BASIC QUALIFYING EXAM

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These are my solutions for the Basic Qualifying Exam at UCLA. The exams can be found *here*. I wrote these solutions up while studying for the Fall 2020 Basic Exam. These solutions should have a majority of the solutions for the basic exam from 2010 Spring to 2020 Spring.

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Contents

1. Spring 2010

Problem 1. Recall that if $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ then AB is invertible if and only if A and B are invertible. Let us define the matrix $U := [u_1, ..., u_n]$ and $Y := [y_1, .., y_n]$ then $U + Y$ is invertible if and only if $U^T(U + Y) = I + U^TY$ is invertible. And $I + U^TY$ is invertible if and only if the columns $\{u_i + y_i\}$ form a basis of $\mathbb{R}^n$.

Notice that $||U^TY||_2^2 := \sum_{i=1,j}^n (U^TY)^2_{ij} = \text{Tr}(Y^TUU^TY) = \text{Tr}(Y^TY) = \sum_{i,j=1}^n Y_{ij}^2 < 1$. So it suffices to show if $||B||_2 < 1$ then $I + B$ is invertible. Indeed, fix x such that $(I + B)x = 0$ then

$$x_i + \sum_{j=1}^n x_j B_{ij} = 0 \text{ for all } i$$

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$$x_i = -\sum_{j=1}^{n} x_j B_{ij}$$

Let $x := (x_1, .., x_n)^T$ and $y := -(\sum_{j=1}^{n} x_j B_{1j}, ..., \sum_{j=1}^{n} x_j B_{nj})^T$ so we get $x = y$. Then by taking norms we get

$$||x||^2 = ||y||^2 = \sum_{i=1}^{n}(\sum_{j=1}^{n} x_j B_{ij})^2 \leq ||x||^2 \sum_{i,j=1}^{n} B_{ij}^2 < ||x||^2$$

where the first inequality is due to Cauchy-Schwarz and the last inequality applies whenever $||x||^2 \neq 0$ due to $||B||_2^2 < 1$. Therefore, we get $x = 0$. so $I + U^T Y$ is invertible so $\{u_1 + y_1, .., u_n + y_n\}$ is linearly independent.

Problem 2. By spectral theorem we can write there exists a basis of orthonormal eigenvectors of A. Write the eigenvectors as $\{v_1, .., v_n\}$ where v_i is associated with λ_i as defined in the problem. Then for any fixed k we have for $U := span\{v_1\} \oplus ... \oplus span\{v_k\}$ which is k dimensional

$$\max_{V, dim(U)=k} \min_{||x||=1, x \in V} (Ax, x) \geq \min_{||x||=1, x\in U} (Ax, x) = \lambda_k$$

where the last inequality follows from

$$(Ax, x) = (\sum_{i=1}^{k} \alpha_i \lambda_i v_i, \sum_{i=1}^{k} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i^2 \lambda_i \geq \sum_{i=1}^{n} \alpha_i^2 \lambda_k = \lambda_k$$

since $\sum \alpha_i^2 = 1$ due to $||x|| = 1$ and $\{v_i\}$ are orthonormal.

For the reverse inequality fix a k dimensional subspace U then we claim that at least k eigenvectors live in U. Indeed, if there are only $\ell < k$ eigenvectors say $v_{i_1}, ..v_{i_\ell}$ then $U \subset span(v_{i_1}) \oplus ... \oplus span(v_{i_\ell})$ so U has at most dimension $\ell < k$ which is a contradiction. So as there exists at least k eigenvectors in U. This implies that $\min_{||x||=1, x\in U}(Ax, x) \leq \lambda_k$ where $||x|| = 1$ since we have at least k eigenvectors. As U is arbitrary we conclude.

Problem 3. If $ST = TS$ and S, T are normal then we have a basis of orthonormal eigenvectors for T i.e. $T(v_i) = \lambda_i v_i$. Then

$$\lambda_i S(v_i) = ST(v_i) = TS(v_i)$$

so $S(v_i)$ is either a eigenvector of T with value λ or $S(v_i) = 0$. In either case we have for $E(\lambda_i, T)$ that $S : E(\lambda_i, T) \to E(\lambda_i, T)$ is a normal operator. So by the spectral theorem there exists a basis of eigenvectors w_j such that $S(w_j) = \alpha_j w_j$. Union all of these eigenvectors in all $E(\lambda_i, T)$ along with using $V = \oplus_{i=1}^{n} E(\lambda_i, T)$ to conclude.

Problem 4a. As A is symmetric and SPD we get all of its eigenvalues are non-negative. But the trace is the sum of the eigenvalues, which implies all of its eigenvalues must be zero. Therefore, by spectral theorem it implies A is similar to the zero matrix, so A is the zero matrix.

Problem 4b. Using TT^* is self adjoint we get $T = T^*$ so we have

$$T^2 = 4T - 3I$$

which implies the minimal polynomial divides $x^2 - 4x + 3 = (x - 1)(x - 3)$ so all of its eigenvalues can be 1 or 3 so it is Positive Definite.

Problem 5. We get that the minimal polynomial $M(t) = \Pi_{i=1}^{n}(t - \lambda_i)^{a_i - 1}$ so both of these matrix have a Jordan Block of size $a_i - 1$ for λ_i. But as $P(t) = \Pi_{i=1}^{n}(t - \lambda_i)^{a_i}$ we get that the total size of the Jordan Blocks of λ_i is a_i. So we must have one Jordan block of size $a_i - 1$ and one of size 1 for λ_i. Therefore, both matrix have the same JCF, so they are similar to one another.

Problem 6a. By direct computation we get

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

Problem 6b. We have

$$A^n = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

take $n = 100$

Problem 6c. By direct computation we get

$$A^n \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

Problem 7. This is a typical diagonalization argument. Indeed enumerate the rationals as $\{q_n\}$. Then $\{f_n(q_1)\}$ is bounded sequence in $\mathbb{R}$ so there is a convergent sub-sequence $n_i^{(1)}$ and a limit $f(q_1)$ and $\{f_{n_i^{(1)}}(q_2)\}$ is also bounded so there exists a sub-sequence $n_i^{(2)}$ of $n_i^{(1)}$ and a limit $f(q_2)$. Repeat this for all n and define the sub-sequence $n_k := n_k^{(k)}$. Then for any j we have as $f_{n_m^{(k)}}(q_j) \to f(q_j)$ so for any fixed $\varepsilon > 0$ the existence of N such that we have for any $m \geq N$

$$|f_{n_m^{(k)}}(q_j) - f(q_j)| \leq \varepsilon$$

By construction we have n_k is a subsequence of $n_m^{(k)}$, so we also have for large enough k that

$$|f_{n_k}(q_j) - f(q_j)| \leq \varepsilon$$

Problem 8. As K is a closed subset of a complete metric space it is easy to see that K is complete. Assume K is also totally bounded. Then let $\{x_n\}$ be an arbitrary sequence in K. Then there exists an integer N such that $K \subset \bigcup_{i=1}^N B_1(z_i)$ for $z_i \in K$. Then if $\{x_n\}$ is a finite set we are done so assume it is infinite this implies there exists a i such that there are infinitely many terms of x_n in $B_1(z_i)$. Let $y_1 := z_i$ and let this new subsequence which has infinitely many terms in $B_1(z_i)$ be defined as $\{x_n^{(1)}\}$. Repeat the argument to find a ball of radius 1/2 with center $y_2 := z_i^{(2)}$ such that there are infinitely many terms $\{x_n^{(1)}\}$ in $B_{\frac{1}{2}}(z_i^{(2)})$ with this new subsequence denoted $\{x_n^{(2)}\}$. We can do this for all n with balls of radius $1/2^n$ and cenreres $y_n := z_i^{(n)}$ and let $w_n := x_n^{(n)}$. Then we claim w_n is Cauchy. Indeed, if $n \leq m$

$$d(w_n, w_m) \leq d(w_n, z_i^{(n)}) + d(z_i^{(n)}, w_m) \leq \frac{1}{2^{n-1}}$$

where the last inequality is due to $w_n, w_m \in B_{\frac{1}{2^n}}(w_n)$. Then completeness implies we have a convergent subsequence.

Problem 9. Since $\nabla f(x_0, y_0, z_0) \neq 0$ we can WLOG assume that $\partial_x f(x_0, y_0, z_0) \neq 0$. Then as $f \in C^1$ with $f : \mathbb{R}1 + 2 \to \mathbb{R}$ such that $f(x_0, y_0, z_0) = 0$ and $\partial_x f(x_0, y_0, z_0) \neq 0$ we can apply the Implicit Function Theorem to find a open neighborhood $U \subset \mathbb{R}^2$ with $(y_0, z_0) \in U$ such that $\partial_x f(x_0, y_0, z_0) \neq 0$ in U and a function $\varphi : U \to \varphi(U)$ such that

$$f(\varphi(s,t), s, t) = 0$$

and $\partial_{x_i} \varphi(x_2, x_3) = -\partial_{x_i} f(\partial_x f)^{-1}$. Take the surface as $(\varphi(s,t), s, t)$ then it is a differentiable surface in U due to the derivative formula above and $f \in C^1$ and $\partial_x f(x_0, y_0, z_0) \neq 0$ in U.

Problem 10a). Fix $\mathbf{u} = (u_1, u_2)$ then $f(t\mathbf{u}) - f(0) = \frac{t^2 u_1 u_2}{t\sqrt{u_1^2 + u_2^2}}$ so we have

$$\frac{f(t\mathbf{u}) - f(0)}{t} = \frac{u_1 u_2}{\sqrt{u_1^2 + u_2^2}}$$

Therefore, the directional derivative exists for all directions at $(0,0)$ and is $\frac{u_1 u_2}{\sqrt{u_1^2 + u_2^2}}$.

Problem 10b. If f was differentiable at $(0,0)$ then the directional derivative for all $\mathbf{u}$ would be given by $Df(0) \cdot \mathbf{u}$ which implies that the directional derivative are linear with respect to the directions. But obviously if $\mathbf{u} \neq \mathbf{v}$ then $Df(0) \cdot \mathbf{u} + Df(0) \cdot \mathbf{v} \neq \frac{f(t\mathbf{u} + t\mathbf{v}) - f(0)}{0}$ so it implies there cannot be differentiable at the origin.

Problem 11. Fix $\varepsilon > 0$ then there exists an N_1 such that if $n \geq N_1$ then if $n \geq N_1$ we have

$$\sum_{k=n}^{\infty} |a_n| < \varepsilon$$

and there exists an a such that $\sum a_n \to a$ so there exists an N_2 such that if $n \geq N_2$ then if $n \geq N_2$

$$\left| \sum_{k=1}^{n} a_k - k \right|$$

Then as σ is a bijection on $\mathbb{N}$ there exists an N_3 such that if $n \geq N_3$ then $\sigma(n) \notin \{1, ..., \max\{N_1, N_2\}\}$. Take any $N \geq \max\{N_1, N_2, N_3\}$ then

$$\left|\sum_{n=1}^{N} a_{\sigma(n)} - a\right| \leq \left|\sum_{n=1}^{N} a_{\sigma(n)} - \sum_{n=1}^{N_2} a_n\right| + \left|\sum_{n=1}^{N} a_n - a\right|$$
$$= |\sum_{i:\sigma(i)\notin\{1,...,N_1} a_{\sigma(i)}| + \varepsilon$$
$$\leq \sum_{i=N_1+1}^{\infty} |a_i| + \varepsilon \leq 2\varepsilon$$

so $a_{\sigma(n)} \to a$.

Problem 12a. False. Take f_n as a triangle on $[0, \frac{1}{n^2}]$ with mass $\frac{1}{n}$. Then $\max_{x\in[0,1]} f_n = n$ and $\int_0^1 f_n(x) = \frac{1}{n}$.

Problem 12b. Type writer function. I.e. $f_1 = 1$, $f_2 = \chi_{[0,1/2]}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4]}, f_5 = \chi_{[1/4,1/2]}, f_6 = \chi_{[1/2,3/4]}, f_{=\chi_{[3/4,1]}}$. This does not converge to 0 anywhere but converges in L^1 to the 0 function. This function is not continuous but we can modify it by making it into tents to get the desired result.

2. Fall 2010

Problem 1a). We first prove that if $\inf_{x\in K, y\in F}\rho(x,y) > 0$ then $K\cap F=\emptyset$. Indeed, assume this was false then there exists a sequence $\{x_j, y_j\}\subset K\times F$ with

$$\lim_{j\to+\infty} d(x_j, y_j) = 0$$

Then as K is compact there exists a sub-sequence $x_{j_k}\subset K$ and $x\in K$ such that $x_{j_k}\to x$. This implies

$$\lim_{k\to+\infty} d(x, y_{j_k}) = 0$$

thanks to the triangle inequality. But this implies x is a limit point in the closed set F, so we must have $x\in F$. Therefore, $x\in K\cap F$ which is a contradiction.

For the reverse direction just note that if $x\in K\cap F$ then

$$0\le \inf_{x\in K, y\in K} d(x,y)\le d(x,x) = 0$$

so we must have $K\cap F=\emptyset$.

Problem 1b. If f is a continuous function then

$$G(f) := \{(x, f(x)) : x\in\mathbb{R}\}$$

is closed subset of $\mathbb{R}^2$. Then let $F := \{(x,0) : x\in\mathbb{R}\}$. Then we have $G(f)$ and F is closed subset of $\mathbb{R}^2$. Then taking the standard metric in $\mathbb{R}^2$ we have $G(\exp(-x^2))$ and F are disjoint since $\exp(-x^2)\neq 0$ for any $x\in\mathbb{R}$. But we have

$$\inf_{x\in G(\exp(-x^2), y\in F} d(x,y) = 0$$

since $d((x,0),(x,\exp(-x^2)) = \exp(-x^2)\to 0$ as $x\to+\infty$.

Problem 2a. We say a bounded function f in $[a,b]$ is Riemann integrable if for any $\varepsilon>0$ we can find a partition such that the lower Riemann sum within epsilon distance of the upper Riemann sum with respect to this partition. I.e. if $\varepsilon>0$ we want to find a partition $P=\{a=x_0<x_1<...<x_N=b\}$ such that

$$\sum_{i=1}^{N}\inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_i + \varepsilon \ge \sum_{i=1}^{N}\sup_{x\in[x_{i-1},x_i]} f(x)\Delta x_i$$

where $\Delta x_i := x_i - x_{i-1}$

Problem 2b. Let f be continous on $[a,b]$ then it is uniformly continuous so there exists a $\delta>0$ such that if $|x-y|\le\delta$ then $|f(x)-f(y)|\le\frac{\varepsilon}{b-a}$. Let $\text{mesh}(P) := \max_{i=1}^{N}\Delta x_i < \delta$ then

$$\sum_{i=1}^{N}|\inf_{x\in[x_{i-1},x_i]} f(x) - \sup_{x\in[x_{i-1},x_i]} f(x)|\Delta x_i \le \sum_{i=1}^{N}\frac{\varepsilon}{b-a}\Delta x_i = \varepsilon$$

This implies

$$\sum_{i=1}^{N}\inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_i + \varepsilon \ge \sum_{i=1}^{N}\sup_{x\in[x_{i-1},x_i]} f(x)\Delta x_i$$

as desired.

Problem 3a. If $f\in C^3(\mathbb{R})$ then we have for any $x,y\in\mathbb{R}$

$$f(x) = f(y) + f'(y)(x-y) + \frac{f''(\xi(y))}{2}(x-y)^2$$

for some $\xi(y)\in(x,y)$ and if $g:\mathbb{R}^2\to\mathbb{R}$ then we must have for any $x,y\in\mathbb{R}^2$

$$g(x) = g(y) + \nabla g(y)\cdot(x-y) + (x-y)^T D^2 g(\xi(y))(x-y))$$

where $\xi(y) = (\xi(y_1),\xi(y_2))$ where $\xi(y_i)\in(x_i,y_i)$ and D^2 is the Hessian Matrix.

Problem 3b. Fix $u, v \in \mathbb{R}^2$ with $u = (u_1, u_2), v = (v_1, v_2($. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by

$$h(t) := g(tu + (1-t)v)$$

then

$$\frac{d}{dt}h(t) = \partial_x g(tu + (1-t)v)(u_1 - v_1) + \partial_y g(tu + (1-t)v)(u_2 - v_2)$$

so

$$\frac{d}{dt}h(t)|_{t=0} = \nabla g(v) \cdot (u - v)$$

and

$$\frac{d^2}{dt^2}h(t) = \partial^2_{xx} g(tu+(1-t)v)(u_1-v_1)^2 + \partial^2_{yy} g(tu+(1-t)v)(u_2-v_2)^2 + 2\partial^2_{xy} g(tu+(1-t)y)(u_1-v_1)(u_2-v_2)$$
$$= (u-v)^T D^2 g(tu + (1-t)v)(u-v)$$

By Taylor Theorem for single variable function with remainder

$$h(1) = h(0) + \nabla g(v) \cdot (u - v) + \frac{(u-v)^T D^2 g(\xi(v))(u-v)}{2}$$

but $h(1) = g(u)$ and $h(0) = g(v)$ so we arrived at the desired result.

Problem 4 a. We claim that the family $\{\sum_{i=1}^N \alpha_i e^{\beta_i x + \gamma_i y}\}$ i.e. finite linear combinations of $e^{\beta_i x + \gamma_i y}$ is dense in $[0,1]^2$. Indeed, this is family is an algebra because $e^{\beta_i x + \gamma_i y} e^{\beta_j x + \gamma_j y} = e^{(\beta_i + \beta_j)x + (\gamma_i + \gamma_j)y}$ and this family is closed under finite linear combinations. This family vanishes nowhere since e^x is never 0 and if $(x_1, y_1) \neq (x_2, y_2)$ then WLOG $x_1 \neq x_2$ then $f(x,y) := e^x$ satisfies $e^{x_1} \neq e^{x_2}$, so it separates points. Therefore, as $[0,1]^2$ is compact Stone Weiestrass implies this family is dense in $C([0,1]^2)$ with the sup norm. This implies if $f \in C([0,1]^2)$ then for any $\varepsilon > 0$ there is an N such that for

$$\sup_{(x,y)\in[0,1]^2} |f(x,y) - \sum_{i=1}^N \alpha_i e^{\beta_i x + \gamma_i y}| = \sup_{(x,y)\in[0,1]^2} |f(x,y) - \sum_{i=1}^N \alpha_i e^{\beta_i x} e^{\gamma_i y}| < \varepsilon$$

Let $g_i(x) := \alpha_i e^{\beta_i x}, h_i(x) := e^{\gamma_i y}$ and we have arrived at the desired conclusion.

Problem 4b. No, if it were true then for any $\varepsilon > 0$ we can find a $\{g_i(x)\}_{i=1}^N$ such that

$$|f(x,y) - \sum_{i=1}^N (g_i(x))^2| < \varepsilon \Rightarrow f(x,y) > \sum_{i=1}^N (g_i(x))^2 - \varepsilon \geq -\varepsilon$$

Letting $\varepsilon \to 0$ we get that $f(x,y) \geq 0$. So if this were true then any continuous function such that $f(x,y) = f(y,x)$ must be positive, but take $f(x,y) = -x^2$ for a counter example. This implies the claim is false.

