A Good Question Won't Go Away: An Example Of Mathematical Research

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Abstract. The story of the question "must commuting maps of the unit interval have a common fixed point" is used to illustrate strategies that advance mathematical research.

1. INTRODUCTION. Some time ago, we were discussing this at a math conference: what would you say if you met someone at a cocktail party and she asked you about your work? If you reply "I teach and I do research" and she asks "what sort of research?" do you just get embarrassed and mumble "it's hard to explain" or do you have something coherent to say?

I do have something to say. "I work with geometric objects. I'm interested in what happens when you move the points around on these objects in a continuous way, which means that nearby points are moved to nearby locations. For example, if you continuously move the points on the sphere, that is, the surface of a ball, just a little bit, it turns out that at least one point has to stay fixed. On the other hand, I can move all the points of a torus, that is, the surface of a donut, a little bit just by rotating it. So that rotation has no fixed points. My specialty is called fixed point theory and what I've just described for you is how fixed point theory reveals a rather subtle mathematical difference between a sphere and a torus."[1]

I probably would have told my cocktail party acquaintance just about as much as she ever wanted to know about mathematical research. I assume that readers of the MONTHLY have a much longer attention span for such topics, especially if they have not (yet) participated in research either in an undergraduate research program or as a graduate student. In this article, I will tell the story of a fixed point question, a story that demonstrates several significant features of mathematical research.

2. THE QUESTION. The question concerns moving points on a geometric object, but one even more basic than the sphere or torus: the interval $I = [0, 1]$ in the real line. A continuous function, for which I'll use the simpler term *map*, f from the interval to itself must have a fixed point. The reason is that the difference function $d(x) = f(x) - x$ is a map because f is, $d(0) \geq 0$, and $d(1) \leq 0$, so the intermediate value theorem tells us that $d(x_*) = 0$ for some $x_* \in I$ and therefore $f(x_*) = x_*$. Now consider two maps f and g of I to itself. Both have fixed points, but in general the fixed points of f have no relationship with those of g. However, suppose the maps f and g are related, are their fixed points also related? Specifically,

The Question: Suppose $f, g\colon I \to I$ *commute*, that is, $f(g(x)) = g(f(x))$ for all $x \in I$. Do their fixed point sets intersect? That is, if f and g are commuting self-maps of the interval, do they have a *common fixed point*: a solution x_* to the equations $f(x_*) = x_* = g(x_*)$?

Although this question was not published, it was informally posed, independently, by three different people: Eldon Dyer in 1954, Alan Shields in 1955, and Lester Dubins

[1] Unfortunately, I have never had the opportunity to say all that. Perhaps I should go to more cocktail parties.

in 1956. The question may have been motivated by the fact that the common fixed point property was known to hold for commuting polynomials. J. F. Ritt [**27**] proved in 1923 that, besides rather obvious examples like $f(x) = (p(x))^m, g(x) = (p(x))^n$ for some polynomial $p(x)$, the only polynomials that commute are the Chebyshev polynomials. This much-studied class was known to have the common fixed point property, see [**3**].[2]

The earliest response to the question, in terms of publication date, was a paper by Ralph DeMarr [**12**], published in this MONTHLY in 1963. The research strategy that DeMarr employed was also used by several other researchers, as we shall see. It appeared that the hypotheses, that f and g are continuous and that they commute, were not strong enough to imply that they shared a fixed point. So DeMarr strengthened the hypotheses to require a more restrictive form of continuity: the maps f and g must be Lipschitz continuous. Recall that a map $f\colon I \to I$ is *Lipschitz continuous* if there exists $\alpha > 0$ such that $|f(x) - f(y)| < \alpha|x - y|$ for all $x, y \in I$. Notice that the number α, called the *Lipschitz constant*, is not unique since the inequality will hold for any larger value. Let β be the Lipschitz constant for g. If the smallest possible values for both α and β is greater than 1, then he showed that the maps have a common fixed point provided these constants satisfy a certain inequality that thus relates the behavior of the two maps.

