UNIQUE CONTINUATION PROPERTY FOR HARMONIC FUNCTIONS VIA CARLEMAN'S METHOD

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Abstract. We first recall the amazing property that holomorphic functions on an open connected subset of $\mathbb{C}$ are uniquely determined by their values on any open sub-domain. The proof of this fact is due to the Taylor Series expansion of holomorphic functions and carries over to harmonic functions since they are also analytic. And as the laplacian is the prototype of uniformly elliptic PDEs, we expect this unique continuation property to extend to uniformly elliptic PDEs. However, solutions to uniformly ellipitic PDEs may not be analytic, so we need to proceed via a new method. So we choose to proceed via Carleman's Method of showing the Unique Continuation Property for Harmonic Functions that generalizes to uniformly ellipitic PDEs, uniformly parabolic PDEs, and even some hyperbolic PDEs.

1. Introduction

A remarkable feature of holomorphic functions on a domain Ω is that they are uniquely determined by their value on any open sub-domain of Ω (throughout this report we assume U, Ω are open and connected subsets of $\mathbb{R}^d$). That is if $U \subset \Omega$ is open (U may not necessarily be Ω) then knowing the holomorphic function on U suffices to understand its global behavior on Ω. In particular, this allows us to uniquely extend holomorphic functions from a sub-domain U to Ω, which is commonly called analytic continuation for holomorphic functions. In fact, an analytic function is uniquely determined by its value on any accumulation point inside Ω since they are analytic.

Since the real parts of holomorphic functions are harmonic functions, an interesting question to pose is can we re-construct a harmonic function $u : \Omega \to \mathbb{R}$ from its data on any sub-domain $U \subset \Omega$. This is true since harmonic functions are analytic. And since harmonic functions are the prototype of the class of uniformly elliptic PDEs, we expect a similar result to hold for this class of PDEs. However, there are solutions to uniformly elliptic PDEs that are not analytic. So we have to use a different proof strategy to show a similar result for uniformly elliptic PDEs. Now we say $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$ has the **unique continuation property** if the values of u on any open sub-domain $U \subset \Omega$ is enough to uniquely determine u on Ω. From our previous discussion, we see that holomorphic or harmonic function has the unique contiuation property. And we expect this result to still be true for a solution to a uniformly elliptic PDE.

In this report we focus on deriving the result for Harmonic functions using the method of Carleman known as Carleman's estimates. This method was first introduced by Carleman in [3] to prove the unique continuation property for a two-dimensional elliptic equation. The first crucial step in the method is to show that if L is our linear uniformly elliptic PDE operator, then we have that the inverse operator L^{-1} is continuous. That is we want to show under some Banach Space X and Y where $L : X \to Y$ that

$$||u||_X \leq C||Lu||_Y$$

this is a common technique known as the method of *apriori* estimates. The motivation for such method comes from the open mapping theorem from functional analysis. This theorem implies that if we had a bijective continuous linear operator L from a Banach Space X to another Banach Space Y, then L^{-1} is continuous. That is there is a $C > 0$ independent of $u \in X$ such that

$$||L^{-1}(L(u))||_X \leq C||Lu||_Y \iff ||u||_X \leq C||Lu||_Y$$

so if we expected our PDE operator was injective (i.e. uniqueness), then it would be a bijection onto its image and we would expect a similar estimate as above. In other words, we expect the inverse to

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be continuous. (See [6] for more discussion on the method of apriori estimates and the open mapping theorem) However, we do not expect the converse continuity i.e.

$$||Lu||_Y \lesssim ||u||_X$$

because differentiation is an operation that reduces regularity, while we expect the reverse to hold since the inverse operator (integration) increases regularity.