Problem 5a. Recall $span(S)$ is defined as the smallest subspace that contains S. Let $V = \mathbb{R}^2$ and $S = \{(x, 2x+1) : x \in \mathbb{R}\}$ and $S' = \{(x, 3x+1) : x \in \mathbb{R}\}$ then $span(S) = span(S') = \mathbb{R}^2$ since the only subspace that contains them is $\mathbb{R}^2$. So $span(S) \cap span(S') = R^2$ but $span(S \cap S') = span(\emptyset) = \{0\}$.

Problem 5b.

Problem 6. By Cayley-Hamilton if p is the characteristic polynomial of T then $p(T) = 0$. And the roots of p are the eigenvalues of T so $p(0) \neq 0$ since T is invertiable. So $p(T) = \sum_{i=1}^n \alpha_i T^i + cI = 0$ where $c \neq 0$ then

$$-T(\sum_{i=1}^n \frac{\alpha_i}{c} T^{i-1}) = I$$

with $T^0 := I$ then $T^{-1} = -\sum_{i=1}^n \frac{\alpha_i}{c} T^{i-1} = q(T)$ for a polynomial q.

Problem 7. Let $\{v_i\}_{i=1}^n$ be an orthonormal basis of V and $\{w_i\}_{i=1}^m$ be an orthonormal basis of W then $n \leq m$ since $\dim(V) \leq \dim(W)$. Let $T(v_i) = w_i$ for $i = 1, .., n$ then

$$(T(v_i), T(v_j))_W = (w_i, w_j)_W = \delta_{ij}$$

and

$$(v_i, v_j)_V = \delta_{ij}$$

so

$$(T(v_i), T(v_j))_W = (v_i, v_j)_V$$

this implies for any $v, v' \in V$ that

$$(T(v), T(v'))_W = (v, v')_V$$

Problem 8. Let $x \in W_1^\perp + W_2^\perp$ then $x = w_1 + w_2$ with $w_i \in W_i^\perp$ then for any $z \in W_1 \cap W_2$

$$(x, z) = (w_1, z) + (w_2, z) = 0$$

so $x \in (W_1 \cap W_2)^\perp$. Now let $e_1, .., e_n$ be a orthonormal basis of $W_1^\perp$ and $v_1, .., v_m$ be a orthonormal basis of $W_2^\perp$. Now we have

$$\begin{aligned}\dim((W_1 \cap W_2)^\perp) &= \dim(V) - \dim(W_1) - \dim(W_2) + \dim(W_1 + W_2)\\ &= \dim(W_1^\perp) + \dim(W_2^\perp) - \dim((W_1 + W_2)^\perp)\end{aligned}$$

and

$$\dim(W_1^\perp + W_2^\perp) = \dim(W_1^\perp) + \dim(W_2^\perp) - \dim(W_1^\perp \cap W_2^\perp)$$

so it suffices to show $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$ since that implies

$$\dim((W_1 \cap W_2)^\perp) - \dim(W_1^\perp + W_2^\perp) = \dim(W_1^\perp \cap W_2^\perp) - \dim((W_1 + W_2)^\perp) \leq 0$$

. Indeed if $x \in (W_1 + W_2)^\perp$ then for any $w_i \in W_i$ $(x, w_i) = (x, w_i + 0) = 0$ since $w_i + 0 \in W_1 + W_2$ so $x \in W_1^\perp \cap W_2^\perp$. This implies

$$\dim((W_1 \cap W_2)^\perp) \leq \dim(W_1^\perp + W_2^\perp)$$

but we already have $W_1^\perp + W_2^\perp \subset (W_1 \cap W_2)^\perp$ so $W_1^\cap + W_2^\cap = (W_1 \cap W_2)^\perp$

Problem 9a. Solving $x = A^{-1}(Bx + c)$ gives $x = (-1, -1)$.

Problem 9b. No, take $x_0 = (0, 0)$ then for all n x_n has positive components so it cannot converge to $(-1, -1)$.

Problem 10. Note that as f is Lipschitz with say constant M then $x_k(t)$ is also Lipschitz with constant M. So the family is equicontinuous. But they are also uniformly bounded on any compact subset since we have $x_k(0) = 0$. So Arzela Ascoli implies the existence of a subsequence that converges uniformly to a limit $x(t)$ on $[-N, N]$. So it suffices to show that

$$x(t) = \int_0^t f(x(t), t)$$

Problem 11. We have due to Jensen's Inequality

$$\int_0^1 |f'(x)|^2 \geq \left(\int_0^1 f'(x)\right)^2 = 1$$

and the min is attained by $f(x) = x$. This min is unique thanks to the strict convexity of $|\cdot|^2$. Indeed, if f and g are both mins then we have for $\lambda \in (0, 1)$ that $|\lambda f'(x) + (1-\lambda)g'(x)|^2 \leq \lambda |f'(x)| + (1-\lambda)|g'(x)|$ with the inequality strict unless $f'(x) = g'(x)$. But since $f \neq g$ and the boundary conditions we know that $f'(x) \neq g'(x)$ for a set of positive measure on $[0, 1]$. This means

$$\int_0^1 |\lambda f'(x) + (1-\lambda)g'(x)|^2 < \int_0^1 \lambda |f'(x)|^2 + (1-\lambda)|g'(x)|^2 = 1$$

which is a contradiction.

Problem 12. Note

$$\int_{D(t)} f(x, t)dx = \int_{\theta=0}^{2\pi} \int_{\rho=0}^{r(t)} \rho f(\rho, \theta, t) d\rho d\theta$$

So one has

$$\begin{aligned}\frac{d}{dt}\int_{D(t)} f(x, t)dx &= \int_{\theta=0}^{2\pi} \frac{d}{dt}\int_{\rho=0}^{r(t)} \rho f(\rho, \theta, t) d\rho d\theta\\ &= \int_{D(t)} f_t dx + \int_{\theta=0}^{2\pi} r(t) f(r(t), \theta, t) r'(t) d\theta\end{aligned}$$

3. Spring 2011

Problem 1. We know that if the eigenvalues of A are $\lambda_1, \lambda_2, \lambda_3$ then the characteristic polynomial is

$$\begin{aligned}\chi(t) &= (t-\lambda_1)(t-\lambda_2)(t-\lambda_3) = (t^2 - t\lambda_2 - t\lambda_1 + \lambda_1\lambda_2)(t-\lambda_3)\\ &= t^3 - (\lambda_1+\lambda_2+\lambda_3)t^2 + t(\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2) - \lambda_1\lambda_2\lambda_3\\ &= t^3 - 4t^2 + t(\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2) - 2\end{aligned}$$

where we solved for the det using the hint. Using the given identities we get $\lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_2 = 5$ so

$$\chi(t) = t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2)$$

Therefore, the minimal polynomial is either

$$(t-1)(t-2) \text{ or } (t-1)^2(t-2)$$

this means either

$$J = \text{diag}(1,1,2) \text{ or } J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 2. If A is diagonalizable then

$$A = S^{-1}DS$$

where D is a diagonal matrix, so

$$A^k = S^{-1}D^kS$$

which means A^k is diagonalizable.

Now assume A^k is diagonalizable. As $\mathbf{F} = \mathbf{C}$ then we can find a Jordan matrix J and an invertible matrix V such that

$$A = V^{-1}JV$$

then

$$A^k = V^{-1}J^kV$$

but as the Jordan form is unique (up to permutation) and A^k is diagonalizable this must mean J^k is a diagonal matrix. This occurs if and only if all there is no $1s$ above the diagonals since we cannot have a zero eigenvalue. So we must have J be a diagonal matrix, so A is diagonalizable.

Problem 3. We claim that when H is Hermitian then we can find a basis of V consisting of orthonormal eigenvectors of H.

We prove the problem by induction on the dimension. It is trivial when the vector space is 1 dimensional. So now assume it holds for any vector space of dimension less than n. Let H be a Hermitian operator on an n dimensional complex inner product vector space V. As the field is complex we know that there exists an eigenvector v_1 with length 1. Let $U := span(v_1)$ then $V = U \bigoplus U^\perp$ and as $H(U) \subset U$ we have $H(U^\perp) \subset U^\perp$ thanks to H being self adjoint. And $dim(U^\perp) = n-1 < n$ so we can consider the restricted operator $H|_{U^\perp}$ and apply the induction hypothesis to find $\{v_2, .., v_n\}$ such that $H|_{U^\perp}(v_i) = \lambda_i v_i$ and $(v_i, v_j) = \delta_{ij}$ and $U^\perp = span(v_2, .., v_n)$. This implies $V = span(v_1, .., v_n)$ and $H(v_i) = \lambda_i v_i$ with $(v_i, v_j) = \delta_{ij}$.

Now we fix a orthonormal basis of V $\{e_1, .., e_n\}$ where we assume every linear operator L matrix form is written as

$$[L(e_1), .., L(e_n)]$$

Then for the unitary operator $U(v_i) = e_i$ (it is unitary since it maps an orthnomrla basis to an orthonormal basis)

$$UHU^{-1}(e_i) = \lambda_i e_i$$

so $UHU^{-1} = diag(\lambda_1, .., \lambda_n)$. But as U is unitary we have $U^{-1} = U^*$.

Problem 4.

Problem 5. If $Ax = b$ then for any $y \in (ker(A^T))$ then

$$(b, y) = (Ax, y) = (x, A^Ty) = 0$$

so $b \in (ker(A^T))^\perp$. And $dim((ker(A^T))^\perp) = dim(range(A))$ which completes the proof.

Problem 6. Let us consider $w = Ay + (w - Ay)$ where $w - Ay \in Range(A)^{\perp}$ then

$$||Ay - w|| \leq ||Ax - Ay|| + ||Ay - w|| = ||Ax - w||$$

where we used $(Ax - Ay) \perp (Ay - w)$. So the minimizers are exactly the y such that $w - Ay \in Range(A)^{\perp}$ i.e. for any $x \in V$

$$0 = (w - Ay, Ax) = (A^*w - A^*Ay, x) = 0$$

or the y such that $A^*Ay = A^*W$ as desired.

Problem 7. Follows by IVT. Indeed, $f(1) = -1$ and $f(0) = 1$ so by IVT there exists a root between 0 and 1.

Problem 8a.

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \cap [0,1] \\ -1 \text{ else} \end{cases}$$

Problem 8b.

$$f_n(x) = \begin{cases} n \text{ for } x \in (0, \frac{1}{n}] \\ 0 \text{ else} \end{cases}$$

then $f_n \to 0$ everywhere but

$$\int_0^1 f_n(x)dx = 1 \neq \int_0^1 f(x) = 0$$

Problem 9. Assume there exists a point $f(x^*) > 0$ then continuity implies there is a δ ball where $f(x) > \frac{f(x^*)}{2}$ so

$$\int_a^b f(x) \geq \int_{x^*-\delta}^{x^*+\delta} \frac{f(x^*)}{2} > 0$$

which is a contradiction

Problem 10a. Let $f : G \subset \mathbb{R}^2 \to \mathbb{R}$. We say f is differentiable at $(x_0, y_0) \in G$ if there exists a linear transformation $Df(x_0, y_0) \in \mathbb{R}^{2\times 1}$ such that for $\boldsymbol{x} := (x_0, y_0)$

$$\lim_{||h|| \to 0} \frac{|f(\boldsymbol{x} + h) - f(\boldsymbol{x}) - Df(\boldsymbol{x}) \cdot h|}{||h||} = 0$$

Problem 10b. Define $Df(\boldsymbol{x}) := (\partial_x f(\boldsymbol{x}), \partial_y f(\boldsymbol{x}))$. Then

$$f(\boldsymbol{x} + h) - f(\boldsymbol{x}) = \sum_{i=1}^{2} f(p_{i+1}) - f(p_i)$$

with $p_1 := \boldsymbol{x}$, $p_2 := (x_0 + h_1, y_0)$ and $p_3 := (x_0 + h_1, y_0 + h_2)$

$$= \sum_{i=1}^{2} h_i \partial_{x_i} f(q_i)$$

with $q_i \to \boldsymbol{x}$ as $||h|| \to 0||$ thanks to MVT. Then

$$\left| \frac{f(\boldsymbol{x} + h) - f(\boldsymbol{x}) - Df(\boldsymbol{x}) \cdot h}{||h||} \right| = \left| \frac{h_1(\partial_x f(q_1) - \partial_x f(\boldsymbol{x})) + h_2(\partial_y f(q_2) - \partial_x f(\boldsymbol{x}))}{||h||} \right|$$

$$\leq |\partial_x f(q_1) - \partial_x f(\boldsymbol{x})| + |\partial_y f(q_2) - \partial_x f(\boldsymbol{x}))|$$

which converges to 0 as $h \to 0$ thanks to continuity of the partial derivatives.

Problem 11a. We claim that all connected sets in $\mathbb{R}$ are intervals. Indeed, let $E \subset \mathbb{R}$ be connected. then the map $f(x) := x$ is continuous so $f(E)$ is connected. Assume for the sake of contradiction that E is not an interval. Then there must exist an $x, y \in E$ and a $z \in E^c$ such that $x < z < y$ but the intermediate value theorem implies $z \in f(E)$. But observe $f(E) = E$ which is a contradiction so all connected sets in $\mathbb{R}$ are intervals so they are arcwise connected.

Problem 11b. Take the topologist sin curve

$$G(f) := \{x, \sin(\frac{1}{x})\} \cup \{0,0\}$$

for $x \in (0,1]$.Note that $\{x, \sin(\frac{1}{x})\}$ is connected since the map $x \mapsto (x, \sin(\frac{1}{x})$ is continuous for $x \in (0,1]$ and $\{0,0\}$ is connected. Then as $\overline{(x, \sin(\frac{1}{x})} \cap \{0,0\} \neq \emptyset$ we get the claim. But it is not path connected so it is not arcwise connected since there is no way to extend $\sin(1/x)$ to a continuous function on $[0,1]$.

Problem 12a. Note that

$$T(f) - T(g) = \int_0^x f(x) - g(x)$$

so

$$||T(f) - T(g)||_{L^\infty} \leq \int_0^c ||f(x) - g(x)||_{L^\infty} = c||f(x) - g(x)||_{L^\infty}$$

so it is a contraction map so we have an f such that $T(f) = f$. But as $f \in C([0,1])$ we actually have

$$g(x) := 1 + \int_0^x f(x) \in C^1([0,1])$$

Indeed, fix $\varepsilon > 0$ then by uniform continuity we can choose a $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f(x), f(y)) < \varepsilon$ so

$$|\frac{g(x+h) - g(x)}{h} - f(x)| = |\frac{1}{h}\int_x^{x+h} f(s) - f(x)| \leq \frac{1}{h}\int_x^{x+h} |f(s) - f(x)| \leq \varepsilon$$

when $h < \delta$. So if

$$f = 1 + \int_0^x f(x) \Rightarrow f' = f$$

but we also have $f(0) = 1$.

Problem 12b. An approximation for $\exp(t)$ thanks to the proof of Banach Fixed Point theorem.

4. Fall 2011

Problem 1. Let (X, d) be a compact metric space. Set

$$g(x) := d(f(x), x)$$

which is continuous, so it attains a min as X is compact at $z \in X$. If $f(z) \neq z$ then we have

$$g(f(z)) = d(f^2(z), f(z)) < d(f(z), z) = g(z)$$

which contradicts the minimality so $f(z) = z$. So we have found a fixed point. But also if $x = f(x)$ and $y = f(y)$ and $x \neq y$ then we have

$$d(z, x) = d(f(z), f(x)) < d(x, z)$$

so it is unique.

Problem 2. As $f \in C^1$ we have for any x, y that

$$f(x) - f(y) = \int_0^1 \nabla f(tx + (1-t)y) \cdot (x - y) dt$$

Let $g(t) := f(tx + (1-t)y) \Rightarrow g'(t) = \nabla f(tx + (1-t)y) \cdot (x - y)$. Then we have for any $t > 0$

$$g'(t) - g'(0) = (\nabla f(tx + (1-t)y) - \nabla f(y)) \cdot (x - y)$$

$$= (\nabla f(tx + (1-t)y) - \nabla f(y)) \cdot \frac{t}{t}(x - y) \geq \frac{c}{t}||t(x - y)||^2 = ct||x - y||^2$$

Therefore, $g'(t) \geq g'(0)$ this implies

$$f(x) - f(y) \geq \int_0^1 g'(0) = \nabla f(y) \cdot (x - y)$$

This condition implies convexity (in fact is equivalent). Indeed, let us fix $\alpha \in [0, 1]$ then let $x := \alpha x + (1-\alpha)y$ then we have

$$\begin{cases} f(x) \geq f(z) + \nabla f(z) \cdot (x - z) \\ f(y) \geq f(z) + \nabla f(z) \cdot (y - z) \end{cases}$$

so we get

$$\alpha f(x) + (1-\alpha)f(y) \geq f(z) + \nabla f(z) \cdot (\alpha x + (1-\alpha)y - z)$$
$$= f(z)$$

so we arrived at

$$\alpha f(x) + (1-\alpha)f(y) \geq f(\alpha x + (1-\alpha)y)$$

i.e. f is convex.

Problem 3.

