = \varphi$ and let $\mathbf{R}$ be the root class of Φ such that $\mathbf{R} \cap (X \times \{0\}) = R$. By the covering homotopy theorem there is a lift $\widehat{\Phi} = \{\hat{\varphi}_t\}\colon \widehat{X} \times I \to F_n(Y)$ of Φ such that $\hat{\varphi}_0 = \hat{\varphi}$. Let $\widehat{\mathbf{R}}_i$ be the root class of $\widehat{\Phi}$ such that $\widehat{\mathbf{R}}_i \cap (\widehat{X} \times \{0\}) = \widehat{R}_i$, then $\widehat{\mathbf{R}}_i \cap (\widehat{X} \times \{1\}) \neq \emptyset$ because $\widehat{R}_i$ is essential. Consequently, $\mathbf{R} \cap (X \times \{1\}) \neq \emptyset$ and we conclude that R is also essential. $\square$

Corollary 5.1. *If a lift $\hat{\varphi}\colon \widehat{X} \to F_n(Y)$ of $\varphi\colon X \to D_n(Y)$ is root-essential then φ is root-essential.*

In Example 2.1, the root class $\{1\}$ of the identity map $\hat{g}_1\colon S^1 \to S^1$ is essential because all self-maps of S^1 homotopic to the identity are onto. Therefore, by Theorem 5.1, $\{1\}$ as a root class of φ is also essential.

Theorem 5.2. *If $\hat{g}_i, \hat{g}_j\colon \widehat{X} \to Y$ are equivalent lift factors, then there is a one-to-one correspondence between the root classes of $\hat{g}_i$ and the root classes of $\hat{g}_j$ such that a root class $\widehat{R}_i$ of $\hat{g}_i$ is essential if and only if the corresponding root class $\widehat{R}_j$ of $\hat{g}_j$ is essential.*

Proof. Let $\sigma = \theta(\alpha)$ then, by Theorem 2.2, $\hat{g}_j = \hat{g}_i h_\alpha$ where h_α is the deck transformation corresponding to $\alpha \in \pi_1(X)$. Let $\widehat{R}_i$ be a root class of $\hat{g}_i$ then also by Theorem 2.2, $\widehat{R}_j = h_\alpha^{-1}(\widehat{R}_i)$ is the corresponding root class of $\hat{g}_j$. Let $\widehat{H} = \{\hat{h}_t\}: \widehat{X} \times I \to Y$ be a homotopy such that $\hat{h}_0 = \hat{g}_j$. Define $\widehat{H}' = \{\hat{h}'_t\}: \widehat{X} \times I \to Y$ by $\widehat{H}'(\hat{x}, t) = \widehat{H}(h_\alpha^{-1}(\hat{x}), t)$. Then $\hat{h}'_0 = \hat{g}_i$ because

$$\widehat{H}'(\hat{x}, 0) = \widehat{H}(h_\alpha^{-1}(\hat{x}), 0) = \widehat{H}(\hat{g}_j h_\alpha^{-1}(\hat{x}), 0) = (\hat{g}_i, 0).$$

Now let $\widehat{\mathbf{R}}_\mathbf{i} \subseteq \widehat{X} \times I$ be the root class of $\widehat{H}$ such that $\widehat{\mathbf{R}}_\mathbf{i} \cap (X \times \{0\}) = \widehat{R}_i$. Let $\mathbf{1}_I$ denote the identity map of I. Repeating the argument above, the homeomorphism $h_\alpha^{-1} \times \mathbf{1}_I: \widehat{X} \times I \to \widehat{X} \times I$ takes the root class $\widehat{\mathbf{R}}_\mathbf{i}$ to a root class $\widehat{\mathbf{R}}_\mathbf{j}$ of $\widehat{H}'$ such that $\widehat{\mathbf{R}}_\mathbf{j} \cap (X \times \{0\}) = \widehat{R}_j$. Moreover, $(h_\alpha^{-1} \times \mathbf{1}_I)(\widehat{\mathbf{R}}_\mathbf{i} \cap (X \times \{1\}) = \widehat{\mathbf{R}}_\mathbf{j} \cap (X \times \{1\})$ so either both are empty or both are not and thus $\widehat{R}_i$ is essential if and only if $\widehat{R}_j$ is essential. □