One of the major difficulties with the method of apriori estimates is figuring out the right Banach space to put our solutions in to make the inverse operator continuous. Common spaces to put the solutions in are Sobolev Space (for a review of Sobolev Spaces see [4] and for L^p spaces see [5]) and in particular $H^1(\Omega) := \{u : u \in L^2(\Omega) \text{ and } \nabla u \in L^2(\Omega)\}$ where the gradient is interpreted as the distributional derivative (where we implicitly assume the distributional derivative is a function). An important distinction of $H^1(\Omega)$ to other Sobolev spaces is that it can be endowed with a Hilbert Space structure by defining its norm as

$$||u||^2_{H^1(\Omega)} := \int_\Omega |u|^2 + |\nabla u|^2 dx$$

However, these are not the only spaces that are appropriate for showing the inverse operator is continuous. We can also define weighted Sobolev spaces where if we denote m as the Lebesgue measure on $\Omega \subset \mathbb{R}^n$ then we integrate with respect to $w(x)dm$ instead of m where $w(x) > 0$ is called the weight function. One important choice of weights are exponentials since the solutions might oscillate very rapidly at a low amplitude near $\partial\Omega$, which will be very hard to detect with a non-weighted norm, but by adding an exponential weight, these details can be picked up thanks to the weight.

In particular, Carleman's method consists of first deriving the weighted H^1 bounds

$$\int_U s(|\nabla u(x)|^2 + u^2)e^{2s\varphi(x)}dx \lesssim \int_U |Lu|^2 e^{2s\varphi(x)}dx$$

where $s > 0$ is a free parameter such that the above inequality holds independent of $u \in C_0^\infty(U)$ for s large enough, $e^{2s\varphi(x)}$ is our weighted function, and L is our uniformly ellipitic PDE operator. Note that Carleman's method extends to parabolic and hyperbolic PDE operators as well. And we closely follow with some slight modifications the proofs in [7] which proves Carleman's estimates for the Heat Equation, then for general uniformly parabolic PDE operators. And the main technique for deriving such estimates is integration by parts, which is why we assume $\partial U \in C^1$.

There is a slight problem with the above Carleman's estimate. The region of validity of s in the above inequality depends on u, so to overcome this a second large parameter $\lambda > 0$ will be introduced along with the weight $\varphi(x)$ also depending on λ. This will allow us to choose the region of validity of the inequality for s provided that λ is sufficiently large. Then this will allow us to derive a stability estimate of the form

$$||u||_{H^1(U(\varepsilon))} \lesssim ||u||^{1-\theta}_{H^1(U(0))} F^\theta + F$$

where $F = ||\Delta u||_{L^2(U)} + ||u||_{H^1(\Gamma)} + ||\partial_\nu u||_{L^2(\Gamma)}$ where $\Gamma \subset \partial U$ is open in ∂U, $\partial_\nu u$ is the normal derivative of u on Γ, $\theta \in (0, 1)$, and $U(\varepsilon) \subset U(0)$ will be appropriate sub-domains and $U(\varepsilon) \to U(0)$ as $\varepsilon \to 0$ in the sense of Hausdroff distance. So we see this stability condition implies by taking $\varepsilon \to 0$ that u being a harmonic function with $u = \partial_\nu u = \partial_\tau u = 0$ (where $\partial_\tau u$ is the tangential component of the derivative) then $u \equiv 0$ a.e. on $U(0)$. This is a weaker condition than the unique continuation property, but is still in the same spirit. Then we conclude by using this inequality will be used to finally prove the unique continuation property of harmonic functions.

Note that we closely follow and slightly modify the proof of [7] and [2] chapter on Carleman Type Estimates and Their Applications to derive the above stability result. However, in [7] even though they proved the stability result in the unique continuation property section, they do not even mention why the stability result implies the unique continuation property.

2. Carleman's Method For UCP applied to the Laplacian Operator

In this report we will focus on the case of our PDE operator being the Laplacian that is $Lu := \Delta u = \sum_{j=1}^n \partial^2_{x_j x_j} u$. Suppose $u \in C^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^d$ such that $u = 0$ on $\mathrm{supp}(u) \subset U$ where U is an open set contained in Ω. Then we want to derive estimates of the form

$$\int_U s(|\nabla u(x)|^2 + u^2)e^{2s\varphi(x)}dx \lesssim \int_U |\Delta u|^2 e^{2s\varphi(x)}dx$$

where $\varphi(x) > 0$ is an appropriate weight function. Let us assume for now we have already chosen such a φ. Then we define

$$w(x) := e^{s\varphi(x)}u(x) \text{ and } P(w) := e^{s\varphi(x)}\Delta(e^{-s\varphi(x)}w(x)) = e^{s\varphi(x)}\Delta u$$