Problem 4a. Note that the sum $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n} := \sum_{n=1}^{\infty} a_n$ converges thanks to Dirichlet's criterion. And it is unconditionally convergent. So we claim for any $\alpha \in \mathbb{R}$ there exists a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{n=1}^{\infty} a_{\sigma(n)}$. Indeed note if we let

$$p_i := \frac{a_i + |a_i|}{2} \quad n_i := \frac{a_i - |a_i|}{2}$$

then p_i is the non-negative terms of a_n and q_i is the non-positive terms of a_n. We also must have $\sum p_i$ and $\sum n_i$ diverge. Therefore, there exists an N_1 such that $\sum_{i=1}^{N_1} p_i \geq \alpha \geq \sum_{i=1}^{N_1-1} p_i$. Note that $p_i = 0$ iff $n_i \neq 0$ and $n_i = 0$ iff $p_i \neq 0$. Let $\{i_1, .., i_N\} \subset \{1, .., N_1\}$ be the index such that $p_i > 0$ then for $1 \leq j \leq N$ define $\sigma(j) = i_j$. Then there exists an N_2 such that $\sum_{i=1}^{N_1} p_i + \sum_{j=1}^{N_2} n_j \leq \alpha \leq \sum_{i=1}^{N_1-1} p_i + \sum_{j=1}^{N_2-1} n_j$. Again let $\{i_1, .., i_{N^{(2)}}\} \subset \{1, .., N_2\}$ such that $n_{i_j} \neq 0$ and define $\sigma(j + N) = i_j$ for $1 \leq j \leq N^{(2)}$. By induction we repeat this procedure for all $\mathbb{N}$ i.e. we find an N_{2n} such that

$$\sum_{i=1}^{N_{2n}} p_i + \sum_{i=1}^{N_{2n-1}} n_i \geq \alpha \geq \sum_{i=1}^{N_{2n}-1} p_i + \sum_{i=1}^{N_{2n-1}} n_i$$

and N_{2n+1} such that

$$\sum_{i=1}^{N_{2n}} p_i + \sum_{i=1}^{N_{2n+1}} n_i \leq \alpha \leq \sum_{i=1}^{N_{2n}} p_i + \sum_{i=1}^{N_{2n+1}-1} n_i$$

and putting $\sigma(i)$ as the index of non-zero terms of p_i from N_{2n-2} to N_2 then of the index of the non-zero terms of q_i from N_{2n-1} to N_{2n+1}. Then we get the following estimate

$$\sum_{i=1}^{N_{2n}} a_{\sigma(n)} \geq \alpha \geq \sum_{i=1}^{N_{2n}-1} a_{\sigma(n)} \Rightarrow 0 \geq \alpha - \sum_{i=1}^{N_{2n}} \geq -a_{\sigma(N_{2n}+1} \to 0$$

since $a_n \to 0$

Problem 4b. This sum converges absolutely by the p-test. So any rearrangement converges to the same sum. Let

$$\alpha := \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

then fix $\varepsilon > 0$ then there exists N_1 such that if $n \geq N_1$ we have

$$\sum_{k=}^{\infty} |a_n| < \varepsilon$$

and as $\sum a_n \to \alpha$ we can find an $N_2 \geq N_1$ such that

$$\left| \sum_{n=1}^{N_2} a_n - a \right| < \varepsilon$$

so for any rearrangement σ there exists an $N_3 \geq N_2$ such that if $n \geq N_3$ then $\sigma(n) \notin \{1, ..., N_2\}$ then for any $n \geq N_3$

$$\begin{aligned} \left| \sum_{k=1}^{n} a_{\sigma(n)} - \alpha \right| &\leq \left| \sum_{k=1}^{n} a_{\sigma(n)} - \sum_{j=1}^{N_2} a_j + \sum_{j=1}^{N_2} a_j - \alpha \right| \\ &\leq \left| \sum_{k=1}^{n} a_{\sigma(n)} - \sum_{j=1}^{N_2} a_j \right| + \left| \sum_{j=1}^{N_2} a_j - \alpha \right| \\ &\leq \left| \sum_{k:\sigma(k) \notin \{1,..,N_2\}} a_{\sigma(k)} \right| + \varepsilon \\ &\leq \sum_{n=N_2}^{\infty} |a_n| + \varepsilon \leq 2\varepsilon \end{aligned}$$

Problem 5. Just take any monotone function with countably many jumps.

Problem 6. See Fall 2012 number 3.

Problem 7. See Fall 2016 number 4.

Problem 8. We will show that for an arbitrary complex valued matrix A then there exists a basis of generalized eigenvectors. But as $null(A-\lambda I) = null((A-\lambda I)^2)$ this implies every generalized eigenvector is an eigenvector. First we show

$$V = range(A^n) \oplus ker(A^n)$$

for $n = dim(V)$ By rank nullity it suffices to show their intersection is the zero element. Let $v \in range(A^n) + ker(A^n)$ then

$$v = A^n x \Rightarrow 0 = A^n v = A^{2n} x \Rightarrow A^n x = 0$$

so the first claim holds. Now fix an eigenvalue λ associated with eigenvector v of A. Let

$$G(\lambda, A) := null((A - \lambda I)^n)$$

then we argue by induction. The case $n = 1$ is trivial, so assume the induction holds true for any subspace withd dimension less than n. Then

$$V = G(\lambda, A) \oplus U$$

for $U := range((A-\lambda I)^n)$. Now we claim $A(U) \subset U$. Indeed, if $x \in U$ then

$$(A-\lambda I)^n Ax = A(A-I)^n x = 0$$

so we can apply our induction hypothesis onto the restricted operator $A|_U$ to find a basis of generalized eigenvectors of $A|_U$ on U. It is clear that these are generalized eigenvectors of A, so we found a basis of eigenvectors of A on V. So we are done.

Problem 9. Let $L : V \to V$ be self adjoint such that there exists a unit vector

$$||Lx - \mu x|| \leq \varepsilon$$

As L is self adjoint there exists a basis of orthonormal eigenvectors. Let us denote the orthonormal eigenvectors with eigenvalue $\lambda_i \in \mathbb{R}$ as v_i. Then

$$x = \sum_{i=1}^{n} (x, v_i) v_i \Rightarrow 1 = ||x||^2 = \sum_{i=1}^{n} (x, v_i)^2$$

Then

$$(Lx - \mu x, Lx - \mu x) = \sum_{i=1}^{n} (\lambda_i - \mu)^2 (x, v_i)^2 \leq \varepsilon^2$$

As $1 = \sum_{i=1}^{n} (x, v_i)^2$ there exists a j such that $(x, v_j)^2 \leq 1$. Then

$$(\lambda_j - \mu)^2 \leq \varepsilon^2$$

This implies

$$|\lambda_j - \mu| \leq \varepsilon$$

as desired.

Problem 10. As A is a real matrix and $A^3 = I$ its eigenvalues must be either 1 repeated with multiplicity 3 or a single eigenvalue 1 with 2 complex conjugate roots of unity (with order 3). In the first case we get A is the identity matrix then our other eigenvalues must be the 2 complex conjugate roots of unity. Let these eigenvalues be denoted as λ and $\overline{\lambda}$ so over $\mathbb{C}$ A is diagonlizable to the form

$$A = S^{-1} \text{diag}(1, \lambda, \overline{\lambda}) S$$

where S may be a complex matrix. Then note that the matrix $diag(\lambda, \overline{\lambda})$ is simialr to

$$R := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

with $\theta = \frac{2\pi}{3}$ since the eigenvalues of R are $\lambda, \overline{\lambda}$. Therefore, there exists U such that

$$\text{diag}(\lambda, \overline{\lambda}) = URU^{-1}$$

so

$$S^{-1} \begin{bmatrix} 1 & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & U^{-1} \end{bmatrix} S = A$$

so A is similar to the desired form with either $\theta = 0$ or $\frac{2\pi}{3}$. Note that if A and B are real matrix such that A is similar to B over $\mathbb{C}$ then they are similar over $\mathbb{R}$.

Problem 11. $dim(ker(S)/im(T)) = dim(ker(S)) - dim(im(T))$. and $dim(Im(T)) = dim(V)$, $dim(W) = dim(U) + dim(null(S))$ so we get both sides of the equality as $dim(ker(S)) - dim(im(T))$.

Problem 12. Note if x satisfies

$$||Ax - b|| \leq ||Ay - b||$$

for all y then $Ax - b \in range(A)^{\perp}$. But $\mathbb{R}^n = range(A) \bigoplus \text{range}(A)^{\perp}$ and $b = Ax + (b - Ax)$ with $Ax \in range(A)$ and $b - Ax \in range(A)^{\perp}$ so Ax must be the same value for any minimizer.

5. Spring 2012

Problem 1. It is clear that $\rho(A,B) \geq 0$ and $\rho(A,B) = \rho(B,A)$. Now if $\rho(A,B) = 0$ then fix $x \in A$ so

$$0 = \sup_{x \in A} \inf_{y \in B} |x-y| \geq \inf_{y \in B} |x-y|$$

so $\inf_{y \in B} |x-y| = 0$ that is there exists a sequence $\{y_n\} \subset B$ such that $y_n \to x$ so $x \in \overline{B} = B$ since B is closed. Therefore, $A \subset B$. The reverse subset follows from $\sup_{y \in B} \inf_{x \in A} |x-y| = 0$. So $\rho(A,B) = 0 \iff A = B$. Now we prove the triangle inequality. Observe for all $a \in A, b \in B$ and $c \in C$ for $A, B, C \in \Omega$ we have

$$|a-b| \leq |a-c| + |c-b|$$
$$\inf_{b \in B} |a-b| \leq |a-c| + \inf_{b \in B} |c-b|$$
$$\inf_{b \in B} |a-b| \leq \inf_{c \in C} \{|a-c| + \inf_{b \in B} |c-b|\}$$
$$\inf_{b \in B} |a-b| \leq \inf_{c \in C} |a-c| + \sup_{c \in C} \inf_{b \in B} |c-b|$$
$$\sup_{a \in A} \inf_{b \in B} |a-b| \leq \sup_{a \in A} \inf_{c \in C} |a-c| + \sup_{c \in C} \inf_{b \in B} |c-b| \leq \rho(A,C) + \rho(B,C)$$

This inequality also holds for $\sup_{b \in B} \sup_{a \in A} |a-b|$ so we have

$$\rho(A,B) \leq \rho(A,C) + \rho(B,C)$$

so ρ is a metric as desired.

Problem 2. Fix $\varepsilon > 0$ then as f is uniformly continuous there exists a $\delta > 0$ such that on $d(x,y) \leq \delta \Rightarrow d(f(x), f(y)) \leq \varepsilon$. Consider a uniform partition of $[a,b]$ by $[a_{i-1}, a_i]$ where $a_i - a_{i-1} \leq \frac{\delta}{2}$. Then as $f_n \to f$ and $\{a_i\}$ is finite we can find an N such that for all $n \geq N$ we have

$$|f_n(a_i) - f(a_i)| \leq \varepsilon$$

for all a_i. Now fix $x \in [a_{i-1}, a_i]$ then we have by uniform continuity that

$$|f(x) - f(a_i)| \leq \varepsilon \quad |f(x) - f(a_{i+1})| \leq \varepsilon$$

Then by convexity we have

$$f(x) \leq \max\{f(a_i), f(a_{i+1})\}$$

so

$$f_n(x) \leq \max\{f_n(a_i), f_n(a_{i+1})\} \leq \max\{f(a_i), f(a_{i+1})\} + \varepsilon \leq f(x) + 2\varepsilon$$

i.e. for all $n \geq N$ we have

$$f_n(x) - f(x) \leq 2\varepsilon$$

For the reverse inequality if $x \in (a_{i-1}, a_i)$ then convexity implies that

$$\frac{f_n(x) - f_n(a_i)}{x - a_i} \leq \frac{f_n(a_{i+1}) - f_n(a_i)}{a_{i+1} - a_i} \leq \frac{f_n(x) - f_n(a_{i+1})}{x - a_{i+1}}$$

so

$$f_n(x) \geq (x - a_{i+1}) \frac{f_n(a_{i+1}) - f_n(a_i)}{a_{i+1} - a_i} + f_n(a_{i+1})$$

and

$$f_n(x) \leq (x - a_i) \frac{f_n(a_{i+1}) - f_n(a_i)}{a_{i+1} - a_i} + f_n(a_i)$$

so we have

$$f_n(x) - f(x) \geq (x - a_{i+1}) \frac{f_n(a_{i+1}) - f_n(a_i)}{a_{i+1} - a_i} + f_n(a_{i+1}) - (x - a_i) \frac{f(a_{i+1}) - f(a_i)}{a_{i+1} - a_i} - f(a_i)$$

By uniform convergence on $\{a_i\}$ and uniform continuity of f we have $f_n(a_{i+1}) - f(a_i) \leq -2\varepsilon$ and the secant slope terms are

$$\frac{x - a_i}{a_{i+1} - a_i} (f_n(a_{i+1}) - f(a_{i+1}) + f(a_i) - f_n(a_i)) - f_n(a_{i+1}) + f_n(a_i)$$

by rewriting $a_{i+1} = a_i + (a_{i+1} - a_i)$. Note $0 \leq \frac{x - a_i}{a_{i+1} - a_i} \leq 1$, then using triangle inequality and $f_n(a_i) \to f(a_i)$ we get that

$$f_n(x) - f(x) \geq -5\varepsilon$$

so we have

$$||f_n(x) - f(x)||_{L^\infty[a,b]} \leq 5\varepsilon$$

for $n \geq N$ so we have uniform convergence.

Problem 3. Bisection Method and completeness of $(\mathbb{R}, |\cdot|)$.

Problem 4. Note that $a_n \geq 0$ implies $s_n \geq 0$. Let $C_n := \sum_{i=1}^n s_n$. We claim $s_n \leq M$ which would imply it converges which s_n is an increasing sequence bounded above. Indeed fix an n then we have as $\frac{C_n}{n} \to s$ there exists an M such that $\frac{C_{2n}}{2n} \leq M$ Then using $C_{2n} \leq (n-1)s_1 + (n+1)s_n$ we get

$$\frac{s_1}{2} - \frac{s_1}{2n} + \frac{s_n}{2} + \frac{s_n}{2n} \leq M$$

so

$$s_n \leq 2M + \frac{s_1}{n} \leq 2M + s_1$$

Therefore, $s_n = \sum_{i=1}^n a_i$. Now let $a := \lim_{n\to\infty} s_n < +\infty$. then we claim $s_n \to a$. Indeed,

$$|\frac{C_n}{n} - a| = |\frac{\sum_{i=1}^n (S_i - a)}{n}|$$

For $\varepsilon > 0$ there exists an N such that if $n \geq N$ then $|S_i - a| \leq \varepsilon$. Then if $n > N$

$$\leq \frac{\sum_{i=1}^n |S_i - a|}{n} = \frac{\sum_{i=1}^N |S_i - a|}{n} + \frac{\sum_{i=N}^n |S_i - a|}{n}$$

$$\leq \frac{2M}{n} + \frac{(n-N)\varepsilon}{n} \leq \frac{2M}{n} + \varepsilon$$

which converges to 0 as $n \to \infty$. Therefore, $\frac{C_n}{n} \to a$ so by uniqueness of limit $a = s$.

Problem 5. Define $T : C([0,1]) \to C([0,1])$ via $T(f) := e^x + \frac{f(x^2)}{2}$. Note that $T(f) \in C([0,1])$ since it is a addition and composition of continuous function and its domain is $[0,1]$ since x^2 is bijection from $[0,1]$ to $[0,1]$. Use Banach Fixed Point Theorem since it's a contraction map with $\alpha = \frac{1}{2}$.

Problem 6. Note that the vector field $(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2})$ is conservative since $\nabla \arctan(\frac{x}{y}) = (\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2})$. However $\arctan(\frac{x}{y})$ is not differentiable on the y-axis. And as our path must start and end at $(1,0)$, we necessarily do not have zero circulation (since the potential cannot be made C^1 on any open neighborhood of the curve). Indeed, we do not have zero circulation since the path $\gamma(t) := (\cos(t), \sin(t))$ we have

$$I(\gamma) = \int_0^{2\pi} \frac{-\sin^2(t) - \cos^2(t)}{\cos^2(t) + \sin^2(t)} = -2\pi$$

Problem 7. We have $A = \begin{bmatrix} 4 & -3 \\ 1 & 0 \end{bmatrix}$ and its eigenvalues are $1, 3$ so it is diagonlizable and we must have $\lim_{n\to\infty} a_n^{1/n} = 3$

Problem 8. As $A \in C^{n\times n}$ there exists an Upper Triangle Matrix T such that

$$A = S^* T S$$

where S is unitary thanks to Schur's Decomposition. As similar matrix share the same eigenvalues and the eigenvalues of the upper triangular matrix T are the diagonal entries, it suffices to find $T_i \to T$ such that T is diagonalizable. Indeed, consider $T_n := T + \text{diag}(h_1, .., h_n)$ where $h_i < \frac{1}{n}$ and are chosen such that $(T_n)_{ii} \neq (T_n)_{jj}$ for any $j \neq i$. Then as T_n has distinct diagonal terms and is upper triangular, it has n distinct eigenvalues, so it is diagonalizable. So $T_n = V_n^* D_n V_n$ where V is unitary. So

$$A_n := S^* T_n S = S^* V_n^* D_n V_n S = (V_n S)^* D_n (V_n S)$$

converges to A entry wise as $n \to \infty$. Note that if A, B are unitary so is AB, therefore $(V_n S)^*$ is the inverse of $(V_n S)$. Therefore,

$$A_n = S^* V_n^* D_n V_n S := B_i L_i B_i^{-1} \to S^* T S = A$$

Problem 9.

Problem 10. It is always equal. Indeed, we have $A = V^*TV$ where T is a upper triangular matrix and V is unitary. Then note that $e^A = V^*e^TV$ which can easily be seen by the definition. So

$$\det(e^A) = \det(e^T)$$

and

$$\exp(Tr(A)) = \exp(\sum_{i=1}^{N} T_{ii}) = \prod_{i=1}^{N} \exp(T_{ii})$$

since similar matrix share the same trace. We also know that $(e^T)_{ii} = 1 + T_{ii} + \frac{T_{ii}^2}{2!} + ... = \exp(T_{ii})$ And $\det(e^T) = (e^T)_{ii}$ since e^T is upper triangular. Therefore, we have $\det(e^A) = \exp(Tr(A))$ for any complex valued matrix.

Problem 11a. By Cayley Hamilton A solve its characteristic polynomial which is of degree 2. Solve for this.

Problem 11b. If $P(A)$ and $Q(A)$ are second degree polynomial such that $P(A) = Q(A) = 0$ make them both monoic. Then $(P - Q)(A) = 0$ and $P - Q$ is a first degree polynomial which is impossible since A is not a constant multiple of the identity matrix.

Problem 12. They are equivalent over via $x' := x + iy$ and $y' := x - iy$ then $Q_1(x', y') = x^2 + y^2 = Q_2(x, y)$ and the transformation is non-singular since

$$\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

and the matrix is of full rank. But by Sylvester Law of Inertia two quadratic forms are equivalent over $\mathbb{R}$ if and only if the associated symmetric matrix A of Q_1 and B of Q_2 have the same number of positive, negative, and zero eigenvalues. We have

$$A = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But notice $1/2$ and $-1/2$ are eigenvalues of B so they cannot be equivalent over $\mathbb{R}$.

6. Fall 2012

Problem 1. We prove the statement by summation by parts. Indeed, let $B_n := \sum_{i=1}^n b_i$ then we have for any $m \geq n$

$$\sum_{i=m}^n a_i b_i = \sum_{i=m}^n a_i(B_i - B_{i-1}) = \sum_{i=m}^n a_i B_i - \sum_{i=m-1}^{n-1} a_{i+1} B_i$$
$$= a_n B_n - a_m B_{m-1} + \sum_{i=m}^{n-1} B_i(a_i - a_{i+1})$$

So

$$\left|\sum_{i=m}^n a_i b_i\right| \leq |a_n B_n| + |a_m B_{m-1}| + \sum_{i=m}^{n-1} |B_i|(a_i - a_{i+1})$$

since a_i is decreasing. Therefore, as $|B_n| \leq M$ we have

$$\left|\sum_{i=m}^n a_i b_i\right| \leq M(|a_n| + |a_{m-1}| + \sum_{i=m}^{n-1}(a_i - a_{i+1})$$
$$= M(|a_n| + |a_{m-1}| + a_m - a_n)$$
$$\leq M(2|a_n| + |a_m| + |a_{m-1})$$

Then as $a_n \to 0$ we can for any $\varepsilon > 0$ find any N such that for $k \geq N$ we have $|a_k| \leq \frac{M}{4}\varepsilon$ then choosing $n, m \geq N$ we have

$$\left|\sum_{i=m}^n a_i b_i\right| \leq \varepsilon$$

so $S_n := \sum_{i=1}^n a_i b_i$ is a Cauchy sequence, so it is convergent.

