MATH 31A: FALL 2020 ALGEBRA REVIEW

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I would highly recommend students to review their algebra and trigonometry skills. Mastery of both algebra and trigonometry are necessary to do well in calculus. I will list a few algebra skills that I believe are important to do well in Calculus along with some examples and explanations of the concepts with additional resources to review the topic in more depth.

Contents

1. Algebra Review

1.1. **Manipulating Expressions:** Students should be adept at manipulating expressions such as

$$x^2 + 3x + 5 = 2x^2 + 4x + 1$$

and simplify it to the following expression

$$x^2 + x - 4 = 0$$

and knowing how to solve for the roots/zeros of a quadratic function i.e.

$$ax^2 + bx + c = 0$$

means that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Exercise 1 Solve for the values of x for which

$$2x - 1 = 3$$

Exercise 2 Solve for the values of x for which the following relation holds

$$x^2 - 2x + 1 = 2x^2 + 3x + 2$$

In Math 31A, you will very often manipulate algebraic expressions, so make sure you feel comfortable manipulating algebraic expressions. See Khan Academy and Paul's Notes for more examples and a more in-depth review.

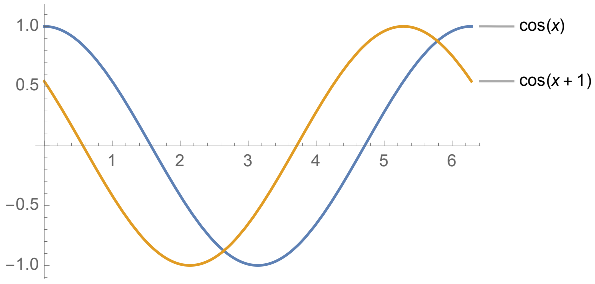

FIGURE 1. $\cos(x)$ compared to $\cos(x+1)$

1.2. **Graphing.** One should review the basic shapes of polynomials such as $ax+b$ and x^2 and trigonometric functions such as $\sin(x)$ and $\cos(x)$. One should also know how if given how the graph of a function $f(x)$ and a real number c how the graphs of $f(x+c)$, $f(x)+c$, $cf(x)$ and $f(cx)$. Such as how $f(x+c)$ induces a horizontal shift of the graph of $f(x)$ by $|c|$ units to the right if $c<0$ and to the left if $c>0$. For example the figure below shows a plot of $\cos(x)$ compared to $\cos(x+1)$ and we can see that $\cos(x+1)$ is shifted to the left by 1 compared to $\cos(x)$.

The textbook covers this in more depth in Chapter 1.1 with more examples.

Exercise Sketch the graph of $\sin(x)$, $\sin(x+2)$, $\sin(x-2)$, $\sin(2x)$, $\sin(-2x)$, $2\sin(x)$, $-2\sin(x)$, $\sin(x)+2$, and $\sin(x)-2$.

Hint It may be useful to use Wolframalpha to double check your solution. For example here is an Example of Sketching on Wolframalpha.

1.3. **Completing the Square and Simplifying Rational Functions.** A rational function is a function of the form of one polynomial divided by another. For example $f(x) := \frac{x}{x+1}$ is a rational function since the numerator and denominator are both polynomials. One will typically have to also manipulate such expressions. Let us do an example.

Simplify the following rational function

$$f(x) := \frac{x^2-4x+7}{x^2-4x+4}$$

The first observation is the denominator is x^2-4x+4 as $(x-2)^2$. And we can rewrite the numerator as $(x-2)^2+3$ so we have

$$f(x) = \frac{(x-2)^2+3}{(x-2)^2} = \frac{(x-2)^2}{(x-2)^2} + \frac{3}{(x-2)^2} = 1 + \frac{3}{(x-2)^2}$$

and the following expressions holds true for any $x \neq 2$. But notice at $x=2$ the numerator and denominator of the expression

$$\frac{(x-2)^2}{(x-2)^2}$$

are both zero. We still want to say this is equal to 1 for any x even $x=2$ since it is one everywhere else, and we will be able to do so later on in the course by defining limits.