Corollary 5.2. *If $\hat{g}_j, \hat{g}_i: \widehat{X} \to T$ are equivalent lift factors and $\hat{g}_i$ is root-essential (resp. root-inessential), then $\hat{g}_j$ is also root-essential (resp. root-inessential).*

6 Maps to closed manifolds

Theorem 6.1. *(Brooks [1]) Let $f: X \to Y$ be a map where X is a connected, locally path connected, semi-locally simply connected Hausdorff space and Y is a connected closed manifold, then f is root-uniform. If f is root-essential, then $NR(f)$ is the order of $\pi_1(X)/f_\#(\pi_1(X))$, that is, the number of left cosets of $\pi_1(X)$ by $f_\#(\pi_1(X))$.*

Let $\varphi: X \to D_n(Y)$ be an n-valued map of those same spaces, then every lift factor $\hat{g}_i: \widehat{X} \to Y$ is root-uniform and thus, by Theorem 6.1 and Theorem 5.2, each μ-orbit consists either of root-essential lift factors or of root-inessential lift factors and thus $\hat{\varphi}$ is root-essential (resp. root-inessential) if and only if all the μ-orbits consist of root-essential (resp. root-inessential) lift factors.

Theorem 6.2. *If Y is a connected closed manifold and the action μ of $\operatorname{im}(\theta)$ on LF_φ is transitive, then $\hat{\varphi}$ is root-uniform. If $\hat{\varphi}$ is root-essential, so also is φ and*

$$NR(\varphi) = RR(\varphi) = \sum_{[\hat{g}_i]} \#(\pi_1(Y)/\hat{g}_{i\#}(\pi_1(\widehat{X})))$$

where $\#(\pi_1(Y))/\hat{g}_{i\#}(\pi_1(\widehat{X})))$ is the number of left cosets of $\pi_1(Y)$ by $\hat{g}_{i\#}(\pi_1(\widehat{X}))$ and the sum is taken over the μ-orbits of lift factors.

Proof. The map $\hat{g}_1 \colon \widehat{X} \to T$ is root-uniform by Theorem 6.1. For $2 \le i \le n$, since the action μ is transitive there exists $\sigma_i \in \mathrm{im}(\theta)$ such that $\hat{g}_i = \hat{g}_1 h_{\alpha_i}$ for $\sigma_i = \theta(\alpha_i)$. Thus, by Theorem 5.2, if $\hat{g}_1$ is root-essential (resp. root-inessential), so also are all $\hat{g}_i$ and thus $\hat{\varphi}$ is root-uniform. Therefore, by Corollary 5.1, if $\hat{\varphi}$ is root-essential, so also is φ. The calculation of $NR(\varphi) = RR(\varphi)$ follows from Theorems 6.1 and 5.2. □

Example 2.1 is a simple instance of the following:

Corollary 6.1. *A non-split 2-valued map $\varphi \colon X \to D_2(Y)$ is root-essential if $\hat{\varphi}$ is root-essential.*

Proof. If a 2-valued map $\varphi \colon X \to D_2(Y)$ is not split, then the homomorphism $\varphi_\# \colon \pi_1(X) \to \pi_1(D_2(Y))$ is non-trivial by Theorem 3.1 of [7]. Thus $\theta \colon \pi_1(X) \to S_2$ is an epimorphism so the action μ is transitive and thus if $\hat{\varphi}$ is root-essential, φ is also root-essential. □

The following example demonstrates that, in contrast to Brooks' theorem, if $n > 1$ then an n-valued map to a closed manifold need not be root-uniform. Some μ-orbits may consist of root-essential lift factors and others consist of root-inessential lift factors. In the example, there is one μ-orbit of each type.