However, if for one of the maps, call it f, we can use $\alpha = 1$, that is, if $|f(x) - f(y)| \le |x - y|$ for all $x, y \in I$ so that f is what is called a *nonexpansive map*, then there is a common fixed point even if g is only continuous rather than Lipschitz continuous. To prove this, let $\Phi(f)$ denote the set of fixed points of f, which is a closed subset of I and thus has a maximum M and minimum m. If $x \in [m, M]$, then

$$f(x) - m \le |f(x) - m| = |f(x) - f(m)| \le |x - m| = x - m,$$

so $f(x) \le x$. In the same way, we can see that $M - f(x) \le M - x$ so $f(x) \ge x$ and we have proved that if f is a nonexpansive map, then $\Phi(f) = [m, M]$. Now if $x \in [m, M]$ then since g commutes with f we have $g(x) = g(f(x)) = f(g(x))$, so the restriction of g is a map of $[m, M]$ to itself and the intermediate value theorem again tells us that $g(x_*) = x_*$ for some $x_* \in [m, M] = \Phi(f)$, that is, x_* is a common fixed point. Thus, if a map commutes with a nonexpansive map, then there is a common fixed point. A few years later, in another MONTHLY paper, Gerald Jungck [**22**] refined DeMarr's conditions so that neither map need actually be Lipschitz continuous, but the maps still must exhibit similar closely related behavior.

Haskell Cohen's paper [**11**] established the existence of a common fixed point by strengthening the hypotheses on the commuting functions f and g in a different way, namely that they are not only continuous but also *open*. In our setting that means that the images of subsets of I that are intersections of open intervals of the real line with I are subsets of the same kind.[3] In particular, homeomorphisms are open maps, so Cohen's result proves that commuting homeomorphisms have a common fixed point. Just as the posing of the problem occurred to several people independently, the same thing happened with the open map condition. Jon Folkman [**14**] and James Joichi [**21**] both improved Cohen's result by showing that if just *one* of the commuting maps f or g is an open map, then they have a common fixed point.

A simple lemma in Folkman's paper can give us a sense of the significance of the commutativity condition. We will not try to prove that a map of the interval that com-

[2]There are more extensive discussions of the background of the question in the introduction to [**5**] and in Section 4 of [**26**].

[3]Cohen stated his condition in different terms, but it was subsequently established that his condition was equivalent to the concept of an open map which is widely employed in topology.

mutes with an open map has a common fixed point with it. Instead we will establish the much weaker result that a map that commutes with a monotone map has a common fixed point with it. Let f be a monotone map of I and g a map that commutes with it. If f is monotone decreasing, it has a single fixed point x_* and since the maps commute, $f(g(x_*)) = g(f(x_*)) = g(x_*)$, so $g(x_*)$ is a fixed point of f and, since there is only one, $g(x_*) = x_*$. If f is monotone increasing, let x_0 be a fixed point of g and define a sequence in I by setting $x_n = f(x_{n-1})$ for $n \geq 1$; then (x_n) is an increasing sequence and therefore it converges to some $x_* \in I$. We can prove by induction that the points (x_n) of the sequence are all fixed points of g: it's true for $n = 0$ and if $g(x_{n-1}) = x_{n-1}$, then by commutativity

$$g(x_n) = g(f(x_{n-1})) = f(g(x_{n-1})) = f(x_{n-1}) = x_n.$$

Consequently, x_* is a fixed point of g because g is continuous, as follows:

$$g(x_*) = g(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} x_n = x_*.$$

But x_* is also a fixed point of the continuous function f because

$$x_* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x_*).$$

A different supplementary hypothesis for our question depends on the concept of a periodic point. Given a function $f\colon X \to X$ let $f^1(x) = f(x)$ and, in general, define $f^n\colon X \to X$ by $f^n(x) = f(f^{n-1}(x))$. A point $x \in X$ is a *periodic point* of f if $f^n(x) = x$. If $f^n(x) = x$ but $f^k(x) \neq x$ for all $k < n$ then n is called the *least period* of the periodic point x. We will simplify the terminology by calling such a periodic point a *period n point*.

Again similar conditions strong enough to imply a common fixed point were discovered independently. John Maxfield and W. J. Mourant [**25**] proved that commuting maps on the unit interval have a common fixed point if one of the maps has no period 2 points. In other words, if $f(f(x)) = x$ implies $f(x) = x$. The condition of Sherwood Chu and R. D. Moyer [**9**] is that there is a subinterval $[a, b]$ on which one of the maps has a fixed point and the other has no period 2 points.

The periodic point concept appears in the somewhat different approach to our question by Arthur Schwartz in [**28**]. In attempting to answer a question like ours, of whether certain hypotheses imply the desired conclusion, rather than strengthening the hypotheses, you can ask whether the given hypotheses are at least sufficient to establish a somewhat weaker conclusion. Rather than ask whether the commuting maps f and g have a common fixed point, we could require only that they have a common periodic point, of possibly different periods. In principle, this should be easier to prove since maps usually have many periodic points that are not fixed points. In [**28**] Schwartz did require a strengthening of the hypotheses, namely, that f is differentiable rather than only continuous. His conclusion is that there is a fixed point of f that is a periodic point of g, but it is not necessarily a fixed point of g.

