TWO IMPORTANT THEOREMS THAT ARE
REALLY ONE

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Section 2.8 of the widely used textbook *Single Variable Calculus*
by John Rogawski and Colin Adams [4] is devoted to a theorem that
the book states this way:

**Theorem 1.** *(Intermediate Value Theorem)* If the function $f(x)$ is
continuous on $[a, b]$ with $f(a) \neq f(b)$ and $y$ is between $f(a)$ and $f(b)$,
then there exists $c \in (a, b)$ such that $f(c) = y$.

The point $y$ is “between” $f(a)$ and $f(b)$ means either that $f(a) < y < f(b)$
or that $f(a) > y > f(b)$, depending on how $f(a)$ and $f(b)$
are related to each other.

In order to avoid “between”, with its two cases, the theorem is
sometimes stated in this neater way:

**Theorem 2.** *(Intermediate Value Theorem)* If the function $g(x)$
is continuous on $[a, b]$ and $g(a)g(b) < 0$, then $g(c) = 0$ for some
$c \in (a, b)$.

What do I mean by claiming that Theorem 2 is also the Inter-
mediate Value Theorem? Theorems 1 and 2 are the same theorem
in the sense that they are equivalent, which means that each can
be proved directly from the other. Certainly Theorem 2 is a special
case of Theorem 1 since one of $g(a)$ and $g(b)$ is positive and the
other is negative. But if you assume that Theorem 2 is true, then
so is Theorem 1 because if $f(x)$ satisfies the hypotheses of Theorem
1, then $g(x) = f(x) - y$ satisfies the hypotheses of Theorem 2 and
$0 = g(c) = f(c) - y$ means that $f(c) = y$.

In that textbook, Problem 32 of Section 2.8 is to prove
Theorem 3. Assume that \( f(x) \) is continuous and that \( a \leq f(x) \leq b \) for \( a \leq x \leq b \). Then \( f(c) = c \) for some \( c \in [a, b] \).\(^1\)

If \( f(a) = a \) or \( f(b) = b \) there is nothing to prove so we can assume \( f(a) > a \) and \( f(b) < b \), then \( g(x) = f(x) - x \) satisfies the hypotheses of Theorem 2 and \( g(c) = 0 \) implies that \( f(c) = c \).

Now we’ll go a bit beyond what is in [4]. Theorem 3 is equivalent to this slightly more general statement.

Theorem 4. If the function \( g(x) \) is continuous on \([a, b]\) with \( g(a) \geq a \) and \( g(b) \leq b \), then there exists \( c \in [a, b] \) such that \( g(c) = c \).

Again there is nothing to prove unless \( g(a) > a \) and \( g(b) < b \). Define \( r(x) \) by setting \( r(x) = a \) for all \( x \leq a \), \( r(x) = x \) for \( a \leq x \leq b \) and \( r(x) = b \) for all \( x \geq b \). Then \( f(x) = r(g(x)) \) is continuous because it is the composition of continuous functions and it satisfied the hypotheses of Theorem 3, so \( f(c) = r(g(c)) = c \) for some \( c \in [a, b] \). But since \( g(a) > a \) and \( g(b) < b \), we can have \( f(c) = c \) only if \( g(c) = c \).

So Theorem 4 is a consequence of the Intermediate Value Theorem because it is equivalent to Theorem 3. The authors of [4] could have gone a step further because we will show that Theorem 4 in turn implies the Intermediate Value Theorem and thus those theorems are equivalent.

For the Intermediate Value Theorem in the form of Theorem 2 we have a function \( g(x) \) continuous on \([a, b]\) with \( g(a)g(b) < 0 \). If \( g(a) > 0 \) and \( g(b) < 0 \), let \( f(x) = g(x) + x \) which satisfies the hypotheses of Theorem 4 so \( f(c) = g(c) + c = c \) and thus \( g(c) = 0 \).