This implies that

$$P(w) = \Delta w - 2s\nabla\varphi \cdot \nabla w + w(s^2|\nabla\varphi| - s\Delta\varphi)$$

And also notice that the right hand side of the inequality is simply

$$\int_U |\Delta u|^2 e^{2s\varphi(x)}dx = \int_U |P(w)|^2 dx$$

so it suffices to find upper and lower bounds of $||P(w)||_{L^2}$. A common technique for finding such a lower bound is to decompose $P = P^+ + P^-$ and to use the inner product structure to get $||P||_{L^2} \geq 2(P^+, P^-)$. So it suffices to find a nice decomposition of P such that we can bound $\int_U s(|\nabla u(x)|^2 + u^2)e^{2s\varphi(x)}dx$ by a constant multiple of (P^+, P^-) for large enough s. In particular, we choose the symmetric decomposition of P that is we decompose

$$P = \frac{(P + P^\top)}{2} + \frac{(P - P^\top)}{2} := P^+ + P^-$$

where $P^\top$ is the adjoint of P in $L^2(U)$ for functions that vanish on $\partial\Omega$. For example, if $L(u) := \Delta u$ then

$$(Lf, g) = \int_U (\Delta f)g dx = \int_U f(\Delta g)dx \Rightarrow L^\top = L$$

where we integrated by parts twice using $f, g = 0$ on ∂U. In particular, this mean $L^\top = L$. Then by direct computation we have

$$P^\top(w) = \Delta w + 2s\nabla w \cdot \nabla\varphi + w(s\Delta\varphi + s^2|\nabla\varphi|^2)$$

from which it follows that

$$P^+ = \Delta w + w(s^2|\nabla\varphi|^2) \text{ and } P^- = -2s\nabla\varphi \cdot \nabla w - ws\Delta\varphi$$

so now it suffices to derive lower bounds of (P^+, P^-). The major advantage of this decomposition is that after some calculus and basic inequalities, one can show that the highest power terms of $|\nabla w|^2$ and w^2 with respect to the parameter s in (P^+, P^-) is positive. This will then let us obtain the desired lower bound. Indeed, observe that

$$\int_U |\Delta u|^2 e^{2s\varphi(x)}dx = \int_U |P(w)|^2 dx = ||P^+(w)||_{L^2(U)} + ||P^-(w)||_{L^2(U)} + 2(P^+(w), P^-(w)) \geq 2(P^+(w), P^-(w))$$

And

$$(P^+(w), P^-(w)) = -\int_U 2s\Delta w(\nabla\varphi \cdot \nabla w) + sw(\Delta w)(\Delta\varphi) + 2s^3 w|\nabla\varphi|^2(\nabla\varphi \cdot \nabla w) + s^3 w^2 \Delta\varphi|\nabla\varphi|^2 dx$$
$$= (1) + (2) + (3) + (4)$$

where (j) corresponds to the integral of the jth integrand. Now we compute to see that

$$(3) = -\sum_{i=1}^n \int_U 2s^3 w|\nabla\varphi|^2 \partial_{x_i}(\varphi)\partial_{x_i}(w)dx = -\sum_{i=1}^n \int_U s^3|\nabla\varphi|^2\partial_{x_i}(\varphi)\partial_{x_i}(w^2) = \sum_{i=1}^n \int_U \partial_{x_i}(s^3|\nabla\varphi|^2\partial_{x_i}(\varphi))w^2$$
$$= \int_U \nabla \cdot (s^3|\nabla\varphi|^2\nabla\varphi)w^2 dx = s^3 \int_U (\nabla(|\nabla\varphi|^2) \cdot \nabla\varphi)w^2 + \Delta\varphi|\nabla\varphi|^2 w^2$$