Example 6.1. *Represent the points of the unit circle S^1 in polar coordinates as $S^1 = \{e^{iu} \colon 0 \le u < 2\pi\}$. We define a split 2-valued self-map $\varphi \colon T \multimap T$ of the torus $T = S^1 \times S^1$ by*

$$\varphi(e^{iu}, e^{iv}) = ((e^{iu}, e^{iv}), (1, e^{i(v+\epsilon)}))$$

for a small $\epsilon > 0$. Let $a = (1, 1)$, then there are two root classes $R_1 = \{(1, 1)\}$ and $R_2 = S^1 \times \{e^{i(-\epsilon)}\}$. The class R_1 is essential because every self-map of the torus homotopic to the identity is onto. To prove that R_2 is inessential, define $\Phi \colon T \times I \multimap T$ by

$$\Phi((e^{iu}, e^{iv}, t) = ((e^{iu}, e^{iv}), (e^{it\epsilon}, e^{i(v+\epsilon)})).$$

7 Maps of closed orientable manifolds

Throughout this section, X and Y will be connected closed orientable m-manifolds. Roots are always defined with respect to a chosen $a \in Y$.

Let $f \colon X \to Y$ be a map and R a root class of f then, since the root classes are a finite set of compact subsets of X, there is a closed neighborhood N of R such that $N \setminus R$ contains no roots of f. As in [2], consider the composition $L_*(f, R) = (f|N)_* e_*^{-1} k_*\colon H_m(X) \to H_m(Y, Y \setminus \{a\})$ where $k\colon X \to (X, X \setminus R)$ is inclusion, the inclusion $e\colon (N, N \setminus R) \to (X, X \setminus R)$ is an excision and $f|N\colon (N, N \setminus R) \to (Y, Y \setminus \{a\})$ is the restriction of f. Choose generators u and v of the infinite cyclic groups $H_m(X)$ and $H_m(Y, Y \setminus \{a\})$, respectively. The *root index* $\lambda(f, R)$ is the integer defined by $L_*(f, R)(u) = \lambda(f, R) \cdot v$.

Let $\varphi\colon X \to D_n(Y)$ be an n-valued map and let R be a root class of φ. By Theorem 2.3(a) there is a lift factor $\hat{g}_i\colon \widehat{X} \to Y$ of φ with a root class $\widehat{R}_i$ such that $p(\widehat{R}_i) = R$. Define the *root index* $\lambda(\varphi, R)$ of R by setting $\lambda(\varphi, R) = \lambda(\hat{g}_i, \widehat{R}_i)$. The root index $\lambda(\hat{g}_i, \widehat{R}_i)$ is well-defined because $\widehat{X}$ is an orientable m-manifold since X is and $p\colon \widehat{X} \to X$ is a finite cover.

Theorem 7.1. *The definition of $\lambda(\varphi, R)$ is independent of the choice of the lift factor $\hat{g}_i$ such that $p(\widehat{R}_i) = R$.*