3. THE ANSWER. It seems appropriate that a question that independently occurred to more than one person should have been answered independently by two people. In 1967 William Boyce and John Huneke constructed, and in 1969 published in [**5**] and [**20**] respectively, examples of commuting maps of the interval that have no common fixed point, so the answer to our question is "no." Before discussing the examples, let's note that this could be viewed as good news for all the people mentioned above. If the answer had been "yes," that is, commuting maps of the interval always have a common fixed point, then all their results would just be partial solutions to a subsequently solved problem. But since the general answer is "no" their contributions are part of a more interesting answer: "not always, but . . . "

In [**20**], Huneke presented two examples. The first is identical to the example in Boyce's paper while the other, obtained in a different manner, is, in Huneke's words "somewhat smoother" with regard to Lipschitz continuity. We will restrict this discussion to the example shared by Boyce and Huneke.

According to Boyce, the key to the construction is contained in a paper of Glen Baxter [**2**] that was inspired by our question and had recently been published. To understand what Baxter did, keep in mind that we can think of the graph of a map $f\colon I \to I$ as a curve in the unit square and the fixed points of f as the intersections of its graph with the diagonal of the unit square. Starting with the usual commuting maps f and g, Baxter noted that the map f permutes the fixed points of the map h defined by $h(x) = f(g(x))$ because if $h(x) = x$, then $f(h(x)) = f(x)$ and by commutativity

$$f(h(x)) = f(f(g(x)) = f(g(f(x)) = h(f(x)),$$

so $h(f(x)) = f(x)$. Assuming that h has a finite number of fixed points, he divided the fixed points of h on $(0, 1)$ into three classes depending on how the graph of h relates to the diagonal: crosses it heading down, crosses it heading up, or hits it but does not cross it. The theorem of Baxter is that f can take a fixed point of h only to a fixed point of the same class. If we number the fixed points of h from left to right as x_1 to x_N, not all permutations of the subscripts are possible since, for instance, the graph of h cannot cross the diagonal heading down at consecutive fixed points. The possible permutations became known as *Baxter permutations*.

The example is based on a Baxter permutation of 13 crossings of the diagonal. Boyce used a computer to reduce the number of possible Baxter permutations that he needed to study to 112 cases and he showed that the only one that could generate the example was number 101.[4] In order to define the commuting functions f and g, Boyce explicitly defined piecewise linear functions f_1, f_2, g_1, and g_2 from I to itself. Thus the graphs of those functions are line segments joined end to end. He then inductively defined sequences of functions (f_n) and (g_n) with the property that $f_n g_{n+1} = g_n f_{n+1}$. He proved that the sequences are uniformly convergent so their limits f and g, which commute, are continuous and the f_n and g_n were defined in such a way that the fixed point sets of f and g are disjoint.

4. RESEARCH STRATEGIES AFTER THE ANSWER.

Like a stone thrown into a still pond that sends out ripples in all directions, this question and its answer were just the starting points for wide-ranging research. Since our focus is on research strategies, I will just discuss some results from this literature that illustrate those strategies.[5]

After the publication of the examples of Boyce and Huneke, there were several improvements to previous results since the negative answer allowed for extensions of what was known. For example, Boyce in [**6**] refined the conditions of Maxfield and Mourant and of Chu and Moyer by proving that if maps of the interval commute and the set of period 2 points of one of them is connected, in particular if there is just one such point, then they still have a common fixed point. Much later, in [**23**], Jungck

[4] Since computers only became generally available to researchers in the 1950s, this 1967 example must have been one of the first uses of a computer to solve an abstract mathematical problem.

[5] A much more thorough exposition of the research inspired by our question is presented as Section 4 of [**26**].

greatly clarified the relationship between fixed and periodic points. The *coincidence set* of f and g is the set of points $x \in I$ such that $f(x) = g(x)$. Jungck proved that if f and g commute on their coincidence set, then they have a common fixed point if and only if g has no periodic points other than fixed points.