Observe that the second term cancels out (4). Then we also have that

$$(2) = -\sum_{i=1}^n \int_U sw(\Delta\varphi)\partial^2_{x_i x_i}(w)dx = \sum_{i=1}^n \int_U \partial_{x_i}(sw(\Delta\varphi))\partial_{x_i} w dx = \int_U s|\nabla w|^2 \Delta\varphi + sw\left(\nabla w \cdot (\nabla\Delta\varphi)\right)dx$$

Now Cauchy-Schwarz combined with $|ab| \leq \frac{a^2+b^2}{2}$ gives

$$\left|\int_U sw(\nabla w \cdot (\nabla \Delta\varphi))dx\right| \geq -s\int_U |w||\nabla w||\nabla \Delta \varphi|dx \gtrsim_\varphi -s\int_U |w||\nabla w|dx \geq -\frac{1}{2}\int_U s^2|w|^2 + |\nabla w|^2 dx$$

And finally

$$(1) = -\sum_{i,j=1}^n \int_U 2s(\partial^2_{x_ix_i} w)(\partial_{x_j}\varphi)(\partial_{x_j} w)dx = 2s\sum_{i,j=1}^n \int_U (\partial_{x_i} w)(\partial^2_{x_jx_i}\varphi)(\partial_{x_j} w) + (\partial_{x_i} w)(\partial_{x_j}\varphi)(\partial^2_{x_ix_j} w)dx$$

$$= 2s\sum_{i,j=1}^n \int_U (\partial_{x_i} w)(\partial^2_{x_jx_i}\varphi)(\partial_{x_j} w) + \frac{1}{2}(\partial_{x_j}\varphi)\partial_{x_j}((\partial_{x_i} w)^2)dx = 2s\sum_{i,j=1}^n \int_U (\partial_{x_i} w)(\partial^2_{x_jx_i}\varphi)(\partial_{x_j} w) - \frac{1}{2}(\partial^2_{x_jx_j})\varphi(\partial_{x_i} w)^2 dx$$

$$= 2s\int_U (\nabla u)^\top D^2\varphi(\nabla u) - \frac{1}{2}\Delta\varphi|\nabla w|^2 dx$$

where $D^2\varphi$ is the Hessian of φ. Note that the Laplacian terms cancels out the previous Laplacian term. Therefore, combining all these terms we arrive at the estimate

$$\int_U |P(w)|^2 \geq \int_U 2s(\nabla u)^\top D^2\varphi(\nabla u) - C(\varphi)s(\frac{|w|}{2} + \frac{|\nabla w|^2}{2}) + s^3(\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi)w^2$$

Now if we assume that $D^2\varphi$ is uniformly positive definite i.e. there is a $\lambda > 0$ such that for any $\xi \in \mathbb{R}^n$ we have $\xi^\top D^2\varphi\xi \geq C|\xi|^2$ and an $r > 0$ such that $(\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi) \geq r$ on $\overline{U}$ then we have

$$\geq s\int_U 2\lambda|\nabla w|^2 - C(\varphi)\left(\frac{sw^2}{2} + \frac{|\nabla w|^2}{2s}\right)dx + s^3\int_U rw^2 dx$$

Since we are considering s large this means we can find an $s_0 = s_0(w)$ so large such that

$$\int_U s^3rw^2 - C(\varphi)s^2\frac{w^2}{2}dx \geq \frac{r}{2}\int_U s^3w^2dx \text{ and } \int_U 2s\lambda|\nabla w|^2 - \frac{C(\varphi)}{2}|\nabla w|^2dx \geq \lambda\int_U s|\nabla w|^2 dx$$

Therefore, there is a constant $K > 0$ such that for s large enough we have

$$\int_U |\Delta u|^2 e^{2s\varphi(x)}dx = \int_U |P(w)|^2dx \geq K\int_U s^3w^2 + s|\nabla w|^2$$

Now recalling that $w = e^{s\varphi}u$ gives for s large enough

$$\int_U |\Delta u|^2e^{2s\varphi(x)} \gtrsim \int_U (s^3u^2 + s|\nabla u|^2)e^{2s\varphi(x)}$$

where the implies constants are independent of $u \in C_0^\infty(U)$. Note if we define $H^2(U)$ as the space of function that are in $L^2(U)$ such that their first and second distributional derivatives are in $L^2(U)$, then the above inequality extends to functions $u \in H_0^2(U)$ since $C_0^\infty(U)$ is a dense subclass. Where $H_0^2(U)$ is the space of $u \in H^2(\Omega)$ such that they have zero boundary data in the sense of trace.