Proof. Let R be a root class of φ, let $\widehat{R}_i$ be the lift class of the lift factor $\hat{g}_i$ such that $p(\widehat{R}_i) = R$, and let $\hat{g}_j$ be another lift factor of φ such that $p^{-1}(R) \cap \mathrm{root}(\hat{g}_j) = \widehat{R}_j$. Therefore, $\hat{g}_j$ is equivalent to $\hat{g}_i$ by Theorem 2.4(a) and consequently, by Theorem 2.2, $\hat{g}_j = \hat{g}_i h_\alpha$ for some deck transformation h_α. Let $\widehat{N}_i$ be a closed neighborhood of $\widehat{R}_i$ such that $\widehat{N}_i \setminus \widehat{R}_i$ contains no roots of $\hat{g}_i$. Since h_α is a homeomorphism that is a lift of the identity map of X, if we let $\widehat{N}_j = h_\alpha^{-1}(\widehat{N}_i)$, then $\widehat{N}_j$ is a closed neighborhood $\widehat{R}_j$ such that $\widehat{N}_j \setminus \widehat{R}_j$ contains no roots of $\hat{g}_j$. Choose generators u and v of the infinite cyclic groups $H_m(\widehat{X})$ and $H_m(Y, Y \setminus \{a\})$, respectively. The root index $\lambda(\hat{g}_i, \widehat{R}_i)$ is the integer defined by $L_*(\hat{g}_i, \widehat{R}_i)(u) = \lambda(\hat{g}_i, \widehat{R}_i) \cdot v$ for $L_*(\hat{g}_i, \widehat{R}_i) = (\hat{g}_i|\hat{N}_i)_* e_{i*}^{-1} k_{i*}\colon H_m(\widehat{X}) \to H_m(Y, Y \setminus \{a\})$ where $k_i\colon \widehat{X} \to (\widehat{X}, \widehat{X} \setminus \widehat{R}_i)$ is inclusion and the inclusion $e_i\colon (\widehat{N}_i, \widehat{N}_i \setminus \widehat{R}_i) \to (\widehat{X}, \widehat{X} \setminus \widehat{R}_i)$ is an excision. In the same way, the root index $\lambda(\hat{g}_j, \widehat{R}_j)$ is the integer defined by $L_*(\hat{g}_j, \widehat{R}_j)(u) = \lambda(\hat{g}_j, \widehat{R}_j) \cdot v$ for $L_*(\hat{g}_j, \widehat{R}_j) = (\hat{g}_j|\hat{N}_j)_* e_{j*}^{-1} k_{j*}\colon H_m(\widehat{X}) \to H_m(Y, Y \setminus \{a\})$ where $k_j\colon \widehat{X} \to (\widehat{X}, \widehat{X} \setminus \widehat{R}_j)$ is inclusion and the inclusion $e_j\colon (\widehat{N}_j, \widehat{N}_j \setminus \widehat{R}_j) \to (\widehat{X}, \widehat{X} \setminus \widehat{R}_j)$ is an excision. By Theorem 2.4(b), $h_\alpha(\widehat{R}_j) = \widehat{R}_i$ and, because h_α is a homeomorphism, $h_\alpha(\widehat{X} \setminus \widehat{R}_j) = \widehat{X} \setminus \widehat{R}_i$ and therefore $h_{\alpha*} k_{j*} = k_{i*}$. Since $h_\alpha e_j = e_i(h_\alpha|\widehat{N}_j)$, then $(h_\alpha|\widehat{N}_j)_* e_{j*}^{-1} = e_{i*}^{-1} h_{\alpha*}$. Moreover,

$\hat{g}_j = \hat{g}_i h_\alpha$ implies that $(\hat{g}_j|\widehat{N}_j)_* = (\hat{g}_i|\widehat{N}_i)_*(h_\alpha|\widehat{N}_j)_*$. Consequently,

$$\begin{aligned} L_*(\hat{g}_j, \hat{R}_j)(u) &= (\hat{g}_j|\widehat{N}_j)_* e_{j*}^{-1} k_{j*}(u) \\ &= (\hat{g}_i|\widehat{N}_i)_*(h_\alpha|\widehat{N}_i)_* e_{j*}^{-1} k_{j*}(u) \\ &= (\hat{g}_i|\widehat{N}_i)_* e_{i*}^{-1} k_{i*}(u) = L(\hat{g}_i, \widehat{R}_i)(u) \end{aligned}$$

and therefore $\lambda(\hat{g}_i, \widehat{R}_i) = \lambda(\hat{g}_j, \widehat{R}_j)$. □

Theorem 7.2. *Let $\varphi, \psi\colon X \to D_n(Y)$ be n-valued maps and $\Phi = \{\varphi_t\}\colon X \times I \to D_n(Y)$ an n-valued homotopy such that $\varphi_0 = \varphi$ and $\varphi_1 = \psi$. Let R_0 be a root class of φ and R_1' a root class of ψ such that R_0 and R_1' are Φ-related, then $\lambda(\varphi, R_0) = \lambda(\psi, R_1')$.*