A common trick to deal with these rational function expressions is that there is a decent chance we can rewrite the numerator or denominator as a square i.e. of the form $(x-a)^p$ for some a and p. Then we want to modify the other expression to be of a similar form to simplify the rational function. For more review on rational functions see Khan Academy Rational Functions.

Factoring One should know how to factor polynomials as this would be likely very useful for the limits section. For example we can write $x^3-6x^2+11x-6 = (x-3)(x-2)(x-1)$. The trick I use to solve these types of problems is to first stare at the expression $x^3-6x^2+11x-6$ and figure out a root of the polynomial. I would usually try nice expressions like $0, 1, 2, -1, -2$ first. Then I see that $x=1$ is a root of the polynomial $x^3-6x^2+11x-6$. This means that the polynomial $(x-1)$ divides $x^3-6x^2+11x-6$ so I would preform polynomial division to get $x^3-6x^2+11x-6 = (x-1)(x^2-5x+6)$ then I would use the quadratic formula to solve for the remaining roots of x^2-5x+6. This gives me all the roots of

$x^3 - 6x^2 + 11x - 6$ are $1, 2$, and 3. And since $x^3 - 6x^2 + 11x - 6$ is a 3rd degree polynomial, we know that it can be factored into

$$x^3 - 6x^2 + 11x - 6 = \alpha(x-1)(x-2)(x-3)$$

for some constant α which by computation gives $\alpha = 1$. It is a general theorem that if $p(x)$ is a degree n polynomial then it has n (potentially complex valued) roots counting multiplicity and if we denote the roots as $\xi_1, .., \xi_n$ then $p(x) = \alpha(x-\xi_1)(x-\xi_2)(x-\xi_3)...(x-\xi_n)$ for some constant α. For example $2x^2 - 6x + 4$ has roots $x = 1$ and $x = 2$ and we have $2x^2 - 6x + 4 = 2(x-1)(x-2)$.

Exercise 3 Factor

$$x^3 - 5x^2 + 6x$$

Exercise 4 Factor

$$5x^4 - 30x^3 + 55x^2 - 30x$$

See Paul Note's Factoring for more practice with factoring.

1.4. **Rationalizing the Denominator or Numerator.** You will often encounter functions that involve square roots and division with another function such as

$$f(x) := \frac{\sqrt{x}+1}{2x^2}$$

and a useful trick to try on these sort of functions is rationalizing the numerator or denominator. For example,

$$f(x) = \frac{\sqrt{x}+1}{2x^2}\frac{\sqrt{x}-1}{\sqrt{x}-1}$$

since $\frac{\sqrt{x}-1}{\sqrt{x}-1} = 1$. Now we observe what multiplying by 1 in this case does. Simplifying the above expression gives

$$f(x) = \frac{x-1}{2x^2(\sqrt{x}-1)}$$

and this expression may be more useful to deal with than the original expression for certain problems.

Exercise Rationalize the numerator of

$$f(x) := \frac{\sqrt{x}-2}{3x^2+1}$$

For more information on rationalizing the numerator or denominator see Khan Academy Rationalizing the Denominator

1.5. **Formula Manipulation.** A student should know how to use formulas well. For example we know that for a circle of radius r that the area A is given by $A = \pi r^2$. Students typically are given the radius and want to solve for the area. But if we had the area, we can also solve for the radius. Indeed, manipulating the expression $A = \pi r^2$ gives

$$\sqrt{\frac{A}{\pi}} = r$$

where we only kept the positive part of the square root since radius can not be a negative quantity. Therefore, we have derived a formula for the radius from the area! For example if the area A is equal to π we would get $r = 1$.

Exercise 5 Given the volume of the sphere, derive a formula for the radius.