So the first immediate question is does there exist a function $\varphi(x)$ with the properties of $D^2\varphi$ being uniformly positive definite with constant λ and $(\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi) \geq r$ for some $r > 0$ on $\overline{U}$. Explicitly fix $x_0 \notin \overline{U}$ then consider

$$\varphi(x) := |x - x_0|^2$$

then $D^2\varphi(x) = 2I_n$ (where I_n is the $n \times n$ identity matrix), so we can take $\lambda = 2$. Also we observe that

$$(\nabla(|\nabla\varphi|^2)\cdot\nabla\varphi) = 2\sum_{i,j=1}^n (\partial^2_{x_ix_j}\varphi)(\partial_{x_i}\varphi)(\partial_{x_j}\varphi) = 2|\nabla\varphi|^2 = 2(\nabla\varphi)^\top D^2\varphi(\nabla\varphi) = 16|x - x_0|^2$$

so in particular $\varphi(x)$ is a valid weight function. In addition, we see that $(\nabla|\nabla\varphi|^2)\cdot\nabla\varphi > 0$ is implied if $|\nabla\varphi|^2 > 0$. Therefore, we have proven the following version of Carleman's estimate for the Laplacian operator:

Theorem 2.1. ***(Local Carleman estimates for the Laplacian)*** *Let $u \in H_0^2(U)$ where $\partial U \in C^1$ then if $\varphi(x) \in C^2(\overline{U})$ is a weight function such that there is a $\lambda > 0$ and $r > 0$ such that*

(1) $\xi^\top D^2\varphi(x)\xi \geq \lambda|\xi|^2$ *for all* $\xi \in \mathbb{R}^n$
(2) $|\nabla\varphi| \geq r > 0$ *on* $\overline{U}$

then there is a constant $C = C(\lambda, r)$ such that for all s sufficiently large we have the following estimate

$$\int_U (s^3u^2 + s|\nabla u|^2)e^{2s\varphi(x)}dx \le C\int_U |\Delta u|^2 e^{2s\varphi(x)}dx$$

Also notice that our estimate could be sharper since we did the crude estimate of dropping the terms $||P^{\pm}(w)||_{L^2(U)}$ when deriving a lower bound. Indeed, with small modifications of our above derivations we can obtain the following version of Carleman's estimate with 2 large parameters s and λ:

Theorem 2.2. ***(Carleman's Estimate with 2 large parameters for Laplacian)*** *Let $\psi \in C^2(\overline{U})$ be such that $|\nabla\psi| > 0$ on $\overline{U}$ with $\partial U \in C^1$. Then set $\varphi(x,t) := e^{\lambda\psi(x,t)}$ then there is a $C, C_0 = C_0(\lambda)$ such that if $\lambda \ge C$ and $s \ge C_0$ then for any $u \in H^2_0(U)$*

$$\lambda\int_U (\lambda s\varphi)^3 e^{2s\varphi}|u|^2 + (\lambda s\varphi)^1 e^{2s\varphi}|\nabla u|^2 + (\lambda s\varphi)^{-1}e^{2s\varphi}|D^2u|^2 \le C\int_U e^{2\tau\varphi}|\Delta u|^2$$

Proof. For a full proof see [2] Chapter 3 Theorem 3.2, but this estimate can also be derived very similarly to the way we derived our original Carleman's estimate. This estimate can also be derived by modifying the proof in [7] from the Parabolic Case to the Elliptic Case.