Proof. Choose points $x_0 \in R_0$ and $x_1' \in R_1'$, then there exists a path $C\colon I \to X$ such that $C(0) = x_0, C(1) = x_1'$ and for $< \Phi, C >= \{f_1, \ldots, f_n\}\colon I \to D_n(Y)$ some $f_k\colon I \to Y$ is a contractible loop at a. Since $x_0 \in \mathrm{root}(\varphi)$, by Theorem 2.3(b)(c) there is a lift factor $\hat{g}_i$ of φ with a root class $\widehat{R}_i$ of $\hat{g}_i$ such that $p(\widehat{R}_i) = R_0$. Choose $\hat{x}_0 \in \widehat{R}_i$ and let $\widehat{C}\colon I \to \widehat{X}$ be the lift of C such that $\widehat{C}(0) = \hat{x}_0$. Set $\hat{x}_1' = \widehat{C}(1)$, then $p(\hat{x}_1') = x_1'$ so $\hat{x}_1'$ is in a root class $\widehat{R}_j'$ of a root factor $\hat{g}_j'$ of ψ such that $p(\widehat{R}_j') = R_1'$. Let $\widehat{\Phi} = \{\hat{\varphi}_t\}\colon \widehat{X} \times I \to F_n(Y)$ be the lift of Φ such that $\hat{\varphi}_0 = \hat{\varphi}$. Since $\widehat{\Phi}$ is a lift of Φ and $\widehat{C}$ is a lift of C, then $q\hat{\varphi}_t(\widehat{C}(t)) = \varphi_t p(\widehat{C}(t)) = \varphi_t(C(t))$ that is, $q(< \widehat{\Phi}, \widehat{C} > (t)) = < \Phi, C > (t)$ for all $t \in I$. Now $< \widehat{\Phi}, \widehat{C} >= (\hat{f}_1, \ldots, \hat{f}_n)\colon \widehat{X} \to F_n(Y)$ and $< \Phi, C >= \{f_1, \ldots, f_n\}$ such that some $f_k\colon I \to Y$ is a contractible loop at a so, since $q\colon F_n(Y) \to D_n(Y)$ induces a monomorphism of fundamental groups, then some $\hat{f}_\ell\colon I \to Y$ is a contractible loop at a. Therefore the points $\hat{x}_0$ and $\hat{x}_1'$ are $\widehat{\Phi}$-related and thus, by Theorem 4.1, the root classes $\widehat{R}_i$ of $\hat{g}_i$ and $\widehat{R}_j'$ of $\hat{g}_j'$ are $\widehat{\Phi}$-related. The homotopy $\widehat{\Phi}$ splits as $\widehat{\Phi} = (\widehat{H}_1, \ldots, \widehat{H}_n)$ where each lift factor $\widehat{H}_k\colon \widehat{X} \times I \to Y$ of $\widehat{\Phi}$ is a homotopy between a lift factor of φ and a lift factor of ψ. In particular, there is a lift factor $\widehat{H}_k$ of $\widehat{\Phi}$ that is a homotopy between $\hat{g}_i$ and $\hat{g}_j'$. Theorems 4.6.1 and 4.10 of [2] then imply that $\lambda(\hat{g}_i, \widehat{R}_i) = \lambda(\hat{g}_j', \widehat{R}_j')$ so, by definition and Theorem 7.1, $\lambda(\varphi, R_0) = \lambda(\psi, R_1')$. □

In particular, if $\varphi = \{f_1, \ldots, f_n\}\colon X \to D_n(y)$ is a split n-valued map, since the root classes are disjoint, the set of root classes of φ is the disjoint union of the root classes of the lift factors f_i. Thus R is a root class of φ if it is a root class of some f_i and $\lambda(\varphi, R) = \lambda(f_i, R)$. Let R_i be a root class of f_i, R_j a root class of f_j and

$H = \{h_t\}: X \times I \to Y$ a homotopy such that $h_0 = f_i$ and $h_1 = f_j$. If $R_i \Phi R_j$, then $\lambda(\varphi, R_i) = \lambda(f_i, R_i) = \lambda(f_j, R_j) = \lambda(\varphi, R_j)$.