Problem 2a. We say a bounded function f is Riemann Integrable on $[0, 1]$ if and only if for all $\varepsilon > 0$ there exists a partition P such that if $P = \{0 = x_0 < x_1 < ... < x_n = 1\}$ with $I_i := [x_{i-1}, x_i]$ and $\Delta x_i := x_i - x_{i-1}$ then for $\omega(f, I_i) := \sup_{x,y \in I_i} |f(x) - f(y)|$ we have

$$\sum_{i=1}^n \omega(f, I_i)\Delta x_i \leq \varepsilon$$

i.e. the upper and lower Riemann sum difference can be made arbitrarily small.

Problem 2b. Fix a uniform partition of size ε i.e. $\delta x_i = \varepsilon$ for all i. Then since f is non-decreasing then $\omega(f, I_i) = f(x_i) - f(x_{i-1})$ so

$$\sum_{i=1}^n \omega(f, I_i)\Delta x_i = \varepsilon(f(b) - f(a))$$

since the sum is telescoping. Then as f is bounded we have

$$\leq 2M\varepsilon$$

for $M := ||f||_{L^\infty}$. Therefore, it is Riemann Integrable.

Problem 3. If $f_n \to f$ uniformly then the 3ε trick shows f is continuous. The converse is known as Dini's Theorem. Indeed as f is continuous any fixed $\varepsilon > 0$ we have

$$G_n := \{x : f(x) - f_n(x) > -\varepsilon\}$$

is open since it is the preimage of an open set on a continuous function. Then as $f_n(x) \to f(x)$ for all $x \in X$ we have

$$X = \bigcup_{n=1}^\infty G_n$$

As X is compact there exists a finite subcover so

$$X \subset \bigcup_{k=1}^N G_{n_k}$$

Note that $f_n(x) \geq f_{n+1}(x)$ implies $f(x) - f_n(x) \leq f(x) - f_{n+1}(x)$. Therefore, we have for all $n \geq \max\{n_1, .., n_N\} := K$

$$f(x) - f_n(x) \geq f(x) - f_K(x) > -\varepsilon$$

where we can ensure $f(x) - f_k(x) > -\varepsilon$ thanks to the monotocity and finite subcover. And we have

$$f_n(x) \geq f(x) \Rightarrow f(x) - f_n(x) \leq 0$$

Therefore, we have for all $n \geq K$ that

$$||f(x) - f_K(x)||_{L^\infty(K)} \leq \varepsilon$$

so we have uniform convergence.

Problem 4. Let F_n be closed sets such that $int(F_n) = \emptyset$ and assume X is complete with

$$X = \bigcup_{n=1}^{\infty} F_n$$

Clearly X cannot have empty interior for $int(X) = X \neq \emptyset$ so there exists an $x \in X \cap F_1^c$. Then we must have an n such that $n \geq 2$ and $B_{\frac{1}{n}}(x) \cap F_1 = \emptyset$ for otherwise we would get x is a limit point which would imply $x \in F_1$ which is a contradiction. Let this ball be denoted as $B_{h_1}(x)$ then we must have $B_{h_1}(x)$ is not contained in F_2 since it has empty interior, so there exists an $x_2 \in B_{h_1}(x) \cap F_2^c$. Similarily we can find an $n \geq 3$ such that $B_{\frac{1}{n}}(x_2) \cap F_2 = \emptyset$. Choosing n smaller we can assume $B_{h_2}(x_2) \subset B_{h_1}(x)$. Proceed inductively to generate points x_n with radius $h_n < \frac{1}{n}$ such that $B_{h_n}(x_n) \subset B_{h_{n+1}}(x_{n+1})$ and $B_{h_n}(x_n) \cap \bigcap_{i=1}^{n} F_n = \emptyset$. Then $\{x_n\}$ forms a Cauchy sequence so there exists an $x \in X$ such that $x_n \to x$. But $x \in B_{h_n}(x_n)$ for all n so $x \notin \bigcup_{n=1}^{\infty} F_n = X$ which is our contradiction. So BCT holds.

Problem 5. An equivalent form of BCT is that if G_n are open dense sets then $\bigcap_{n=1}^{\infty} G_n$ is dense in a complete metric space. This implies is not a G_δ for we could define $H_n := (-\infty, q_n) \cup (q_n, \infty)$ for an enumeration of q_n. Then this is an open dense set, so $G_n \cap H_n$ is an open dense set (density is due to G_n is open). But $\bigcap_{n=1}^{\infty}(G_n \cap H_n)$ is the empty set but BCT says it is dense, which is a contradiction. In fact it shows any countable set in a complete metric space cannot be a G_δ.

Problem 6a. Assume for sake of contradiction that there exists an (x^*, y^*) such that $F(x^*, y^*)$ is non-zero (wlog it is positive). Then by continuity there is a small square with (x^*, y^*) at the center such that $F(x, y) \geq \frac{F(x^*,y^*)}{2}$. But then for this square we have the integral mass is positive since

$$0 = \int_S F(x, y) \geq \int_S \frac{F(x^*, y^*)}{2} > 0$$

so $F \equiv 0$.

Problem 6b. We have for any square $\ell_1 \leq x \leq \ell_2$ with $\ell_3 \leq y \leq \ell_4$

$$\int_{x=\ell_1}^{\ell_2} \int_{y=\ell_3}^{\ell_4} \partial^2_{x,y} f(x, y) \mathrm{d}y \mathrm{d}x = \int_{y=\ell_3}^{\ell_4} \int_{x=\ell_1}^{\ell_2} \partial^2_{x,y} f(x, y) \mathrm{d}y \mathrm{d}x = \int_{y=\ell_3}^{\ell_4} \partial_x f(x, \ell_4) - \partial_x f(x, \ell_3)$$

$$f(\ell_2, \ell_4) - f(\ell_1, \ell_4) - f(\ell_2, \ell_3) + f(\ell_1, \ell_3)$$

$$= \int_{x=\ell_1}^{\ell_2} \int_{y=\ell_3}^{\ell_4} \partial^2_{y,x} f(x, y)$$

so by $6a$) we must have $\partial^2_{y,x} f(x, y) = \partial^2_{x,y} f(x, y)$.

Problem 7. This means there exists an N such that $A^N = A$. Therefore, if we let $\mu(x)$ be the minimal polynomial of A we have the existence of a polynomial p such that

$$p(x)\mu(x) = x(x^{N-1} - 1)$$

$$= x \prod_{i=1}^{N-1} (x - \lambda_i)$$

where λ_i are the $(N - 1)$ roots of unity. In particular this implies that $\mu(x)$ has no repeated root which is equivalent with A being diagonalizable.

Problem 8. Note that $[w_1, w_2] = (Hw_2, w_1)$. Then $w \in W$ iff for all $v \in W$ we have $(Hw, v) = 0$ i.e. $H(W) \subset W^\perp$. Then the restricted operator satisfies $H|_W : W \to W^\perp$. As $det(H) \neq 0$ we must have H is injective i.e. $dim(W) = rank(H|_W)$. Then

$$n = dim(W) + dim(W) \geq dim(W) + rank(H|_W) = 2dim(W)$$

where we used $Im(H|_W) \subset W^\perp$ i.e.

$$\frac{n}{2} \geq dim(W)$$

For the examples take $H = diag(1, -1, 1, -1, ...)$.

Problem 9. Note that we have $\mathbb{R}^m = Im(A) \oplus (Im(A))^\perp$ so there exists $b_1 \in Im(A)$ and $b_2 \in (Im(A))^\perp$ such that $b = b_1 + b_2$. Then

$$\begin{aligned} ||b_1 - b||^2 &\leq ||b_1 - b||^2 + ||Ax - b_1||^2 \\ &= ||b - Ax||^2 \end{aligned}$$

where for the equality we used Pythagerous theorem since $b_1 - b \in (Im(A))^\perp$ and $Ax - b_1 \in Im(A)$. Then as $f(x) := ||Ax - b||^2$ is convex the minimum is unique i.e. $Ax = b_1$ is the unique min. Then we have for any $x, y \in M$ $A(x - y) = b_1 - b_1 = 0$ so $x - y \in N$. Then fix an $x_0 \in M$ then for any $x \in M$ we have $x = x_0 + (x - x_0)$ where $x_0 \in M$ and $x - x_0 \in N$ so $M \subset x_0 + N$. Choosing the same x_0 as before we have $Ax = b_1$ then we have for any $y \in N$ $A(x + y) = b_1$ i.e. it minimizes the problem so $x_0 + N \subset M$ i.e. $M = x_0 + N$.

Problem 10. Note that $P(A) = (A + I)^3(A - I) = 0$ so as the minimal polynomial divides $P(A)$ all of A eigenvalues are -1 or $+1$. As $rank(B) = 2$ we have $nullity(B) = 2$ i.e. the eigenspace of -1 has dimension 2 so there are two Jordan Blocks with eigenvalue -1. As $|Tr(A)| = 2$ we must have 3 eigenvalues of -1 and 1 eigenvalue of 1. But as we only have two eigenvalues, we must have a 2×2 Jordan Block of -1, a 1×1 Jordan Block of -1 and a 1×1 Jordan block of 1

Problem 11.

Problem 12. We have $rank(A) \geq r$ if and only if there exists a $r \times r$ sub-matrix such that it has invertible. Then we have for any linear operator L that L is invertible if and only if L^T is invertible since $ker(L) = range(L^T)^\perp$. This implies $rank(A) = rank(A^T)$.

7. Spring 2013

Problem 1a. See 2012 Fall Problem 2a)

Problem 1b. See 2012 Fall Problem 2b)

Problem 1c. Observe that $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ and let $S_n := \sum_{k=1}^{n} \frac{1}{2^k}$ and let $I_k := [S_{k-1}, S_k]$ with $S_0 := 0$. Then $[0,1] = \bigcup_{k=1}^{\infty} I_k$ is a disjoint union except at the end points. Let

$$f := \begin{cases} S_k \text{ for } x \in I_k \end{cases}$$

then this is a monotone function with infinitely many jumps i.e. it is discontinuous on a countable set but it is Riemann Integrable thanks to monotocity.

Problem 2.

Problem 3. We will show sequentially compact implies complete and totally bounded first. Given any Cauchy sequence the sequential compactness implies there is a sub-sequence that converges, but a cauchy sequence with a convergent sub-sequence is convergent. Hence it is complete. It is totally bounded since if not there exists an $\varepsilon_0 > 0$ such that if the space is denoted as X we have

$$X \not\subseteq B_{\varepsilon_0}(x_1)$$

thus there is an x_2 such that $d(x_2, x_1) > \varepsilon_0$ but being not totally bounded implies

$$X \not\subseteq B_{\varepsilon_0}(x_1) \cup B_{\varepsilon_0}(x_2)$$

similarily we can find an x_3 such that $d(x_1, x_3) > \varepsilon_0$ and $d(x_2, x_3) > \varepsilon_0$ due to X not being totally bounded such that

$$X \not\subseteq B_{\varepsilon_0}(x_1) \cup B_{\varepsilon_0}(x_2) \cup B_{\varepsilon_0}(x_3)$$

Thus proceeding by induction we can find a sequence $\{x_n\}$ such that for any m we have

$$d(x_n, x_m) \geq \varepsilon_0$$

which means there cannot be a convergent sub sequence. So X must be totally bounded.

Now assume X is totally bounded and complete. Fix a sequence $\{x_n\} \subset X$ and assume x_n has infinitely many distinct values for otherwise the sequence will have a convergent sub-sequence and there will be nothing to prove. As X is totally bounded we have $y_1, .., y_N$ such that

$$X \subset B_{\frac{1}{2}}(y_1) \cup ... \cup B_{\frac{1}{2}}(y_N)$$

Thus there exists an $1 \leq i \leq N$ such that there are infinitely many values of $x_n \in B_{\frac{1}{2}}(y_i)$. Denote this subsequence as $x_n^{(1)}$ then there exists $z_1, .., z_M$ such that

$$X \subset \bigcup_{i=1}^{M} B_{\frac{1}{4}}(z_i)$$

and again there exists a subsequence $x_n^{(2)}$ of $x_n^{(1)}$ such that they are infinitely many terms of $x_n^{(2)}$ in $B_{\frac{1}{4}}(z_j)$ for some j. Proceeding inductively we can find a sequence $x_n^{(k)}$ such that $x_n^{(k)}$ is a susbequence of $x_n^{(k-1)}$ and there are infinitely many terms of $x_n^{(k)}$ in $B_{\frac{1}{2^k}}(w_j^{(k)})$ for some $w_j^{(k)}$. Let the subsequence $y_n := x_n^{(n)}$ i.e. the diagonal susbequence then for $n \geq m$

$$d(y_n, y_m) \leq d(y_n, w_j^{(m)}) + d(w_j^{(m)}, y_m) \leq \frac{1}{2^m} + \frac{1}{2^m}$$

since as $n \geq m$ we have $\{x_k^{(n)}\} \subset B_{\frac{1}{2^m}}(w_j^{(m)})$ since it is a subsequence of $\{x_k^{(m)}\}$. Therefore, it is Cauchy then by completeness there exists a limit. So the space is sequentially compact.

Problem 4. We prove the stronger general result: If $f : [1, \infty) \to [0, \infty)$ such that f is decreasing and $\lim_{x \to +\infty} f(x) = 0$ then

$$\sum_{i=1}^{N} f(i) - \int_{1}^{N+1} f(x) \mathrm{d}x$$

converges to a finite limit. Indeed, as $f(x) \to 0$ we have for all $\varepsilon > 0$ an $M > 0$ such that for $x > M$ such that $f(x) \leq \varepsilon$. Then let $a_N := \sum_{i=1}^{N} f(x) - \int_1^{N+1} f(x)\mathrm{d}x$ then we have for $N \geq M$

$$\begin{aligned} |a_N - a_M| &= \left| \sum_{i=M+1}^{N} f(i) - \int_{M+1}^{N+1} f(x)\mathrm{d}x \right| \\ &= \left| \sum_{i=1}^{N-M} \int_{M+i}^{M+1+i} f(M+i) - f(x)\mathrm{d}x \right| \\ &= \sum_{i=1}^{N-M} \int_{M+i}^{M+1+i} f(M+i) - f(x)\mathrm{d}x \\ &\leq \sum_{i=1}^{N-M} \int_{M+i}^{M+1+i} f(M+i) - f(M+i+1)\mathrm{d}x \\ &= \sum_{i=1}^{N-M} f(M+i) - f(M+i+1) \\ &= f(M+1) - f(N+1) \\ &\leq 2\varepsilon \end{aligned}$$

for M, N large. So it is a Cauchy Sequence and we conclude by the completeness of $\mathbb{R}$. The third equality we used $f(M+i) \geq f(x)$ on $[M+i, M+1+i]$ so that the term is already positive. Note that

$$h_n := \sum_{j=1}^{n} f(j) - \int_1^n f(x)\mathrm{d}x$$

for $f(x) := \frac{1}{x}$. By our result we have

$$\sum_{j=1}^{n} f(j) - \int_1^{n+1} f(j)$$

converges so

$$h_n = \sum_{j=1}^{n} f(j) - \int_1^{n+1} f(j) - \int_n^{n+1} f(j)$$

and since f decreases to 0 that $\lim_{n\to+\infty} \int_n^{n+1} f(j) = 0$. So h_n converges to the limit of $\sum_{j=1}^{n} f(j) - \int_1^{n+1} f(j)$.

Problem 5a. There is a typo and it should be $U_n(\cos(\theta)) = \frac{\sin(n(\theta))}{\sin(\theta)}$ base case is trivial since $U_1 = 1$. Then we have

$$\begin{aligned} \sin(\theta)U_{n+1}(\cos(\theta)) &= 2\cos(\theta)\sin(n\theta) - \sin((n-1)\theta) \\ &= 2\cos(\theta)\sin(n\theta) - (\sin(n\theta)\cos(-\theta) + \sin(-\theta)\cos(n\theta)) \\ &= \cos(\theta)\sin(n\theta) + \sin(\theta)\cos(n\theta) = \sin((n+1)\theta) \end{aligned}$$

so induction holds.

Problem 5b. Consider $x \mapsto \cos(\theta)$ then we get

$$\int_{-1}^{1} U_m(x)U_n(x)\sqrt{1-x^2}\mathrm{d}x = \int_0^{\pi} \sin(n\theta)\sin(m\theta)\mathrm{d}\theta$$

Problem 6a. Note by Schur's Decomposition we have that any complex matrix is unitary equivalent to an upper triangular matrix i.e.

$$A = U^T T U$$

where T is upper triangular and $U^{-1} = U^T$. Then we recall if an operator has only distinct eigenvalues then it is diagonalizable and that the eigenvalues of an upper triangular matrix are its diagonals. Then consider

$$A_k := U^T(T + \mathrm{diag}(h_1, .., h_n))U$$

where $\sqrt{h_1^2 + ... + h_n^2} \leq \frac{1}{k}$ and h_i are chosen such that $T_{ii} + h_i \neq T_{jj} + h_j$ for all $i \neq j$. Letting $k \to \infty$ gives $A_k \to A$ and each A_k is diagnolizable since it has distinct eigenvalues.

Problem 6b. Note that $f(A) := \det(A - \lambda I)$ is continuous from $\mathbb{R}^{n\times n} \to \mathbb{R}$ isnce it is a polynomial of the coefficients of A. So if $A_n \to A$ we have $f(A_n) \to f(A)$. Let

$$A := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

with $\theta = \frac{\pi}{2}$ then this has only complex eigenvalues. So if $A_n \to A$ we must have for large n that A_n has complex eigenvalues. Therefore, there does not exist a sequence of real diagonolizable matrix A_n such that $A_n \to A$. So they are not dense.

Problem 7a. We define

$$||A|| := \sup_{||x||=1} ||A(x)||$$

Note that

$$||A(x)|| \leq ||A|| \, ||x||$$

so

$$||A^2(x)|| \leq ||A|| \, ||A(x)|| \leq ||A||^2 ||x||$$

so

$$||A^2|| \leq ||A||^2$$

Problem 7b. By the observation above we have for when $|x| = 1$

$$\exp(A)(x) := x + Ax + \frac{A^2(x)}{2!} + ..$$

$$\leq 1 + ||A|| + \frac{||A||^2}{2} + ...$$

$$= \exp(||A||)$$

so we have

$$\exp(A)(x) \leq |x| \exp(||A||) < \infty$$

so the series makes sense everywhere.