The other possibility is that \( g(a) < 0 \) and \( g(b) > 0 \) in which case let \( f(x) = x - g(x) \) and then \( f(c) = c - g(c) = c \) means \( g(c) = 0 \). Therefore Theorem 4 implies Theorem 2 and thus these theorems are equivalent.

Theorem 3 is a special case of our first “important theorem”, as promised in the title: the Brouwer fixed point theorem. Instead of the function \( f(x) \) that takes the real line to itself, the theorem discovered by L. E. J. Brouwer early in the 20th century concerns a function that takes Euclidean \( n \)-space \( \mathbb{R}^n \) to itself.

Theorem 5. (Brouwer Fixed Point Theorem) Let

\[
B^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a_i \leq x_i \leq b_i, i = 1, 2, \ldots, n\}.
\]

Suppose \( f(x) \) is continuous and \( f(x) \in B^n \) for all \( x \in B^n \), then there exists \( c \in B^n \) such that \( f(c) = c \).

\(^1\)In [4] the problem is stated for \( a = 0 \) and \( b = 1 \), but this more general notation is consistent with the way that we are stating the other results and the solution is really the same either way.
The point \( c \) such that \( f(c) = c \) is called a fixed point of the function \( f(x) \). Here is some evidence that the Brouwer fixed point theorem is important: it has been mentioned in more than 500 research papers. Since that list includes more than 50 papers published since 2015, we can see that this theorem is related to research topics of current interest and that is why it is important.

Theorem 3, the solution to Problem 32, is the \( n = 1 \) case of the Brouwer fixed point theorem. The Intermediate Value Theorem is also the \( n = 1 \) case of a more general theorem, one that was stated by Henri Poincaré in 1883. It also concerns a function \( g(x) \) that takes \( \mathbb{R}^n \) to itself. We can think of such a function as a vector-valued function \( g(x) = (g_1(x), g_2(x), \ldots, g_n(x)) \).

**Theorem 6.** Suppose \( g(x) = (g_1(x), g_2(x), \ldots, g_n(x)) \) is continuous on \( B^n \) such that

\[
g_i(x_1, \ldots, x_i-1, a_i, x_{i+1}, \ldots, x_n) \leq 0
\]

and

\[
g_i(x_1, \ldots, x_i-1, b_i, x_{i+1}, \ldots, x_n) \geq 0
\]

where \( a_i < b_i \) for \( i = 1, 2, \ldots, n \), then \( g(c) = (g_1(c), g_2(c), \ldots, g_n(c)) = (0, 0, \ldots, 0) = 0 \) for some \( c \in \mathbb{R}^n \).

In 1940, Carlo Miranda [3] proved that Poincaré’s theorem is equivalent to the Brouwer fixed point theorem. As a result, Theorem 6 became known as the Poincaré-Miranda Theorem\(^2\), and that is our other “important theorem”. The paper [3] has appeared in the list of references of over 120 research papers, more than half of them published since 2015, so it is very much a part of contemporary mathematical research.

For the proof that the Poincaré-Miranda Theorem implies Brouwer’s Theorem, if \( f(x) \) satisfies the hypotheses of Theorem 5, it can be shown that \( g(x) = f(x) - x \) satisfies the hypotheses of Theorem 6, so \( g(c) = 0 \) which again implies \( f(c) = c \). The proof of the converse result is more difficult, but the paper [2] contains an elegant demonstration.

Thus, by proving that the fixed point theorem of Problem 32 is equivalent to the Intermediate Value Theorem, we have not only an interesting exercise that is related to that familiar theorem, but we have actually proved the \( n = 1 \) case of Miranda’s surprising discovery that the Brouwer Fixed Point Theorem and an \( n \)-dimensional version of the Intermediate Value Theorem are equivalent. Those

\(^2\)To find out more about the Poincaré-Miranda Theorem, in particular how it is proved, see [1]
two important theorems state the same mathematical fact in different forms and so, as our title states, they really are one.\footnote{I thank Phoenix Kim for suggesting improvements in the exposition of this note.}

References