Our survey of the literature prior to the answering of the question did not illustrate the research strategy that is probably the one most frequently employed: generalization. The definitions of commuting maps and common fixed points make sense for any maps $f, g \colon X \to X$ of any topological spaces, though in general you would not expect the first to imply the second even in the presence of rather strong supplementary conditions. What is known about maps of the unit interval certainly extends to any *arc*, that is, a space X for which there is a homeomorphism $h \colon I \to X$, because in that case a map $f \colon X \to X$ can be translated to the map $h^{-1}fh \colon I \to I$. The question becomes more interesting if X is an *arc-like continuum*, that is, a compact connected space that is a limit, in an appropriate sense, of arcs. Jan Boronski in [**4**] recently proved that there are even commuting homeomorphisms, not just maps, of an arc-like continuum that have no common fixed points. In a different direction of generalization, suppose Y is a simple triad, that is, three arcs that intersect only at a common endpoint so it looks like the letter "Y." If $f, g \colon Y \to Y$ are commuting maps, do they at least have a coincidence point $y_* \in Y$, that is, $f(y_*) = g(y_*)$, even though it may not be a fixed point? In 2009, Eric McDowell published an extensive discussion [**26**] of coincidence points. But as yet the simple triad question remains unanswered.

Although commuting maps of the interval may fail to have a common fixed point, we have seen that additional hypotheses do imply its existence, in particular if one of the maps is open [**14, 21**]. Suppose that f and g are commuting open and onto maps of a space that has many of the topological properties of the interval, must they have a common fixed point? In 1975, William Gray and Carol Smith discussed this question in [**15**] and proposed the properties that the space should share with the interval, but it seems that no progress has been made in answering their question.

A natural direction of generalization concerns dimension. The unit interval is one-dimensional; what happens if we consider commuting maps in the next dimension, that is, maps of the unit square I^2? Martin Grinc [**16**] studied what he called *triangular maps*, that is, maps $F \colon I^2 \to I^2$ of the form $F(x, y) = (f_1(x), f_2(x, y))$ for continuous f_1 and f_2. He found conditions under which commuting triangular maps of the unit square have common fixed points and, in particular, he and Lubomir Snoha [**17**] proved that Jungck's theorem, that a map with no periodic points other than fixed points has a common fixed point with every map that commutes with it, extends to triangular maps. Antonio Linero in [**24**] considered a class of maps F, G of the square that are of the form $F(x, y) = (f_2(y), f_1(x)), G(x, y) = (g_2(y), g_1(x))$ where the f_i and g_i are continuous. He called them *Cournot maps*, after the economist who introduced them in the 19th century. They are a significant class of functions in mathematical economics. He was able to extend results of Grinc to Cournot maps under suitable hypotheses.

Another quite different way to generalize results to two dimensions is to exploit the fact that, just as the points of the line can be viewed as the real numbers, the points of the plane represent the complex numbers, so a map defined on the plane can be viewed as a complex function. Dan Eustice in [**13**] proved that commuting maps of the 2-disc have a common fixed point if they are holomorphic, that is, complex differentiable, on the interior. Eustice's theorem was extended to holomorphic maps of all cartesian products of discs by Roberto Tauraso in [**29**].

There is a generalization that concerns maps on the unit interval, but now with respect to periodic points as a conclusion rather than a hypothesis. Must the sets of all the periodic points of commuting maps have a nonempty intersection? As noted above, Schwartz proved that the answer is "yes" if one of the maps is differentiable, but Aliasghar Alikhani-Koopaei conjectured that it's "no" if only continuity is assumed [**1**]. Jose Canovas and Antonio Linero proved in [**8**] that the answer is "yes" provided the set of periodic points of one of the maps is a closed subset of I, and in fact then a fixed point of that map is a periodic point of the other one. However, it appears that Alikhani-Koopaei's conjecture is still unsettled.

Recall that the Baxter permutations arose as part of the investigation of our question and were the key to its negative answer. Of the $N!$ permutations of the numbers 1 to N, how many of them are Baxter permutations? In what the reviewer in [**19**] described as "a gem of enumeration," Fan Chung, Ronald Graham, Verner Higgett, and Mark Kleiman obtained the desired formula in terms of binomial coefficients in [**10**]. But that was by no means the end of the interest in these combinatorial objects, which, as its extensive literature demonstrates, have close connections with many other topics in that area of mathematical research. For example, in [**18**], Olivier Guibert and Svante Linusson related the Baxter permutations to the Catalan numbers that were defined in the 19th century and in 2015, Benjamin Caffrey and his coauthors [**7**] introduced "snow leopard permutations" as a useful type of Baxter permutation.

I have assumed that readers of the MONTHLY have a greater attention span for a discussion of mathematical research than my hypothetical cocktail party companion and, if you are still reading, you have demonstrated it. By now I hope that I have convinced you: a good question just won't go away.

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