Problem 7c. Note if $||A|| < 1$ then for $|x| = 1$ we have

$$\log(I + A)(x) \leq 1 + \sum_{n=1}^{\infty} \frac{||A||^n}{n} \leq \sum_{n=0}^{\infty} ||A||^n < +\infty$$

where the last line is justified via $||A|| < 1$.

Problem 7d. No thank you.

Problem 8a.

$$(Tx, y) = (x, T^* y)$$

Problem 8b. Typo it should be transpose of the conjugate matrix. But it follows from writing out the inner products.

Problem 8c. We have $x \in ker(T)$ iff for all $y \in V$

$$0 = (Tx, y) = (x, T^* y)$$

i.e. $x \in Im(T^*)^{\perp}$ so $Ker(T) \subset Im(T^*)^{\perp}$. Then fix $y \in Im(T^*)^{\perp}$. So for all $x \in V$ we have

$$0 = (y, T^* x) = (Ty, x)$$

so $Ty = 0$ Therefore, $Im(T^*)^{\perp} = Ker(T)$.

Problem 8d. This implies that if L is an operator then it is invertible if and only if T^* is invertible. Use that $rank(T) \geq r$ iff there exists an $r \times r$ submatrix such that the submatrix is invertible. This implies that $rank(T^*) \geq rank(T)$ by choosing $r = rank(T)$ and the other inequality follows from replacing T with T^* and using $T^{**} = T$.

Problem 9a. Observation 1: If

$$A := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

then its eigenvalues are $\cos(\theta) + i\sin(\theta)$ and $\cos(\theta) - i\sin(\theta)$ and any $z \in \mathbb{C}$ such that $|z| = 1$ can be represented as $\cos(\theta) + i\sin(\theta)$.

Observation 2: As A is orthogonal so its eigenvalues λ satisfy $|\lambda| = 1$. But as A is real if λ is complex valued then $\overline{\lambda}$ is an eigenvalue since eigenvalues come in complex conjugate pairs for real valued matrix.

Observation 3: A is normal, so it is diagonalizable over $\mathbb{C}$.

By observation 1 as A is diagnolizable over it has a basis of eigenvectors. Order the eigenvalues such that all the real ones are from 1 to j i.e. $\lambda_1, .., \lambda_j$ are real. Then for λ_{j+1} to λ_n these are the complex valued eigenvalues, but by observation 2 we have for any $\lambda_{j+1} = \overline{\lambda_{j+\ell}}$ for some $\ell > 1$. Order the eigenvalues so $\lambda_{j+2} = \overline{\lambda_{j+1}}$, $\lambda_{j+3} = \overline{\lambda_{j+4}}$, ... till n. Then we have by observation 3 that

$$A = U^*DU$$

where $U^{-1} = U^*$ and for $D = diag(\lambda_1, .., \lambda_n)$ then the complex conjugate eigenvalues i.e. $\lambda_{j+1}, \lambda_{j+2}$ and $\lambda_{j+3}, \lambda_{j+4}$,.. till λ_{n-1}, λ_n are similar to a rotation matrix since rotation matrix are diagnolizable over $\mathbb{C}$ since they are unitary i.e. we have

$$\begin{bmatrix} \lambda_i & 0 \\ 0 & \overline{\lambda_i} \end{bmatrix} = V_i^T \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} V_i$$

where $V_i^{-1} = V_i^*$ for any $i \geq j$. Then notice we have $D = A_1 \bigoplus A_2 \bigoplus ... \bigoplus A_{j+1} \bigoplus A_{j+3} \bigoplus .. \bigoplus A_{n-1}$ where $A_i = [\pm 1]$ for $i \leq j$. And $A_{j+1} = \begin{bmatrix} \lambda_j & 0 \\ 0 & \overline{\lambda_j} \end{bmatrix}$ and similarily till A_{n-1} and we have $V_j A_{j+1} V_{j+1}^*$ equal to some rotation matrix. Therefore, D is similar to $A_1 \bigoplus ... \bigoplus R_{j+1} \bigoplus ... \bigoplus R_{n_1}$ where $R_k := B$ is some rotation matrix and the change of basis matrix is a block matrix along the diagonals with unitary blocks so its inverse is its conjugate transpose. Then as multiplication of unitary matrix is unitary we have shown

$$A = V^*BV$$

where B is of the desired form. However, the similarity is over $\mathbb{C}$ to get similarity over $\mathbb{R}$ note that $V = V_1 + iV_2$ where V_1 and V_2 are real matrix, so

$$(V_1 + iV_2)A = B(V_1 + iV_2)$$

but as A and B are real we must have

$$V_2 A = BV_2$$

and $V_1 A = BV_1$, therefore for any $r \in \mathbb{C}$ we have

$$(V_1 + rV_2)A = B(V_1 + rV_2)$$

Then as we have for $f(r) := \det(V_1 + rV_2)$ and $f(i) \neq 0$ since it is invertible then $f(r)$ is a non-zero polynomial. So there are only finitely many roots so there exists an $r \in \mathbb{R}$ such that $f(r) \neq 0$ so $(V_1 + rV_2)$ is invertible. So

$$A = (V_1 + rV_2)^{-1} B (V_1 + rV_2)$$

so they are similar over $\mathbb{R}$.

Problem 9b. As n is odd there must exist a real eigenvalue which is either -1 or 1. Let this eigenvector associated to it be denoted by v then $A^2 v = \lambda^2 v = v$. We note v is real since A is real so $A - \lambda I$ is also real so its Kernal is real.

Problem 10a. By computation we get C is of the form

$$\begin{bmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then $G = Id$ is a subspace of the set of 4×4 matrix.

Problem 10b. It is 3 dimensional since there are 3 free parameters a, b, and c.

Problem 11a. Note that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_{n-2} \end{bmatrix} = \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$$

so in particular we get $F_n = \frac{1}{\sqrt{5}}(\lambda_1^n - \lambda_2^n)$ by diagnolizing the matrix where $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$. Then

$$\frac{F_n}{F_{n-1}} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^{n-1} - \lambda_2^{n-1}} = \lambda_1 - \lambda_2 + \frac{\lambda_1 \lambda_2^{n-1} - \lambda_2 \lambda_1^{n-1}}{\lambda_1^{n-1} - \lambda_2^{n-1}}$$

Since $\lambda_2^n \to 0$ as $n \to \infty$ we have the second term approaches λ_2 so the it approaches λ_1.

Problem 11b. Note that playing with the explicit solution we have

$$F_{2n+3}F_{2n+1} - F_{2n+2}^2 = (\lambda_1^2)(\lambda_2^2)(F_{2n+1}F_{2n-1} - F_{2n}^2)$$

and $\lambda_1^2 \lambda_2^2 = 1$ so the result follows from induction.

Problem 12. Note

$$\frac{1}{x^2+1} = \sum_{n=1}^{\infty} (-1)^n x^{-2n}$$

for $x \in (-1, 1)$. So for any $\varepsilon > 0$ with $\varepsilon < 1$ we have

$$\int_0^{1-\varepsilon} \frac{1}{1+x^2} = \int_0^{1-\varepsilon} \sum_{n=1}^{\infty} (-1)^n x^{2n}$$

since

$$\left| \sum_{n=1}^{\infty} (-1)^n x^{-2n} \right| \leq \sum_{n=1}^{\infty} (1-\varepsilon)^{2n} = C(\varepsilon) < +\infty$$

so it is uniformly convergent on $[0, 1-\varepsilon]$ So by uniform convergence we can swap the integral and sum

$$= \sum_{n=1}^{\infty} \int_0^{1-\varepsilon} (-1)^n x^{2n}$$

$$= \sum_{n=1}^{\infty} (-1)^n \frac{(1-\varepsilon)^{2n+1}}{2n+1}$$

Note that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$$

converges due to summation by parts since $|\sum (-1)^n| < M$ and $\frac{1}{2n+1}$ monotonically goes to zero. So Abel's Theorem says

$$\lim_{\varepsilon \to 0} \sum_{n=1}^{\infty} (-1)^n \frac{(1-\varepsilon)^{2n+1}}{2n+1} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$$

and

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1} = \lim_{\varepsilon \to 0} \int_0^{1-e} \frac{1}{1+x^2} = \frac{\pi}{4}$$

where in the last inequality we used that $f(\varepsilon) := \int_0^{1-e} \frac{1}{1+x^2}$ is continuous to get $\lim_{\varepsilon \to 0} f(\varepsilon) = f(0) = \frac{\pi}{4}$.

8. Fall 2013

Problem 1. Fix $\{a_n\}$ such that a_n is positive. Assume

$$P_n := \prod_{i=1}^{n}(1+a_i)$$

converges to a non-zero limit a. Then as $P_n > 0$ for all n we can take the log of it to see

$$\log(P_n) = \sum_{i=1}^{n} \log(1+a_i)$$

So as $\log(x)$ is continuous on $(0,\infty)$ and $P_n \to a > 0$ we have

$$\lim_{n\to\infty} \log(P_n) = \log(\lim_{n\to\infty} P_n) = \log(a) > 0$$

Therefore, $\sum_{i=1}^{n} \log(1+a_i)$ converges so $\log(1+a_i) \to 0$ so $a_i \to 0$. Then as we have

$$\lim_{x\to 0} \frac{\log(x+1)}{x} = 1$$

we conclude there exists an $\delta > 0$ such that if $|x| < \delta$ then

$$\frac{1}{2} \le \frac{\log(x+1)}{x} \le 2$$

Then as $a_i \to 0$ there exists an N such that for $i \ge N$ we have $|a_i| = a_i < \delta$ so for any $M \ge N$ we have

$$(*) \qquad \frac{1}{2}\sum_{i=N}^{M} a_i \le \sum_{i=N}^{M} \log(a_i+1) \le 2\sum_{i=N}^{M} a_i$$

Therefore, for any fixed $\varepsilon > 0$ choosing N, M large enough from convergence of $\sum_{i=N}^{M} \log(a_i+1)$ we have

$$\sum_{i=N}^{M} \log(a_i+1) \le \frac{\varepsilon}{2}$$

In particular,

$$\left|\sum_{i=N}^{M} a_i\right| = \sum_{i=N}^{M} a_i \le \varepsilon$$

so $\{\sum_{i=1}^{N} a_i\}$ forms a Cauchy sequence so it converges.

Now assume $\sum_{n=1}^{N} a_n$ converges then by $(*)$ $\sum_{i=1}^{N} \log(a_i+1)$ converges. So

$$P_n := \sum_{i=1}^{N} \log(a_i+1) = \log(\prod_{i=1}^{N} 1+a_i)$$

Therefore, there exists an a such that

$$\log(\prod_{i=1}^{N} 1+a_i) \to a$$

Then we have by taking exponentials and using that it is a continuous map to see

$$\prod_{i=1}^{N} 1+a_i \to \exp(a)$$

which is strictly bigger than 0. And the equivalence is proved.

Problem 2a. Let

$$A := \{x : f(x) \text{ is not continuous } \}$$

we claim that

$$A = \{x : \lim_{y\to x^-} f(y) \ne \lim_{y\to x^+} f(y) := B$$

Note for any x that the left and right limits of f are well defined since f is monotone and locally bounded (by the right and left end points of the interval). So B makes sense and it is clear $B \subset A$. But if $x \in A$

then the left and right limits are well defined so we must have $\lim_{y\to x^-} f(y) \neq \lim_{y\to x^+} f(y)$ for otherwise f would be continuous at x. Then for each $x \in A$ we can pick $q \in (\lim_{y\to x^-} f(y), \lim_{y\to x^+} f(y)) \cap \mathbb{Q}$. But as f is monotone we have for any $z \in$ that $q \notin (\lim_{y\to z^-} f(y), \lim_{y\to z^+} f(y))$ since f is monotone. Therefore, we have found an injection from A to . So A is countable.

Problem 2b.

Problem 3a. For any partition of $[0,1]$ we have

$$\sum_{i=1}^{n-1} |\gamma(t_{j+1}) - \gamma(t_j)| = \sum_{i=1}^{n-1} \sqrt{(t_{j+1}-t_j)^2 + (f(t_{j+1}) - f(t_j)^2}$$

$$\leq \sum_{i=1}^{n-1} |t_{j+1} - t_j| + |f(t_{j+1}) - f(t_j)| = \sum_{i=1}^{n-1} (t_{j+1} - t_j) + (f(t_{j+1}) - f(t_j))$$

$$= 1 + f(1) - f(0)$$

where for the second equality we used f is increasing and $t_{j+1} > t_j$ and the last equality we used it was two telescoping sums.

Problem 3b.

Problem 4. See 2012 Fall number 6 a).

Problem 5. See Fall 2011 number 2.

Problem 6. I do not think compactness is needed. Indeed, assume $\{x_n\}$ does not converge to x then there exists an $\varepsilon_0 > 0$ such that for any N there is an $n(N) \geq N$ such that

$$d(x_{n(N)}, x) \geq \varepsilon_0$$

Take $N = 1, 2, 3, ..$ then there is a sequence $x_{n(N)}$ such that

$$d(x_{n(N)}, x) \geq \varepsilon_0$$

But as $x_{n(N)}$ is a sub-sequence we can find a further sub-sequence that converges but this is a contradiction since

$$d(x_{n(N)}, x) \geq \varepsilon_0$$

for all N.

Problem 7. Let P_N denote the $N+1$ dimensional space of polynomials of degree N and define the linear map $\psi : P_N \to \mathbb{R}^{N+1}$ via $\psi(P) = (P(z_1), .., P^{(m_1)}(z_1), P(z_2), .., P^{(m_2)}(z_2), ..., P(z_n), ..., P^{(m_2)}(z_n))$. Then it suffices to show ψ is bijective which is equivalent to showing it is injective since $dim(P_N) = dim(R^{N+1})$. If $\psi(P) = (0, ..., 0)$ then z_i is a $(m_i + 1)$ root of ψ so ψ has $N+1$ roots which implies by the fundamental theorem of algebra that $P \equiv 0$. Therefore, this map is bijective so the desired result holds.

Problem 8. As P is an orthogonal projection withe trace 2, we have the existence of a unitary matrix U such that

$$P = U^T \text{diag}(1,1,0)U$$

Therefore,

$$P - I = U^T \text{diag}(1,0,0)U$$

i.e. it has rank 1. So we must have the existence of $p, q \in \mathbb{R}^3$ such that

$$P - I = pq^T$$

and as $P - I$ is self adjoint and as $(P-I)^2 = P - I$ we have

$$P - I = pq^T pq^T = qp^T pq^T = \alpha qq^T$$

for some $\alpha = ||p||^2$. We can assume $\alpha = 1$ since we can put the put the constant into q. So

$$P - I = qq^T$$

for some q and we have $\text{diag}(P - I) = (q_1^2, q_2^2, q_3^2)$ so we have

$$q_1 = \pm\frac{\sqrt{2}}{\sqrt{3}} \quad q_2 = \pm\frac{1}{\sqrt{2}} \quad q_3 = \pm\frac{\sqrt{5}}{\sqrt{6}}$$

so

$$P = I + qq^T$$

for any choice of the q with the above signs.

Problem 9. Fix $v \in V$ such that $v \neq 0$. Then consider

$$W := \{v, Av, .., A^{k-1}v\}$$

where $k-1 \leq d-1$ is the largest integer such that the above list is linearly independent. So we have $\alpha_0, .., \alpha_{k-1}$ such that

$$\alpha_0 v + \alpha_1 Av + ... + \alpha_{k-1}A^{k-1}v + A^k v = 0$$

this implies $T(W) \subset W$. Therefore, for $A|_W$ in the basis of $\{v, Av, .., A^{k-1}v\}$ we have $A(A^m v) = A^{m+1}(v)$ for $m = 0$ till $k-2$ and $A(A^{k-1}v) = -\alpha_0 v - \alpha_1 Av - ...\alpha_{k-1}A^{k-1}v$ i.e. it is a companion matrix so $T|_W$ characteristic polynomial if denoted $g(t)$ is

$$(-1)^k(\alpha_0 + \alpha_1 t + ...\alpha_{k-1}t^{k-1} + t^k)$$

Then we claim this divides the characteristic polynomial of T. Indeed fix a basis of $w = \{v, Av, .., A^{k-1}v\}$ and extend it to a basis of V $\beta := \{v, Av, .., A^{k-1}v, w_1, ..w_n\}$ then in this basis we have

$$[T]_\beta = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

so its characteristic polynomial $f(t)$

$$f(t) = \det(A - tI) = \det \begin{bmatrix} B_1 - tI & B_2 \\ 0 & B_3 - tI \end{bmatrix}$$
$$= \det(B_1 - tI)\det(B_3 - tI) = g(t)\det(B_3 - tI)$$

so the characteristic polynomial of A divides the 0 characteristic polynomial of $A|_W$. But as $f(T) = p(T)g(T)$ for some polynomial p we have $f(T)v = p(T)g(T)v = 0$ since $g(T)v = 0$. As v is arbitrary it implies $f(T)v = 0$ for all $v \in V$ so T satisfies its own characteristic polynomial. And the characteristic polynomial is of degree d so we are done.

Problem 10.

Problem 11. We say T is normal iff $T^*T = TT^*$ where T^* is the adjoint of T then we have

$$(Tx, Tx) = (x, T^*Tx) = (x, TT^*x) = (T^*x, T^*x)$$

so $||Tx|| = ||T^*x||$. Now by Schur's Decomposition as T is complex there is a unitary matrix U such that

$$T = U^T AU$$

where A is an upper triangular matrix. We will show from $||Tx|| = ||T^*x||$ that A is in fact diagonal. Note that unitary equivalence preserves normal operators so $||A|| = ||A^*||$. Then

$$||A(e_1)||^2 = |a_{11}|^2$$
$$||A^*(e_1)||^2 = |a_{11}|^2 + |a_{12}|^2 + ... + |a_{1n}|^2$$

so $a_{12} = ... = a_{1n} = 0$. Then using $a_{12} = 0$

$$||A(e_2)||^2 = |a_{22}|^2$$
$$||A^*(e_2)||^2 = |a_{22}|^2 + |a_{23}|^2 + ... + |a_{2n}|^2$$

so $a_{2j} = 0$ for $j \neq 2$. We can proceed inductively to show all the non-diagonal terms are zero. So A is a diagonal matrix. So T is unitarily equivalent to a diagonal matrix. This means there exists an orthonormal basis such that $Tv = \lambda v$ for some λ. Indeed,

$$TU^T = U^T A$$
$$TU^T = [Tu_1, .., Tu_n] \quad U^T A = [\lambda_1 u_1, ..., \lambda_n u_n]$$

where u_i is the ith column of U^T so $T(u_i) = \lambda_i u_i$ and we have u_i are an orthonormal basis since U is unitary. So we have a orthnormal basis of eigenvectors.