□

Remark 2.1. *Notice by introducing a second large variable λ we are now able to choose s independent of $u \in H^2_0(U)$. This is a key feature that will be crucial for the stability estimate that we are about to prove.*

But for the next stability result we will need to include boundary terms. These terms were neglected earlier since we assumed $u \in C_c^\infty(U)$ so the boundary terms went away when we integrated by parts. However, when u does not vanish on ∂U we have the following Carleman's estimate

Theorem 2.3. ***(Carleman's Estimate with 2 large parameters for Laplacian with Boundary Terms)*** *Let $\psi \in C^2(\overline{U})$ be such that $|\nabla\psi| > 0$ on $\overline{U}$ with $\partial U \in C^1$. Then set $\varphi(x,t) := e^{\lambda\psi(x,t)}$ then there is a $C, C_0 = C_0(\lambda)$ such that if $\lambda \ge C$ and $s \ge C_0$ then for any $u \in H^2(U)$*

$$\lambda\int_U \sum_{|\alpha|\le 1} s^{3-2|\alpha|}e^{2s\varphi}|\partial^\alpha u|^2 \le C\left(\int_U e^{2s\varphi}|\Delta u|^2 + \int_{\partial U}\sum_{|\alpha|\le 1} s^{3-2|\alpha|}e^{2s\varphi}|\partial^\alpha u|^2 dS\right)$$

where we are using multi-index notation.

Remark 2.2. *In the above inequalities, we can also replace Δu with a uniformly elliptic PDE operator Lu under some structural conditions and even under some geometric constraints on U we my replace Δu with a hyperbolic operator. See [2] and [7] for more details. One major difference in the approach of our harmonic function and a general parabolic or ellipitic operator is that one decomposes $P = P_1 + P_2$ where P_1 has all the first order terms (with respect to s) elements of P, while P_2 has the second and zeroth order. And one estimates $||P_1||_{L^2(U)} + 2(P_1, P_2)$ instead, but the details are very similar. (Note that one cannot use the same symmetric decomposition as before since the operator may not be in divergence form, so figuring out the adjoint is much more difficult).*

Note that either of these Carleman's estimate already implies a weak unique continuation property. That is if $\Delta u = 0$ and $u \in H^2_0(U)$ then $u = 0$ a.e. on U thanks to the inequalities. The main purpose of unique continuation property is to extend u from being zero on U to all of Ω, which is not at all clear from this inequality. But it turns out that Carleman's inequalities imply stability inequalities that make the unique continuation property easier to deduce.

Theorem 2.4. ***(Stability Estimate)*** *Let $\psi \in C^2(\overline{U})$ be such that $|\nabla\psi| > 0$ on $\overline{U}$ with $\partial U \in C^1$ and define $U(\varepsilon) := U \cap \{x : \psi(x) > \varepsilon\}$ then let u be $C^\infty(U) \cap H^2(U)$ solve*

$$\begin{cases} \Delta u = f \text{ in } U \\ u = g \text{ on } \Gamma \\ \partial_\nu u = h \text{ on } \end{cases}$$

where $f \in L^2(U), h \in L^2(\Gamma), g \in H^1(\Gamma)$ *and if we assume*

$$\overline{U(0)} \subset U \cup \Gamma$$

where $\Gamma \subset \partial U$ *is* C^1 *and is open in* ∂U*. Then we have for some* $\theta \in (0,1)$ *that*

$$||u||_{H^1(U(\varepsilon))} \lesssim \left(||u||^{1-\theta}_{H^1(U(0))} F^\theta + F\right)$$

where

$$F = ||f||_{L^2(U)} + ||g||_{H^1(\Gamma)} + ||h||_{L^2(\Gamma)} = ||\Delta u||_{L^2(U)} + ||u||_{H^1(\Gamma)} + ||\partial_\nu u||_{L^2(\Gamma)}$$

where θ *and the implied constant in the inequality both depend on* $\varepsilon > 0$*.*