1.6. **Exponential, Logs, and Exponents.** The natural log function $\ln(x)$ is defined as the inverse of the exponential function e^x i.e. for any x we have $\ln(e^x) = x$ and for any $x > 0$ $e^{\ln(x)} = x$. Recall $\ln(x)$ is only defined for $x > 0$. Some important properties of e^x is it turns addition to multiplication i.e.

$$e^{x+y} = e^x e^y$$

so the inverse of the exponential turns multiplication to addition i.e.

$$\ln(xy) = \ln(x) + \ln(y)$$

and we have

$$\ln(x^a) = a\ln(x)$$

See Khan Academy Exponential and Log for more information about $\exp(x)$ and $\ln(x)$. Note that $\ln(x)$ is also commonly written as $\log(x)$.

One should also know the following properties of exponents:

$$(x^a)(x^b) = x^{a+b} \text{ and } (x^a)^2 = (x^a)(x^a) = x^{2a}$$

And in general we have

$$(x^a)^p = x^{ap}$$

See Khan Academy Algebraic Expression with Exponents for more review.

1.7. **Piece-wise Functions.** Let us say we have two functions say $f(x)$ and $g(x)$ and a real number α the we can define a new function $h(x)$ via

$$h(x) := \begin{cases} f(x) \text{ if } x \leq \alpha \\ g(x) \text{ if } x > \alpha \end{cases}$$

i.e. $h(x)$ is just $f(x)$ when $x \leq \alpha$ and is $g(x)$ when $x > \alpha$. This is called a piece-wise function because there are multiple pieces to the function (the pieces are $f(x)$ and $g(x)$). There is nothing precluding f or g from also being a piece-wise function so an expression such as

$$F(x) := \begin{cases} 1 \text{ if } x \leq 2 \\ 2 \text{ if } 2 < x < 3 \\ 3 \text{ if } x \geq 3 \end{cases}$$

is also a piece-wise function. In addition, we say a piece-wise function is a step function is it is of the form $F(x)$ i.e. the pieces of the piece-wise function F are constant functions. To graph piece-wise function $h(x)$ we just graph f for $x \leq \alpha$ and then graph g for $x > \alpha$. For example, this is how we would graph

$$H(x) := \begin{cases} x \text{ if } x < 0 \\ \sin(x) \text{ if } x \geq 0 \end{cases}$$

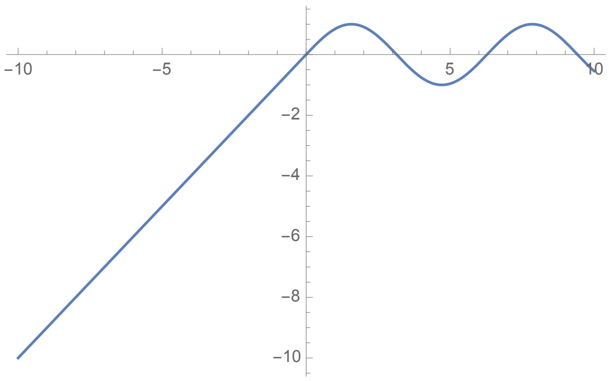

FIGURE 2. Graph of $H(x)$ for x in $[-10, 10]$

In addition if our piece-wise function $h(x)$ is given by

$$h(x) := \begin{cases} f(x) \text{ for } x \leq \alpha \\ g(x) \text{ for } x > \alpha \end{cases}$$

where f and g are continuous functions then h is a continuous function if and only if $f(\alpha) = g(\alpha)$ i.e. there is no jump between the transition from f to g. For more review on piece-wise functions see Piece-Wise Functions.

1.8. **Trigonometry.** One should know the trig functions $\sin(x)$, $\cos(x)$, $\tan(x)$, $\sec(x)$, $\csc(x)$ and $\cot(x)$ and how they are all related to one another i.e. $\tan(x) := \frac{\sin(x)}{\cos(x)}$. One should also know the various values of these trig functions at values such as $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ etc. The following expressions are also useful to know

$$\begin{cases} \sin^2(x) + \cos^2(x) = 1 \\ \sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a) \\ \cos(a)\cos(b) - \sin(a)\sin(b) \end{cases}$$

the product to sum and sum to product formulas i.e. see the following sheet Trig Identities for mode identities. These identities will probably be helpful throughout the course. In addition, section 1.4 of the textbook reviews trigonometric functions.