Problem 12. Note

$$C_A(X) = (X-1)^2(X-2)^2$$

and that A is similar to B if and only if they have the same Jordan Canonical Form. We can either have the Jordan form as 4 1×1 block, or one 2×2 block with two 1×1 block, or two 2×2 blocks so there are a total of 4 similarity/congruence classes.

9. Spring 2014

Problem 1a. Note that

$$t^4 = -1 \Rightarrow t = \cos(\theta) + i\sin(\theta)$$

such that $\cos(4\theta) = -1$ and $\sin(4\theta) = 0$. So $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$. Note that $\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}) = \cos(\frac{7\pi}{4}) - i\sin(\frac{7\pi}{4})$ and $\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}) = \cos(\frac{5\pi}{4}) - i\sin(\frac{5\pi}{4})$ so the matrix

$$A := \begin{bmatrix} R(\frac{\pi}{4}) & 0 \\ 0 & R(\frac{3\pi}{4}) \end{bmatrix}$$

with

$$R(\theta) := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

has characteristic polynomial $t^4 + 1$. But as all the eigenvalues of A are distinct we have the minimal polynomial is the characteristic polynomial.

Problem 1b. This question is false. But it can be shown that all sub-spaces are of even dimension i.e. for A take $W = \text{span}\{a, b, 0, 0\}$ for $a, b \in \mathbb{R}$ or $\text{span}\{0, 0, a, b\}$. To see why it has to be two dimensional. Fix $W \subset \mathbb{R}^4$ such that W is a subspace and let $A(W) \subset W$ where A is defined in part a. Assuming $W \neq \{0\}$ this means if we fix an orthonormal basis $\{w_1, .., w_m\}$ of W and extend it to a orthonormal basis of $\mathbb{R}^n$ i.e. $\beta = \{w_1, .., w_m, v_1, .., v_{n-m}\}$ then A written in this basis takes the form

$$[A]_\beta = \begin{bmatrix} B_1 & B_2 \\ 0 & B_3 \end{bmatrix}$$

In particular this implies the restricted operator $A|_W$ characteristic polynomial divides the characteristic polynomial of A since if we let $g(t)$ be the characteristic polynomial of A and $f(t)$ be the characteristic polynomial of $A|_W$ we have

$$g(t) = \det(A - tI) = \det(B_1 - tI)\det(B_3 - tI) = f(t)\det(B_3 - tI)$$

And since $A|_W$ is a real operator all of its eigenvalues come in complex conjugate pairs so either the characteristic polynomial of B_1 is $t^4 + 1$, $(t - \lambda_1)(t - \overline{\lambda_1})$, or $(t - \lambda_2)(t - \overline{\lambda_2})$ for $\lambda_i = \cos(\theta_i) + i\sin(\theta_i)$ with $\theta_1 = \frac{\pi}{4}$ or $\theta_2 = \frac{3\pi}{4}$. If the characteristic polynomial of B_1 is $t^4 + 1$ then we are done since B_1 will be a 4×4 matrix i.e. the basis of W has dimension 4. So WLOG assume that the characteristic polynomial of B_1 is $(t - \lambda_1)(t - \overline{\lambda_1})$. Then B_1 is similar to the rotation matrix $R(\frac{\pi}{4})$ which implies B_1 is a 2×2 matrix. So W has dimension 2 so either $dim(W)$ is 2 or 4.

Problem 2. Note that

$$\begin{aligned} rank(ST) + nullity(ST) = dim(V) \Rightarrow rank(ST) &= dim(V) - nullity(ST) \\ &\geq dim(V) - nullity(S) - nullity(T) \end{aligned}$$

so

$$\begin{aligned} rank(ST) &\geq rank(S) + nullity(S) - nullity(S) - nullity(T) \\ &= rank(S) - nullity(T) \end{aligned}$$

which is equivalent to the desired inequality.

Problem 3. Assume for the sake of contradiction that A^{-1} exists then

$$B - A^{-1}BA = I$$

which implies $n = tr(I) = tr(B - A^{-1}BA) = tr(B) - tr(B) = 0$ which is a contradiction.

Problem 4. We claim this holds for all invertible matrix B indeed

$$det(BA - \lambda I) = det(B^{-1}(BA - \lambda I)B) = det(AB - \lambda I)$$

where we used $det(AB) = det(A)det(B) = det(B)det(A) = det(BA)$. Now we claim the set of invertible matrix is dense in $\mathbb{R}^{n\times n}$. Indeed, given any matrix $A \in \mathbb{R}^{n\times n}$ we can extend it to an operator over ^{n}. So by Schur's Decomposition we have A is unitarily equivalent to an upper triangular matrix i.e.

$$A = U^T TU$$

and the eigenvalues of A are the diagonal terms of T. So consider

$$A_n := U^T(T + \frac{1}{n}I)U$$

then as there are only finitely many eigenvalues there exists an N such that for $n \geq N$ we have $\text{diag}(T+\frac{1}{n}I)$ have no zero entries. Therefore, A_n is invertible and clearly as $n \to \infty$ we have $A_n \to A$ and A_n is real valued since we are adding a real valued matrix to a real valued matrix. Then since the determinant is a continuous function since its a polynomial of the coefficients of the matrix we have for a given B there exists $B_n \to B$ where B_n are invertible so

$$\lim_{n\to\infty} \det(AB_n - tI) = \lim_{n\to\infty} \det(B_n A - tI)$$

so continuity lets us put the limit inside so

$$\det(AB - tI) = \det(BA - tI)$$

Problem 5. Note $V = range(L) \bigoplus (range(L))^{\perp}$ and we see that for any b there exists unique $b_1 \in range(L)$ and $b_2 \in (range(L))^{\perp}$ such that $b = b_1 + b_2$. Then $L(x)$ minimizes

$$||L(x) - b||$$

if and only if $L(x) = b_1$ since

$$||b_1 - b||^2 \leq ||b_1 - b||^2 + ||L(x) - b_1||^2 = ||L(x) - b||^2$$

where the last line we used Pythagoras theorem since $b_1 - b \in (range(L))^{\perp}$ and the other term is in $range(L)$. So b_1 is a min but the convexity of $f(x) := ||L(x) - b||^2$ tells us the minimum is unique since $L(x) - b$ is affine and $||\cdot||$ is convex. Therefore, all minimizes x satisfy $L(x) = b_1$ so if x and y minimize it then $L(x) = L(y)$.

Problem 6. Note the spectral theorem implies that

$$A = U^* D U$$

where U is unitary and D is a diagonal matrix since A is a normal operator so

$$A^* = U^* D^* U$$

Note for any given polynomial P we have

$$P(A) = U^* P(D) U$$

so it suffices to show there exist a polynomial such that $P(D) = D^*$. Note that if $P(x) = \sum_{i=1}^n \alpha_i x^i$ we have $P(D) = \sum_{i=1}^n \alpha_i D^i$. If we let $diag(D) = (\lambda_1, ..., \lambda_n)$ and $diag(D^*) = (\beta_1, .., \beta_n)$ then it suffices to show there exists a P such that $P(\lambda_i) = \beta_i$ for all $i = 1, .., n$. Indeed this will just be the usual Lagrnage Polynomials indeed fix a j and note

$$P_j(x) := \prod_{i \neq j} \frac{x - \lambda_i}{\lambda_j - \lambda_i}$$

satisfies

$$P_j(\lambda_k) = \begin{cases} 1 \text{ if } k = j \\ 0 \text{ else} \end{cases}$$

So let

$$P(x) := \sum_{j=1}^n \beta_i P_j(x)$$

then it satisfies

$$P(\lambda_k) = \beta_k$$

for all k. Therefore, $P(D) = \sum_{i=1}^n \alpha_i D^i = \text{diag}(\sum_{i=1}^n \alpha_i \lambda_1, .., \sum_{i=1}^n \alpha_i \lambda_n) = \text{diag}(\beta_1, .., \beta_n) = D^*$ so $P(A) = A^*$ as desired.

Problem 7. We will write out our counter example $\{a_{nm}\}$ in matrix form:

$$\begin{bmatrix} 1 & -1 & 0 & ... & 0 & 0 & 0 & 0 & & \\ -1 & 1 & -1 & 1 & 0... & 0 & 0 & 0 & & \\ 0 & 0 & 1 & -1 & 1 & -1 & ... & 0 & .. & 0 \end{bmatrix}$$

and so on. Then summing along each row we get zero and summing column we get zero since there are only finitely many $1s$ and $-1s$. In particular, $\sum_m a_{nm} = \sum_n a_{nm} = 0$ and these sums converge absolutely since there are only finitely many terms. However, $\sum_{n,m} |a_n| = +\infty$ since there are infinitely many $1s$ and $-1s$.

Problem 8a. Note that by induction one easily sees that we have

$$f^{(n)}(t) = \begin{cases} Q_n(t)e^{-\frac{1}{t}} \text{ for } t > 0 \\ 0 \text{ for } t \le 0 \end{cases}$$

where Q_n is a rational function so it suffices to show $\lim_{t\to 0^+} f^{(n)}(t) = 0$. Then as we have exponentials e^{-t} decay faster than any rational functions at $t = +\infty$ we have the limit is zero so it is smooth.

Problem 8b. Note that we have $f(t^2-1) = \begin{cases} e^{-\frac{1}{t^2-1}} \text{ for } -1 < t < 1 \\ 0 \text{ else} \end{cases}$ is smooth since it is the composition of two smooth functions. In particular in $\mathbb{R}^d$ we have

$$f(|x|^2 - 1) = \begin{cases} e^{-\frac{1}{|x|^2-1}} \text{ for } |t| < 1 \\ 0 \text{ else} \end{cases}$$

is smooth since its the composition of two functions. Then as this function is strictly positive we can divide by its L^1 mass to find a function as desired in the problem statement

Problem 9. See Fall 2010 number 11.

Problem 10. This is one side of Arzela-Ascoli. Enumerate $\mathbb{Q}\cap[0,1] = \{q_n\}_{n\in\mathbb{N}}$ then for any $\{f_n\} \subset \mathcal{F}$ we have from uniform bound of the family

$$|f_n(q_1)| \le M$$

so by compactness of $[0,1]$ we find a limit $f(q_1)$ along the subsequence $n_k^{(1)}$ such that $f_{n_i^{(1)}}(q_1) \to f(q_1)$. Then by induction for any k we find have that

$$|f_{n_i^{(k-1)}}(q_k)| \le M$$

so we find a limit $f(q_k)$ and a subsequence $n^{(k)} \subset n^{(k-1)}$ such that $f_{n_i^{(k)}}(q_k) \to f(q_1)$. Let the subsequence $m_k := n_k^{(k)}$ then we have for any $n \in \mathbb{N}$ that $f_{m_k}(q_n)$ converges. Fix $\varepsilon > 0$ then by equicontinuity there is a $\delta > 0$ such that if $d(x,y) < \delta$ then for any $f \in \mathcal{F}$ we have $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Then as $\mathbb{Q}\cap[0,1]$ is dense we have

$$[0,1] \subset_{n=1}^{\infty} B_\delta(q_n)$$

so compactness gives us a finite subcover say $q_1,..,q_N$ are the centers. Then as $f_{m_k}(x) \to f(q_i)$ for all $1 \le i \le N$ we can find an M such that if $k,m \ge M$ then

$$|f_{m_n}(q_i) - f_{m_k}(q_i)| < \frac{\varepsilon}{3}$$

for any $1 \le i \le N$. Then for any x there exists a q_i such that $x \in B_\delta(q_i)$ so

$$|f_{m_k}(x) - f_{m_n}(x)| \le |f_{m_k}(x) - f_{m_k}(q_i)| + |f_{m_k}(q_i) - f_{m_n}(q_i)| + |f_{m_n}(q_i) - f_{m_n}(x)|$$

so the first and last term are controlled by $\varepsilon/3$ due to equicontinuity while the second term if less than $\varepsilon/3$ if $k,n \ge M$ so we have for $k,m \ge M$

$$|f_{m_k}(x) - f_{m_n}(x)| \le \varepsilon$$

so it is a uniformly cauchy subsequence of $C([0,1])$ which by completeness implies the existence of a limit f.

Problem 11. We note that this means $\overline{\mathcal{F}}$ is a compact subset of $C([0,1])$. In particular, $\overline{\mathcal{F}}$ is totally bounded. We claim this implies $\mathcal{F}$ is totally boubded. Indeed, fix $\varepsilon > 0$ then there exists $f_1,..,f_N \in \overline{\mathcal{F}}$ such that

$$\overline{\mathcal{F}} \subset \bigcup_{i=1}^{N} B_{\varepsilon/2}(f_i)$$

then as $f_i \in \overline{\mathcal{F}}$ there exists an $g_i \in \mathcal{F}$ such that $d(g_i, f_i) < \frac{\varepsilon}{2}$. Then for any $f \in \mathcal{F}$ there is an i such that $d(f, f_i) < \frac{\varepsilon}{2}$ so $d(f, g_i) \le d(f, f_i) + d(f_i, g_i) \le \varepsilon$ i.e.

$$\mathcal{F} \subset \bigcup_{i=1}^{N} B_\varepsilon(g_i)$$

for $g_i \in \mathcal{F}$ so $\mathcal{F}$ is totally bounded. Then we have the existence of $g_1, .., g_N \in \mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{N} B_1(g_i)$$

so for any $f \in \mathcal{F}$ we have $||f|| \leq 1 + \max_{1 \leq i \leq N} ||g_i||$ so it is uniformly bounded. For equicontinuity fix $\varepsilon > 0$ then we have the existence of $g_1, .., g_N \in \mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{N} B_{\varepsilon/3}(g_i)$$

Then there exists a $\delta > 0$ such that if $d(x, y) < \delta$ then for all $1 \leq i \leq N$ that $d(g_i(x), g_i(y)) < \varepsilon/3$ due to uniform continuity of g_i since $[0, 1]$ is compact. Then for any $f \in \mathcal{F}$ there is a g_i such that $||f - g_i|| < \varepsilon/3$ so if $d(x, y) \leq \delta$ then

$$|f(x) - f(y)| \leq |f(x) - g_i(x)| + |g_i(x) - g_i(y)| + |g_i(y) - f(y)| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

so the family is equicontinuous.

Problem 12a. Note that $E^c \cap [0, 1] = \bigcup_{n=1}^{\infty} \text{int}(I_n)$ so E^c is open in $[0, 1]$.

Problem 12b. We need the following lemma: If $A \subset \mathbb{R}$ is a perfect set i.e. it is a closed set that has no isolated points then A is uncountable. Indeed we first show A is complete. Indeed given a Cauchy sequence $\{x_m\} \subset A$ we have that there exists a limit in $\mathbb{R}$ which implies $x \in A$. Now this means Baire Category Theorem can be applied. Indeed, as A is countable we have

$$A = \bigcup_{n=1}^{\infty} \{a_n\}$$

but each $\{a_n\}$ is closed with empty interior so BCT says there exists an n such that $\{a_n\}$ has non-empty interior which is our contradiction.

Now we have 4 cases either 0 is an isolated point or a limit point, or 1 is an isolated point or a limit point. WLOG assume that 0 and 1 are limit points for if say 1 is an isolated point then $\tilde{E} := E - \{1\}$ would be closed and we can repeat the proof below. Now fix $x \in E$ that is not 0 or 1 and we claim x is a limit point. Indeed assume not then x is isolated because E is closed then there exists an $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \cap E = \{x\}$$

This is our contradiction. Indeed, let $x \in I_i$ then WLOG x is a left end point then $(x - \varepsilon, x) \notin I_i$ but as there is no interval end points in $(x - \varepsilon, x)$ and I_n cover $[0, 1]$ we must have an I_k such that $(x - \varepsilon, x) \subset I_k$ but as it does not have a end point in $(x - \varepsilon, x)$ its right boundary point must be greater than or equal to x hence we must have $I_k \cap I_i \neq \emptyset$ which is a contradiction. Therefore, E is a countable perfect set (since each I_n has two points), which is a contradiction.

10. Fall 2014

Problem 1. Let $v := x - y, w := x + y$ then

$$H(x,y) = f(v,w) = \frac{v^2 + w^2}{2} + \frac{1}{|v|}$$

Fix $\varepsilon > 0$ and $R > \varepsilon$ and let the annul us with outer radius R and inner radius be defined as e $A_{R,\varepsilon} := B_R(0) - B_\varepsilon(0)$. Then $\overline{A_{R,\varepsilon}}$ is compact and f is continuous so f attains a min over $\overline{A_{R,\varepsilon}}$. Our goal is to show there exists an R and ε such that the min becomes a global min. Indeed, on $B_\varepsilon(0)$ we have $f(v,w) \geq \min_{v \in B_\varepsilon(0)} \frac{1}{|v|} = \frac{1}{\varepsilon}$ and on $\mathbb{R}^2 - B_R(0)$ we have $f(v,w) \geq \frac{R^2}{2}$. Then as $f(1,1) = 2$ we see by taking R big enough and ε small enough that

$$\min_{v \in \mathbb{R}^2 - \overline{A_{R,\varepsilon}}} f(v,w) \geq \min\{\frac{R^2}{2}, \frac{1}{\varepsilon}\} > 2 = f(1,1)$$

and $(1,1) \in A_{R,\varepsilon}$. Therefore, if $z := \min_{(v,w) \in \overline{A_{R,\varepsilon}}} f(v,w)$ then

$$z \leq f(1,1) < \min_{v \in \mathbb{R}^2 - \overline{A_{R,\varepsilon}}} f(v,w)$$

Therefore, z is a global minimum and it is attained at a point $v \neq 0 \iff x \neq y$ $\square$.

Problem 2. Claim: If A is closed and the union of two disconnected sets X, Y then X and Y are closed. Indeed, let $x \in \overline{X}$ then $x \in \overline{A} = A$ implies that $x \in X$ or Y. So $x \in X$ or Y, but as we have $\overline{X} \cap Y = \emptyset$ and $x \in \overline{X}$ implies x is not in Y i.e. $x \in X$, so X is closed.

Now assume for the sake of contradiction that A is disconnected so there exists closed sets X, Y such that

$$A = X \cup Y \quad X \cap Y = \emptyset$$

then

$$A \cap B = (X \cap B) \cup (Y \cap B)$$

and

$$(X \cup Y) \cap B \subset X \cap Y = \emptyset$$

so it follows that $X \cap B$ or $Y \cap B = \emptyset$ since $A \cap B$ is connected. Assume $X \cap B = \emptyset$, then

$$A \cup B = X \cup (Y \cup B)$$

and

$$X \cap (Y \cup B) = (X \cap Y) \cup (X \cap B) = \emptyset$$

therefore, we have $A \cup B$ is disconnected which is a contradiction.