Proof. We closely follow the proof in [2]. for this stability result. The idea is to let χ be a smooth cut off function that is identically 1 on $U(\varepsilon/2)$ and 0 on $U \setminus U_0$. This gives $u\chi$ vanishes on $\partial U \setminus \Gamma$, so in the Carleman estimate with boundary terms (Theorem 2.3), we will only keep the boundary terms over Γ. Applying Theorem 2.3 onto $u\chi$ yields combined with shrinking the integration domain of the left hand side of the inequality to $U(\varepsilon/2)$ where $\chi \equiv 1$ gives

$$s\lambda \sum_{|\alpha|\le 1} ||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(\varepsilon/2))} \le \lambda \sum_{|\alpha|\le 1} s^{3-2|\alpha|}||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(\varepsilon/2))} \le C\left(\int_{U(0)} e^{2s\varphi}|\Delta(u\chi)|^2 + \int_\Gamma \sum_{|\alpha|\le 1} s^{3-2|\alpha|}e^{2s\varphi}|\partial^\alpha(u\chi)|^2\right)$$

for $C \le \lambda$ and $s \ge C_0(\lambda)$ as defined in the theorem. Now we use

$$\Delta(u\chi) = \chi\Delta u + u\Delta\chi + 2\nabla u \cdot \nabla\chi \Rightarrow |\Delta(u\chi)|^2 \lesssim_\chi |\Delta u|^2 + |u|^2 + |\nabla u|^2$$

Also as

$$\nabla(u\chi) = u\nabla\chi + \chi\nabla u \Rightarrow |\nabla(u\chi)|^2 \lesssim_\chi |u|^2 + |\nabla u|^2 = |u|^2 + |\partial_\nu u|^2 + |\partial_\tau u|^2$$

where we recall that $\partial_\tau u$ is the tangential component of the derivative of u. Therefore, on Γ we have $|\partial_\tau u|^2 = |\nabla g|^2$, so we have the estimates

$$\int_\Gamma s^3 e^{2s\varphi}|u\chi|^2 + se^{2s\varphi}|\nabla(u\chi)|^2 dS \lesssim_\chi \int_\Gamma e^{2s\varphi}(s^3|u|^2 + s(|\partial_\nu u|^2 + |\partial_\tau u|^2))ds$$

Let $\Phi := \sup_x \varphi = \sup_x e^{\lambda\psi(x)}$ yields

$$\lambda s \sum_{|\alpha|\le 1} ||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(\varepsilon/2))} \le C(\lambda,\chi)e^{2s\Phi}\left(\int_{U(0)}(|\Delta u|^2) + \int_\Gamma s^3(|u|^2 + |\partial_\nu u|^2 + |\partial_\tau u|^2)\right) + C(\lambda,\chi)\sum_{|\alpha|\le 1}||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(0)}$$

Recalling from the theorem that we can take $\lambda > C$ lets us choose $\lambda > 2C$, so we obtain

$$sC(\lambda,\chi)\sum_{|\alpha|\le 1}||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(\varepsilon/2)} \le C(\lambda,\chi)e^{2s\Phi}s^3F^2 + C(\lambda,\chi)\sum_{|\alpha|\le 1}||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(0)\setminus U(\varepsilon/2))}$$

That is if we define $\gamma := e^{\lambda\varepsilon}$ and $\beta := e^{\lambda\varepsilon/2}$ then we have from $U(\varepsilon) \subset U(\varepsilon/2)$ that

$$e^{2s\gamma}||u||_{H^1(U(\varepsilon))} \le \sum_{|\alpha|\le 1}||e^{s\varphi}\partial^\alpha u||^2_{L^2(U(\varepsilon/2)}$$

so we have

$$e^{2s\gamma}||u||_{H^1(U(\varepsilon))} \le e^{2s\Phi}s^2F^2 + \frac{1}{s}e^{2s\beta}||u||_{H^1(U(0))\setminus U(\varepsilon/2)} \le e^{2s\Phi}s^3F^2 + \frac{1}{s}e^{2s\beta}||u||_{H^1(U(0))}$$

so we have

$$||u||_{H^1(U(\varepsilon)} \le e^{2s\Phi-2s\gamma}s^2F^2 + \frac{1}{s}e^{2s(\beta-\gamma)}||u||_{H^1(U(0))} \lesssim e^{2s\Phi-2s\gamma}F^2 + e^{2s(\beta-\gamma)}||u||_{H^1(U(0))}$$

Now noting that if $||u||_{H^1(U_0)} \le F$ implies the desired inequality, so if this is not the case then define

$$s := \log(\frac{||u||_{L^2(U(0))}}{F})/(\Phi + 1 - \beta) > 0$$

then we get

$$\theta = \frac{\gamma - \beta}{\Phi + 1 - \beta}$$

in the above inequality as desired.