Lemma 7.1. *Let $f: X \to Y$ be a map and R an inessential root class of f at a, then $\lambda(f, R) = 0$.*

Proof. Since R is inessential, there is a homotopy $H = \{h_t\}: X \times I \to Y$ such that $h_0 = f$ and $\mathbf{R} \cap (X \times \{1\}) = \emptyset$ for $\mathbf{R}$ the root class of H such that $\mathbf{R} \cap (X \times \{0\}) = R$. Let $\mathbf{N}$ be a closed neighborhood of $\mathbf{R}$ such that $\mathbf{N} \setminus \mathbf{R}$ contains no roots of f at a. Let $N_t = \mathbf{N} \cap (X \times \{t\})$ and $R_t = \mathbf{R} \cap (X \times \{t\})$. Let $k_t: X \to (X, X \setminus R_t)$ be inclusion, $e_t: (N_t, N_t \setminus R_t) \to (X, X \setminus R_t)$ be inclusion which is an excision, and $f|N_t: (N_t, N_t \setminus R_t) \to (Y, Y \setminus \{a\})$ be the restriction of f. Then $L_*(f, R) = (f|N_t)_*(e_t)_*^{-1} k_{t*}: H_m(X) \to H_m(Y, Y \setminus \{a\})$ is independent of t because the homomorphisms are induced by homotopic maps. Since $R_1 = \emptyset$, then $H_m(N_1, N_1 \setminus R_1) = H_m(N_1, N_1) = 0$, so $L_*(f, R)$ is the zero homomorphism and therefore $\lambda(f, R) = 0$. □

Theorem 7.3. *Let R be a root class of $\varphi: X \to D_n(Y)$ such that $\lambda(\varphi, R) \neq 0$, then R is essential.*

Proof. To prove the contrapositive statment, let R be an inessential root class of φ. Therefore, there is a homotopy $\Phi = \{\varphi_t\}: X \times I \to D_n(Y)$ such that $\varphi_0 = \varphi$ and $\mathbf{R} \cap (X \times \{1\}) = \emptyset$ for $\mathbf{R}$ the root class of Φ such that $\mathbf{R} \cap (X \times \{0\}) = R$. Let $\widehat{\Phi} = \{\hat{\varphi}_t\} = (\widehat{H}_1, \ldots, \widehat{H}_n): \widehat{\Phi} \times I \to F_n(Y)$ be the lift of Φ such that $\hat{\varphi}_0 = \hat{\varphi}$ and let $\widehat{R}_i$ be a root class of $\hat{g}_i$, a lift factor of φ, such that $p(\widehat{R}_i) = R$. Let $\widehat{H}_k$ be the factor of $\widehat{\Phi}$ such that $\widehat{H}_k \cap (\widehat{X} \times \{0\}) = \hat{g}_i$ and let $\widehat{\mathbf{R}}$ be the root class of $\widehat{H}_k$ such that $\widehat{\mathbf{R}} \cap (\widehat{X} \times 0) = \widehat{R}$. Since $\mathbf{R} \cap (X \times \{1\}) = \emptyset$, then $\widehat{\mathbf{R}} \cap (\widehat{X} \times 1) = \emptyset$ and therefore $\widehat{R}_i$ is an inessential root class of $\hat{g}_i$. By Lemma 7.1 and the definition of the root index, we have $\lambda(\varphi, R) = \lambda(\hat{g}_i, \widehat{R}_i) = 0$. □

For Example 2.1 we have $\lambda(\varphi, \{1\}) = \lambda(\hat{g}_1, \{1\})$ and since $\hat{g}_1: S^1 \to S^1$ is the identity function, we can choose homology generators so that $\lambda(\hat{g}_1, \{1\}) = 1$ and therefore since $\lambda(\varphi, \{1\}) \neq 0$ we conclude that $NR(\varphi) = 1$.

From the contrapositive statement of Theorem 7.3, we have the following consequence.

Corollary 7.1. *Let R be a root class of $\varphi: X \to D_n(Y)$ and let $\Phi = \{\varphi_t\}: X \times I \to D_n(X)$ a homotopy such that $\varphi_0 = \varphi$. If R is not Φ-related to any class of φ_1, then $\lambda(\varphi, R) = 0$.*

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