Problem 3. As f is continuous on $[0,1]$ compact we know that f is uniformly continuous. So for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $d(x,y) < \delta$ then $d(f(x), f(y)) < \delta$. Note that

$$[0,1] \subset \bigcup_{x \in [0,1]} B_\delta(x)$$

so by compactness there exists a finite subcover say $\bigcup_{i=1}^N B_\delta(x_i)$ where we ordered the centers such that $x_{i-1} \leq x_i \leq x_{i+1}$. Then from pointwise convergence we can find an N such that if $n \geq N$ then $|f_n(x_i) - f(x_i)| < \varepsilon$ for all $1 \leq i \leq N$. Then observe for any $y \in (x_{i-1}, x_i)$ that form monotonicity

$$f_n(x_{i-1}) \leq f_n(y) \leq f_n(x_i)$$

In particular, we have for $n \geq N$ that

$$f_n(x_{i-1}) - f(x_i) \leq f_n(y) - f(x_i) \leq f_n(x_i) - f(x_i) \leq \varepsilon$$

In addition we have

$$|f_n(x_{i-1}) - f(x_i)| \leq |f_n(x_{i-1}) - f(x_{i-1})| + |f(x_{i-1}) - f(x_i)| \leq \varepsilon + \varepsilon = 2\varepsilon$$

where the first ε is due to pointwise convergence and the second ε is due to uniform continuity of f. So putting these inequalities together gives

$$|f_n(y) - f(x_i)| \leq 2\varepsilon$$

Then we have

$$|f_n(y) - f(y)| \le |f_n(y) - f(x_i)| + |f(x_i) - f(y)| \le 3\varepsilon$$

In particular, this means that if $n \ge N$ that

$$\sup_{x\in[0,1]} ||f_n(x) - f(x)|| \le 3\varepsilon$$

so it converges uniformly.

Problem 4. We will show that the family is uniformly Lipschitz on $[-1,1]$, which thanks to the equibound on f_n implies by Arzela-Ascoli the desired result. Indeed fix $y < x \in [-1,1], z \in (1,2)$ then we claim we have

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(z) - f(y)}{z - y}$$

Then for any $h > 0$

$$f(x) = f(\lambda y + (1-\lambda)z) \le \lambda f(y) + (1-\lambda)f(z)$$

for $\lambda := \frac{x-z}{y-z} < 1$ and plugging in this inequality gives the desired bound. This implies

$$\frac{f(x) - f(y)}{x - y} \le \frac{2||f||_{L^\infty([-2,21])}}{z - y} \le \frac{2||f||_{L^\infty([-2,2])}}{z - 1} := C$$

and a similar argument shows if $w \in (-2,-1)$ then

$$\frac{f(x) - f(w)}{x - w} \le \frac{f(x) - f(y)}{x - y}$$

which implies

$$\frac{f(x) - f(y)}{x - y} \ge C||f||_{L^\infty([-2,2]}$$

which implies convex functions are locally Lipschitz with a constant that only depends on the max of f over the domain. Therefore, since our family is uniformly bounded by 1 the claim follows from Arzela-Ascoli.

Problem 5. We claim for all n we have $a_n \le 2$. Indeed, $a_1 = \sqrt{2} < 2$ then by induction we have

$$a_{n+1}^2 = 2 + a_n \le 4 \Rightarrow a_{n+1} \le 2$$

so we have that a_n is a bounded sequence. Now we claim a_n is a monotone increasing sequence. Indeed, for any n we have

$$a_n^2 = 2 + a_{n-1} \ge 2a_{n-1} \ge a_{n-1}^2$$

which shows that a_n is a monotone increasing sequence bounded above by 2, so it converges. To find the limit note that we just need to solve

$$x = \sqrt{2+x} \Rightarrow x = \lim_{n\to+\infty} a_n = 2$$

Problem 6. Note that

$$\sum_{k=0}^{n-1} |f(\frac{k+1}{n}) - f(\frac{k}{n})| = \frac{1}{n}\sum_{k=0}^{n-1} |f'(y_k^{(n)})|$$

where $y_k^{(n)} \in (\frac{k}{n}, \frac{k+1}{n})$ due to the MVT. So as $|f'| \in C([0,1])$ we have Riemann's criterion that for the partition $P_n := \{x_0 := 0 < x_1 := \frac{1}{n} < x_2 := \frac{2}{n} < ... < x_n :=\}$ and any $y_k \in [x_{k-1}, x_k]$ that

$$\lim_{n\to+\infty} \sum_{k=0}^{n-1} |f(\frac{k+1}{n}) - f(\frac{k}{n})| = \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} |f'(y_k^{(n)})| = \int_0^1 |f'(x)|\mathrm{d}x$$

as desired.

Problem 7. Computation gives that solutions are of the form

$$\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \alpha\begin{bmatrix}2\\-3\\1\\0\end{bmatrix} + \beta\begin{bmatrix}4\\-5\\0\\1\end{bmatrix}$$

Then the norm squared is given by

$$f(\alpha,\beta) = (1+2\alpha+4\beta)^2 + (-3\alpha-5\beta)^2 + (1+\alpha)^2 + \beta^2$$

and we want to minimize this over all α, β so we find the points where

$$\partial_\alpha f = \partial_\beta f = 0$$

and compare their values. This gives the min is at

$$\alpha = -\frac{34}{19} \quad \beta = \frac{13}{19}$$

Problem 8. We have an eigenvalue of $n+k-1$ with the vector $(1, 1.., 1)^T$. We also $(n-1)$ eigenvalues $k-1$ with the eigenvector $e_1 - e_2$, $e_2 - e_3$ and we have $(n-1)$ of these and they are linearly independent. So our determinant is $(k-1)^{n-1}(n+k-1)$.

Problem 9. We know that as $A \in \mathbb{C}^{n\times n}$ that there exists an invertible matrix S such that

$$A = S^{-1}(J_1 \bigoplus J_2 ... \bigoplus J_k)S$$

where J_i are Jordan Block. WLOG put $\ell, \ell+1, .., k$ as the Jordan Blocks with zero diagonal. Then as $A \neq 0$ we know that $\ell > 1$. Then for $1 \leq i \leq \ell - 1$ let the diagonal terms of each block be denoted as $\lambda_1, .., \lambda_{\ell-1}$ and $\lambda_i \neq 0$ for $1 \leq i \leq \ell-1$ so there exists a α_i such that $\lambda_i + \alpha_i \neq 0$ and λ_i. So let $B_i = \alpha_i Id$ for $1 \leq i \leq \ell-1$ with the same size as J_i. Then fix any $\alpha_\ell, ..., \alpha_k \in \mathbb{C}$ such that $\alpha_{\ell+k}^{\frac{1}{n_{\ell+k}}} \neq \alpha_i$ or zero where n_i denote the size of J_i. Define

$$B_{\ell+k} := \begin{bmatrix} 0 & 0 \\ \alpha_{\ell+k} & 0 \end{bmatrix}$$

where $B_{\ell+k}$ is of size $n_{\ell+k}$ and $B_{\ell+k}$ is zero everywhere except the $(n_{\ell+k}, 1)$ entry. Then $B_{\ell+k}^2 = 0$ so its only eigenvalues are zero. But we have

$$J_{\ell+k} + B_{\ell+k} = \text{super diagonal}(1, .., 1) + B_{\ell+k}$$

In particular the transpose of $J_{\ell+k} + B_{\ell+k}$ is the companion matrix with characteristic polynomial $x^{n_{\ell+k}} - \alpha_i$. So we have the eigenvalues of

$$A + B := S^{-1}((J_1 + B_1) \bigoplus (J_2 + B_2) \bigoplus ... \bigoplus (J_k + B_k))S$$

are $\lambda_i + \alpha_i$ for $1 \leq i \leq \ell - 1$ and $\alpha_{\ell+k}^{\frac{1}{n_{\ell+k}}}$ while the eigenvalues of B are α_i and 0. And by construction these are different.

Problem 10. We can have at most $n^2 - (n-1)$ $1s$ since if we had more than that there would be at least two rows with all ones. Then let

$$A := \begin{bmatrix} 1 & 1 & 1 & 1... & 1 \\ 0 & 1 & 1 & ... & 1 \\ 1 & 0 & 1 & ... & 1 \end{bmatrix}$$

i.e. it is one everywhere except the first subdiagonal. Then this is invertible since if

$$Ax = 0$$

then we get from the first two equations

$$\sum_{i=1}^n x_i = 0 \text{ and } \sum_{i=2}^n x_i = 0$$

which gives $x_1 = 0$. Repeating a similar argument usign the $1st$ row and jth column for $j > 1$ gives $x_{j-1} = 0$. But then this means $x_1, .., x_{n-1} = 0$ but using the first equation again gives $x_n = 0$. So its kernal is only the zero vector so it is invertible.

Problem 11. Note that A^2 is still a integer matrix so $Tr(A^2) \in \mathbb{Z}$ but $Tr(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2$.

Problem 12. Note that $a_{ij} = \int_0^1 x^{i-1}x^{j-1}$ i.e. A is a gram matrix. Let $(f,g) := \int_0^1 f(x)g(x)$ then this is an inner product so if $\xi \in \mathbb{R}^n$ we have

$$\sum_{i,j} a_{ij}\xi_i\xi_j = \sum_{i,j}(x^{i-1}, x^{j-1})\xi_i\xi_j$$

$$= \sum_{i,j}(\xi_i x^{i-1}, \xi_j x^{j-1}) = (\sum_i \xi_i x^{i-1}, \sum_j \xi_j x^{j-1}) = (\sum_i \xi_i x^{i-1}, \sum_i \xi_i x^{i-1})$$

$$= ||\sum_i \xi_i x^{i-1}||^2 = \int_0^1 \sum_i \xi_i^2 x^{2i-2} > 0$$

with equality iff $\xi = 0$ so the quadratic form is positive. And we also have $a_{ij} = a_{ji}$ so A is symmetric so it is positive definite.

11. Spring 2015

Problem 1. We claim $f < 2$. Indeed, observe that

$$1 + \frac{4}{10} < 2$$

so inequality fails at $x = 2$. But as $f(0) = 0$ and f is continuous we see if there exists a point y such that $f(y) \geq 2$ then IVT implies there exists a point where $f(x^*) = 2$ which contradicts the inequality.

Problem 2. Define

$$[f]_\alpha := \sup_{\substack{x,y \in [0,1] \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

Then let $\mathcal{F} \subset C^\alpha([0,1])$ be a bounded sequence i.e. for all $f \in \mathcal{F}$ we have $||f||_{C^\alpha} := ||f||_{L^\infty} + [f]_{C^\alpha} \leq C$ where C does not depend on f. Then the family is totally bounded since $||f||_{L^\infty} \leq C$ and as $[f]_{C^\alpha} \leq C$ it is equicontinous with modulus of continuity $C|x-y|^\alpha$. So by Arzela-Ascoli there exists a uniformly convergent subsequence which we denote by $\{f_n\}$ to a limit function $g \in C^\alpha([0,1])$. We know $g \in C^\alpha([0,1])$ since $C^\alpha([0,1])$ is a closed subset of $C([0,1])$. Now we want to show that for any $\beta < \alpha$ we have

$$||f_n - g||_{C^\beta([0,1])} \to 0$$

As $f_n \to f$ in $C([0,1])$ it suffices to control $[f_n - g]_{C^\beta}$. Indeed observe that if we let $f := f_n - g$

$$\frac{|f(x)-f(y)|}{|x-y|^\beta} = \left(\frac{|f(x)-f(y)|^{\frac{\alpha}{\beta}}}{|x-y|^\alpha}\right)^{\frac{\beta}{\alpha}} = \left(|f(x)-f(y)|^{\frac{\alpha}{\beta}-1}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right)^{\frac{\beta}{\alpha}} = |f(x)-f(y)|^{1-\frac{\beta}{\alpha}}\left(\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right)^{\frac{\beta}{\alpha}}$$

$$\leq 2^{1-\frac{\beta}{\alpha}}||f||_{L^\infty}^{1-\frac{\beta}{\alpha}}[f]_{C^\alpha}$$
$$\leq C2^{2-\frac{\beta}{\alpha}}||f||_{L^\infty}^{1-\frac{\beta}{\alpha}}$$

Note that $1 - \frac{\beta}{\alpha} > 0$ since $\alpha > \beta$. This implies $\mathcal{F} \subset C^\beta$ since we can have $[f]_\beta \leq 2^{1-\frac{\beta}{\alpha}}||f||_{L^\infty}^{1-\frac{\beta}{\alpha}}[f]_{C^\alpha}$. This inequality also implies that $[f]_{C^\beta} = [f_n - g]_{C^\beta} \to 0$ as $n \to +\infty$. So they converge in C^β. The problem statement has $\alpha = \frac{1}{2} > \frac{1}{3} = \beta$.

Problem 3. Fix $0 < |h| < 1$ then notice for any $n \in \mathbb{N}$ that

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x+h) - f(x+\frac{1}{n}) + f(x+\frac{1}{n}) - f(x)}{h}$$
$$= \frac{f(x+h)-f(x+\frac{1}{n})}{h-\frac{1}{n}}\frac{h-\frac{1}{n}}{h} + \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}\frac{\frac{1}{n}}{h}$$
$$= (I) + (II)$$

Note to make (II) become zero in the limit we just need to choose a sequence n such that $\frac{1}{nh}$ is bounded. Indeed, assume that $0 < h < 1$ then there exists an N such that

$$\frac{1}{N+1} \leq h \leq \frac{1}{N}$$

so

$$N+1 \geq \frac{1}{h} \geq N$$

So we get

$$\frac{N+1}{N} \geq \frac{1}{Nh} \geq 1$$

so along this sub-sequence of N we get $(II) \to 0$. Also note that since f is Lipschitz that

$$\frac{f(x+h)-f(x+\frac{1}{n})}{h-\frac{1}{n}} \leq C$$

so it suffices to show that along this same subsequence of N that

$$\frac{h-\frac{1}{N}}{h} \to 0$$

Indeed, we have the estimate

$$\frac{-1}{N(N+1)} \leq h - \frac{1}{N} \leq 0$$

so

$$\frac{-1}{h(N^2+N)} \leq \frac{h-\frac{1}{N}}{h} \leq 0$$

so w e get

$$-\frac{(N+1)}{N^2+N}\frac{h-\frac{1}{N}}{h} \leq 0$$

Therefore, (I) and (II) both converge to 0 as $h \to 0$. Therefore, f is differentiable and $f' \equiv 0$ on $\mathbb{R}$ which is connected so we also have that f is constant.

Problem 4. We first need the following lemma: Let f be a function with the IVT property then if f is discontinuous at x_0 then there exists an $\varepsilon_0 > 0$ such that we have a sequence $\{x_n\} \to x_0$ and $f(x_n) = f(x_0) + \frac{\varepsilon_0}{2}$ or $f(x_n) = f(x_0) - \frac{\varepsilon_0}{2}$. For now assuming the lemma is true, we have our contradiction since there exists a sub-sequence with $f(x_{nk}) = f(x_0) + \frac{\varepsilon_0}{2}$ or $f(x_{n_k}) = f(x_0) - \frac{\varepsilon_0}{2}$ for all k. Say $f(x_{n_k}) = f(x_0) - \frac{\varepsilon_0}{2}$ for all k then $A := f^{-1}(f(x_0 - \frac{\varepsilon_0}{2}))$ is a closed set so as $x_{n_k} \to x$ we have $x_0 \in A$ but this implies $f(x_0) = f(x_0) - \frac{\varepsilon_0}{2}$ which is a contradiction. So it suffices to prove the lemma.

Let f have the IVT property and assume it is discontinuous at x_0 then there exists an $\varepsilon_0 > 0$ such that there exists an y with $|x_0 - y| < \frac{1}{n}$ and

$$|f(x_0) - f(y)| \geq \varepsilon_0$$

Assume that $f(y) > f(x_0)$ then we have $f(y) \geq \varepsilon_0 + f(x_0)$. In particular, this means $f(x_0) + \frac{\varepsilon_0}{2} \in [f(x_0), f(y)]$ so by the IVT property there exists an $x_n \in (x_0, y)$ such that $f(x_n) = f(x_0) + \frac{\varepsilon_0}{2}$. Note if $f(x_0) \geq f(y_0)$ an identical argument yields the existence of an x_n such that $f(x_n) = f(x_0) - \frac{\varepsilon_0}{2}$ and the lemma holds. Which concludes the problem.

Problem 5. We proved this in Spring 2013 Number 4 and used it to prove that problem.

Problem 6. Let the operator $T : C([0,1]) \to C([0,1])$ be defined via

$$T(f) := e^{t^2} + \frac{1}{2}\int_0^1 \cos(s)f(s)\mathrm{d}s$$

Then we have

$$||T(f) - T(g)||_{L^\infty} \leq \frac{1}{2}\int_0^1 ||f-g||_{L^\infty} = \frac{1}{2}||f-g||_{L^\infty}$$

where we used $|\cos(s)|$ is bounded by 1. Then Banach Fixed Point Theorem implies that there exists a unique fixed point of $T(f)$ i.e. $f(t) = e^{t^2} + \frac{1}{2}\int_0^1 \cos(s)f(s)\mathrm{d}s$ and $f \in C([0,1])$.

Problem 7. This quadratic form is associated to the following symmetric matrix

$$A := \begin{bmatrix} 9 & 6 & -5 \\ 6 & 6 & -1 \\ -5 & -1 & 6 \end{bmatrix}$$

which has a negative eigenvalue so there exists an (x, y, z) such that $[x, y, z]A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = f(x, y, z) < 0$

Problem 8a. This is false. Let

$$A := \mathrm{diag}(2, 2, 2)$$

then $\det(A) = 8$. If $A_n \to A$ then we have each entry of A_n converges to each entry of A. But observe that $f : \mathbb{R}^{3\times 3} \to \mathbb{R}$ defined via

$$f(A) = \det(A)$$

is a continuous map since it is a polynomial int he coefficients of A. So in particular if $A_n \to A$ then $f(A_n) \to f(A)$ but for all n we have $f(A_n) = 1$ and $f(A) = 8$ which is a contradiction.

Problem 8b. This is true. Indeed, by Schur Decomposition we have

$$A = U^T T U$$

where T is an upper triangular matrix. Fix $k \in \mathbb{N}$ and let $h_k := \{h_1, .., h_n\}$ such that $||h|| < \frac{1}{k}$ such that $T_{ii} + h_i \neq T_{jj} + h_j$ for all $j \neq i$. This is possible for any k since we have a finite number of eigenvalues then let us define

$$A_n := U^T (T + \text{diag}(h_1, .., h_n))U$$

then A_k has distinct eigenvalues and $A_n \to A$ since $d(A, A_k) = (\sum_{i=1}^n h_k^2)^{\frac{1}{2}} < \frac{1}{k}$ for any k.