Now that we have the stability inequality, we are ready to show the unique continuation property of harmonic functions. The major achivement in this theorem compared to the previous inequality, is that we only need a harmonic function u and its derivatives to vanish on Γ (which we do not necessarily assume to be all of ∂U) to conclude $u \equiv 0$ on $U(0)$.

Theorem 2.5. ***(Unique Continuation Property Of Harmonic Functions)*** *Let Γ be any nonvoid open subset of $\partial\Omega$. Then if $\Delta u = 0$ in Ω with $u \in H^1(\Omega), u = \partial_\nu u = 0$ on Γ then $u \equiv 0$ on Ω.*

Proof. We refer to [2] corollary 4.2 for a full proof of this, but we will give an outline of a different proof based on chaining domains. By the previous theorem, we know that if $\psi(x) \in C^2(\overline{\Omega})$ such that $|\nabla\psi| > 0$ and if we define $\Omega_\psi(0) := \{x : \psi(x) > 0\}$ then if $\overline{\Omega_\psi(0)} \subset \Omega \cup \Gamma$ that $u \equiv 0$ on $\Omega_\psi(0)$. Then the idea is after we construct such a ψ to construct another function ϕ such that $\partial\Omega_\phi(0) \cap \partial\Omega_\psi(0)$ is open to reapply the previous theorem (since we can now take $\Gamma := \partial\Omega_\phi(0) \cap \partial\Omega_\psi(0)$) to get $u \equiv 0$ on $\Omega_\phi(0)$. Then we want to keep constructing such functions till we fill up the domain to get $u \equiv 0$ on Ω. (Note that we expect that since Ω is connected that we can chain our domains in this way to fill up Ω).

□

Remark 2.3. *Note that Theorem 2.5 is named Unique Continuation Propert of Harmonic Functions since if a harmonic function vanishes on an open set $U \subset \Omega$ then it vanishes on a ball. Then we want to chain balls like in the previous argument to deduce that $u \equiv 0$ on $\overline{\Omega}$ (since Ω is connected we can chain balls). And we only need to chain countably many balls since we can write $\overline{\Omega} = \bigcup_{n\in\mathbb{N}} \overline{\Omega} \cap \overline{B_n(0)}$ and the each element in the union is compact.*

Also note that this gives an alternative proof that holomorphic functions on a connected open domain are uniquely determined by their value on any sub-domain without using their analyticity properties. This is due to their real and imaginary parts are harmonic functions, which implies holomorphic functions have the unique continuation property.

□

3. Conclusion

In summary, the observation that holomorphic functions on a connected domain are uniquely determined by their values on any open sub-domain leads to the observation that an interesting class of functions have the unique continuation property. However, the usual proof of holomorphic functions having the unique continuation property is due to their analyticity is too strong for a large class of interesting solutions to PDEs. And a popular method of extending this unique continuation property to a large class of solutions to uniformly elliptic, parabolic, and even hyperbolic PDEs is Carleman's Method. This method consists of first deriving weighted L^2 norms bounds on u by Lu where L is the PDE operator. And the main trick for deriving such bounds is integration by parts. Then after deriving these weighted L^2 norms, we then extend it next to a stability result like Theorem 2.4. And finally the stability result will usually be strong enough to imply the unique continuation principle.

And the great flexibility of this method is that it can also be applied to other uniformly elliptic PDE operators, uniformly parabolic PDE operators, and even some hyperbolic PDE operators under suitable constraints. We refer to [3], [7], and [1] for further discussions of such extensions.

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