Problem 9. Fix a basis of $U_1 \cap W_1$ i.e. $\{v_1, .., v_d\}$ extend it to a basis of $U_1 + W_1$ with the first $d - \ell$ elements being from U_1 and the next $d\ell$ from W_1 and extend it to a basis of $\mathbb{R}^n$ i.e. $\{v_1, .., v_d, u_1, ..., u_{d-\ell}, w_1, .., w_{d-\ell}, q_1, .., q_m\}$. Now do the same i.e. start with a d element basis of $U_2 \cap W_2$ extend it to a basis of $U_2 + W_2$ with the first $d - \ell$ elements from U_2 and the next from W_2 and the finish the rest to form a basis of $\mathbb{R}^n$. Denote this basis of $\{v_1^{(1)}, ..., v_d^{(1)}, u_1^{(1)}, .., u_{d-\ell}^{(1)}, w_1^{(1)}, .., w_{d-\ell}^{(1)}, q_1^{(1)}, .., q_m^{(1)}\}$. Define T via $T(u_i) = T(u_i^{(1)}), T(v_i) = T(v_i^{(1)}), T(w_i) = T(w_i^{(1)}), T(q_i) = T(q_i^{(1)})$. This is possible since $U1, V1, U2, V2$ all have the same dimension and their intersections do as well. Therefore, we have found an operator such that $T(U_1) = U_2$ and $T(W_1) = W_2$.

Problem 10. Note that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & S \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}$$

so we have $det(M)det(S) = det(A)$ as desired.

Problem 11. Let $\theta_n := \frac{2\pi n}{11}$ then consider

$$R_n := \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n) \end{bmatrix}$$

these are 11 commuting matrix with order 11. Note that we have $R_n^{(11)} = Id$ and no smaller number k such that $R_n^k = Id$ since

$$R_n^k := \begin{bmatrix} \cos(k\theta_n) & -\sin(k\theta_n) \\ \sin(k\theta_n) & \cos(k\theta_n) \end{bmatrix}$$

and $k\theta_n = \frac{2\pi nk}{11}$ and nk divides 11 iff $k = 11m$ for some $m \in \mathbb{N}$ since 11 is prime.

Problem 12a. We have

$$M = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{bmatrix}$$

so we have

$$\exp(M) = \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^4 & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 1/6 & 1/6 \\ 1/6 & -5/6 \end{bmatrix}$$

Problem 12b. We claim given any matrix A that $\exp(A)$ has positive eigenvalues. Indeed given any matrix $A \in \mathbb{C}^{n \times n}$ we can find a unitary $S \in \mathbb{C}^{n \times n}$ such that

$$A = S^*(T)S$$

where T is upper triangular by Schur Decomposition. Then

$$\exp(A) = S^*(\exp(T))S$$

Note that applying any polynomial to T still results in an upper triangular matrix. In particular $\exp(T)_{ii} = \exp(T_{ii})$ so all the eigenvlaues of $\exp(A)$ are of the form $\exp(T_{ii}) > 0$. And the claim is proved But M has an eigenvalue -2 which implies there is no map A such that $\exp(A) = M$.

12. Fall 2015

Problem 1. Fix $\varepsilon > 0$ and $N \in \mathbb{N}$ then for any $n \geq N$ we have $n = \alpha N + r$ where $\alpha \in \mathbb{N}$ and $r \in \{0, 1, .., N-1\}$ then

$$\frac{a_n}{n} = \frac{a_{\alpha N + r}}{\alpha N + r} \leq \frac{a_{\alpha N} + a_r}{\alpha N + r} \leq \frac{a_{\alpha N}}{\alpha N} + \max_{i=1,..,N-1} \frac{a_i}{n}$$

$$\leq \frac{\alpha a_N}{\alpha N} + \max_{i=1,..,N-1} \frac{a_i}{n}$$

$$= \frac{a_N}{N} + \varepsilon$$

for n large enough where we used $\max_{i=1,..,N-1} \frac{a_i}{n} \to 0$. So it follows that

$$\lim_{n\to\infty} \frac{a_n}{n} \leq \inf_{n\in\mathbb{N}} \frac{a_n}{n}$$

this implies

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n}$$

as desired □

Problem 2. Observe that as $h \geq 0$

$$\min_{\xi\in[a,b]} g(\xi) \int_a^b h(x)\mathrm{d}x \leq \int_a^b g(x)h(x) \leq \max_{\xi\in[a,b]} g(\xi) \int_a^b h(x)\mathrm{d}x$$

So for the continous function

$$F(y) := g(y) \int_a^b h(x)\mathrm{d}x$$

we can apply IVT to find a ζ such that

$$F(\zeta) = \int_a^b g(x)h(x)\mathrm{d}x = g(\zeta) \int_a^b h(x)\mathrm{d}x$$

as desired □

Problem 3. By Dini's Theorem since f_n is non-increasing, $[-1, 1]$ is compact, and $f_n \to 0$ continuous we have $f_n \to 0$ uniformly. Now we sum by parts i.e. for $B_n := \sum_{i=1}^n b_i$ we have

$$\sum_{i=m}^n a_i b_i = \sum_{i=m}^n (B_i - B_{i-1})a_i = \sum_{i=m}^n B_i a_i - B_{i-1}a_i = \sum_{i=m}^n B_i a_i - \sum_{j=m-1}^{n-1} B_j a_{j+1}$$

$$= B_n a_n - B_{m-1}a_m + \sum_{j=m}^{n-1} B_j(a_j - a_{j+1})$$

So we have for $n \geq m$ for $B_n := \sum_{j=1}^n (-1)^j$

$$|g_m(x) - g_n(x)| \leq |B_n f_n(x) - B_{m-1} f_m(x)| + |\sum_{j=m}^{n-1} B_j(f_j(x) - f_{j+1}(x)|$$

using that $|B_n| \leq C$ we have

$$\leq C(|f_n(x)| + |f_m(x)|) + \sum_{j=m}^{n-1} C|f_j(x) - f_{j+1}(x)|$$

using f_j is non-increasing we have

$$= C(|f_n(x)| + |f_m(x)|) + \sum_{j=m}^{n-1} C(f_j(x) - f_{j+1}(x))$$

$$= C(|f_n(x)| + |f_m(x)|) + \sum_{j=m}^{n-1} C(f_n(x) - f_m(x))$$

Then as $f_n(x)$ is a montone sequence this is equation to

$$= C(|f_n(x)| + |f_m(x)| + f_{m-1}(x) - f_{n-1}(x))$$

then using $\sup_x |f_j(x)| \to 0$ we have that the sequence $\{g_m(x)\}$ is Cauchy in $C([-1,1])$ we have by completeness of $C([-1,1])$ the existence of $g \in C([-1,1])$ such that $g_n(x) \to g(x)$ uniformly.

Problem 4. Let us define the operator from $T : C([0,\infty)) \to C([0,\infty))$ by

$$T(f) := e^{-2x} + \int_0^x f(t)e^{-2t}dt$$

then

$$d(T(f),T(g)) \leq ||f-g||_{L^\infty} \int_0^\infty e^{-2t}dt = \frac{1}{2}d(f,g)$$

Therefore, T is a contraction mapping so there exists a unique fixed point of T i.e. there is some $f \in C([0,\infty))$ such that $T(f) = f$. By Banach's Fixed Point if we start with any $f \in C([0,\infty))$ then define $f_{n+1} := T(f_n)$ then f_{n+1} converges to the unique fixed point i.e. f. To explicitly find f note that

$$f = e^{-2x} + \int_0^x f(t)e^{-2t}dt$$

and differentiate to find f, which converts this integral equation into a differential equation.

Problem 5. By the implicit function theorem we have

$$\begin{cases} \partial_y x(y,z) = -\partial_y F(\partial_x F)^{-1} \\ \partial_z y(x,z) = -\partial_z F(\partial_y F)^{-1} \\ \partial_x z(x,y) = -\partial_x F(\partial_x F)^{-1} \end{cases}$$

so multiplying them we get

$$\partial_y x \partial_z y \partial_x z = -1$$

Problem 7. By Sylvester Rank Theorem we have

$$rank(T) - ker(S) \leq rank(ST) \leq \min\{rank(S), rank(T)\}$$

Take $S = A^T$ and $T = B$ then

$$1 \leq rank(A^T B) \leq 3$$

Problem 8. Follows from direct computation.

Problem 9. Since we have

$$det(A - \lambda I) = det((A-\lambda I)^T) = det(A^T - \lambda I)$$

we conclude A and A^T have the same eigenvalues. Therefore, as

$$A^T = -A$$

we must have for every positive eigenvalue a negative eigenvalue. Therefore, the product of the eigenvalues must be non-negative i.e. $det(A) \geq 0$.

Problem 10a. Note that $\exp(A)$ is absolutely convergent for all x since

$$||\exp(A)|| \leq \sum_{n=0}^\infty \frac{||A||^n}{n!} \leq \exp(||A||)$$

then if $AB = BA$ we can apply binomial theorem to see

$$\exp(A+B) = \sum_{n=0}^\infty \frac{(A+B)^n}{n!} = \sum_{n=0}^\infty \sum_{k=0}^n \binom{n}{k} \frac{A^{n-k}}{(n-k)!} \frac{B^k}{k!}$$
$$= \exp(A)\exp(B)$$

since by Cauchy Product and $AB = BA$ we have

$$\left(\sum_{n=0}^\infty a_n A^n\right)\left(\sum_{k=0}^\infty b_k B^k\right) = \sum_{n=0}^\infty (\sum_{k=0}^n a_{n-k} A^{n-k} b_k x B^k)$$

Problem 10b. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$\exp(A) = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix} \quad \exp(B) = \begin{bmatrix} 0 & 0 \\ 0 & e \end{bmatrix}$$

and

$$\exp(A)\exp(B) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \exp(A+B) = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

Problem 11a. Note that for any $m \geq 0$

$$ker(A^{\mathrm{dim(V)}}) = ker(A^{\mathrm{dim(V)+m}})$$

so if there exists a square root we must have

$$S^{10} \neq 0$$

but $S^{12} = 0$ but as $dim(V) = 6$ we must have $ker(S^6) = ker(S^{10}) = ker(S^{12})$ but the last two do not agree so no square roots exist.

Problem 11b. Consider

$$S_{i,i+1} = 1 \text{ and } 0 \text{ else}$$

where $S \in \mathbb{R}^{12\times 12}$ then $S^{10} \neq 0$ but $S^{12} = 0$ so define $A := S^2$ then A has such a square root.

Problem 12. We know $M = \mathrm{diag}(1, 2, 3, 4, .., n) + A$ where A is a matrix of all ones let $\Lambda := \mathrm{diag}(1, 2, 3, 4, .., n)$ Then

$$\begin{aligned}(Mx, x) &= (\Lambda x, x) + (Ax, x) \\ &= \sum_{j=1}^{n} jx_j^2 + (\sum_{k=1}^{n} x_k)^2 \geq 0\end{aligned}$$

so M is positive definite.

13. Spring 2016

Problem 1. Let $x_n \to x$ then we either have

$$f(x,x) - f(x_n,x) \le f(x,x) - f(x,x_n) \le f(x,x) - f(x,x_n)$$

or

$$f(x,x) - f(x,x_n) \le f(x,x) - f(x,x_n) \le f(x,x) - f(x_n,x)$$

in either case we have as $n\infty$ that $|f(x,x) - f(x_n,x_n)| \to 0$ so $g(x) := f(x,x)$ is continuous.

Problem 2a. We say f is Riemann Integrable if for any $\varepsilon > 0$ there exists a partition P of $[a,b]$ with $P = \{a = x_0 < x_1 < ... < x_n = b\}$ and $I_i := [x_{i-1}, x_i]$ and $\omega(f, I_i) := \sup_{x,y \in I_i} |f(x) - f(y)|$ with $\delta x_i := x_i - x_{i-1}$

$$\sum_{i=1}^{n} \omega(f, I_i)\Delta x_i$$

Problem 2b. Fix $\varepsilon > 0$ and as $x_n \to x$ there exists an N such that if $n \ge N$ then $x_n \in B_{\frac{\varepsilon}{2}}(x) := I_1$. Then for $i = 1,..,N$ let $I_{i+1} := B_{\frac{\varepsilon}{2^{i+1}}}(x_i)$. If $I_i \cap I_j \neq \emptyset$ for $i \neq j$ then we can make the radius of each ball smaller to ensure they are disjoint so WLOG assume that $I_i \cap I_j = \emptyset$ whenever $j \neq i$. Then consider any partition that contains $I_1,..,I_{N+1}$ call the remaining intervals $I_{N+2},...,I_M$ then observe

$$\omega(f, I_i) = \begin{cases} 1 \text{ if } i = 1,..,N+1 \\ 0 \text{ else} \end{cases}$$

so

$$\sum_{i=1}^{M} \omega(f, I_i)\Delta x_i \le \sum_{i=1}^{N+1} \frac{\varepsilon}{2^{i+1}} \le \varepsilon$$

so f is Riemann Integrable.

Problem 3. Note that

$$\sum_{k=0}^{n-1} f(\frac{k}{n}) - n\int_0^1 f(x) = \sum_{k=0}^{n-1} f(\frac{x}{n}) - n\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)$$

and we have

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x) = (x+A)f(x)|_{\frac{k}{n}}^{\frac{k+1}{n}} - \int_{\frac{k}{n}}^{\frac{k+1}{n}} (x+A)f'(x)$$

Taking $A = -\frac{k+1}{n}$ gives along with the MVT of integrals that

$$= \frac{1}{n}f(\frac{k}{n}) - f'(\xi_k)\int_{\frac{k}{n}}^{\frac{k+1}{n}} (x - \frac{k+1}{n})$$

$$= \frac{1}{n}f(\frac{k}{n}) + \frac{f'(\xi_k)}{n^2}$$

where $\xi_k \in (\frac{k}{n}, \frac{k+1}{n})$. So

$$\sum_{k=0}^{n-1} f(\frac{k}{n}) - n\int_0^1 f(x) = -\sum_{k=0}^{n-1} \frac{f'(\xi_k)}{n}$$

This is a Riemann Sum so it converges to

$$-\int_0^1 f'(x) = f(0) - f(1)$$

So

$$\sum_{k=0}^{n} f(\frac{k}{n}) - n\int_0^1 f(x) \to f(0)$$

In particular, this implies

$$n(\sum_{k=0}^{n} f(\frac{k}{n}) - n\int_0^1 f(x))$$

diverges. I assume the correct question was to find the limit of

$$\sum_{k=0}^{n} f(\frac{k}{n}) - n\int_0^1 f(x)$$

Problem 4. As β is a continuous map from $[0,1] \to [0,1)$ we know from continuity that there exists an $x^* \in [0,1]$ such that $\beta(x^*)$ is the max. In particular, $\beta \leq \beta(x^*) < 1$. So define the map $T : C([0,1]) \to C([0,1])$ defined via

$$T(f) := \alpha(x) + \int_0^1 \beta(t) f_n(t)\text{dt}$$

is a contraction map on the complete metric space $C([0,1])$. Consider the iteration scheme

$$f_0(x) \equiv 0$$

$$f_{n+1}(x) = \alpha(x) + \int_0^1 \beta(t) f_n(t)\text{dt}$$

Then we have for any $n \geq m$ we have

$$T(f_n) - T(f_m) = \int_0^1 \beta(t)(f_n(t) - f_m(t))$$

So

$$||T(f_n) - T(f_m)||_{L^\infty(0,1)} \leq \gamma ||f_n(t) - f_m(t)||_{L^\infty(0,1)}$$

for $\gamma := \beta(x^*) < 1$. In particular, iterating this inequality gives

$$||T(f_n) - T(f_m)|| \leq \gamma^m ||f_0(t) - f_{n-m}(t)|| = \gamma^m ||f_{n-m}(t)||$$
$$\leq \gamma^m (||f_{n-m} - f_{n-m-1}|| + ||f_{n-m-1} - f_{n-m-2}|| + ...||f_1(t)||)$$
$$\leq \gamma^m (\gamma^{n-m}||f_1|| + \gamma^{n-m-1}||f_1|| + ... + ||f_1||)$$
$$= ||f_1|| \sum_{k=n}^{m} \gamma^n$$

so it is Cauchy and completeness implies the existence of a limit f. Then $f_n \to f$ uniformly so

$$\lim_{n\to\infty} T(f_n) = T(\lim_{n\to\infty} f_n) = T(f)$$

and

$$\lim_{n\to\infty} T(f_n) = \lim_{n\to\infty} f_{n+1} = f$$

i.e. $f = T(f)$. So the limit is a fixed point. To find it explicitly we differentiate the integral equation

$$T(f) = f$$

and solve the ODE.

Problem 5. As $\nabla g(x_0, y_0) \neq 0$ we can assume WLOG that $\partial_y g(x_0, y_0) \neq 0$. Then the Implicit Function Theorem implies there exists an open neighborhood $U \subset \mathbb{R}^1$ containing x_0 and a map $\varphi : U \to \varphi(U)$ satisfies

$$g(x, \varphi(x)) = g(x_0, y_0) = 0$$

for $x \in U$. Then we get

$$0 = \frac{d}{dx} g(x, \varphi(x)) = \partial_x g(x, \varphi(x)) + \varphi'(x)\partial_y g(x, \varphi(x))$$

i.e.

$$\varphi'(x) = -\frac{\partial_x g(x, \varphi(x))}{\partial_y g(x, \varphi(x))}$$

Then let us define

$$\psi(x) := f(x, \varphi(x))$$

then ψ has a minimum at $x = x_0$ so we have

$$0 = \frac{d}{dx}\psi(x)|_{x=x_0} = \partial_x f(x_0, \varphi(x_0)) + \partial_y f(x_0, \varphi(x_0))\varphi'(x_0)$$

Putting these together we get and that $\varphi(x_0) = y_0$ gives

$$\frac{\partial_x f(x_0, y_0)}{\partial_x g(x_0, y_0)} = \frac{\partial_y f(x_0, y_0)}{\partial_y g(x_0, y_0)}$$

so it follows that for $\lambda := \frac{\partial_y f(x_0,y_0)}{\partial_y g(x_0,y_0)} = \frac{\partial_x f(x_0,y_0)}{\partial_x g(x_0,y_0)}$ that $\partial_x f(x_0, y_0) = \lambda \partial_x g(x_0, y_0)$ and $\partial_y f(x_0, y_0) = \lambda \partial_y g(x_0, y_0)$ i.e. $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$

Problem 6. Let us consider an open ball $B_r(x)$. Let us assume that y is a limit point then there exists an $y_n \subset B_r(x) \to y$. Then there exists an N such that for $n \geq N$ we have

$$d(y, y_n) \leq \frac{r}{2}$$

for any $n \geq N$. So we have for any $n \geq N$

$$d(x, y) \leq \max\{d(x, y_n), d(y_n, y)\} \leq \max\{d(x, y_n), \frac{r}{2}\} < r$$

since $y_n \in B_r(x)$ so it follows $y \in B_r(x)$ i.e. the open ball is also closed.

Now consider a closed ball. Let $\varepsilon := \frac{r}{2}$. Then if $y \in \{y : \rho(x, y) \leq r\}$ then for any $z \in B_e(y)$ we have

$$d(z, x) \leq \max\{d(z, y), d(y, x)\} \leq \max\{\frac{r}{2}, d(x, y)\} \leq r$$

i.e. $z \in \{y : \rho(x, y) \leq r\}$ so $B_\varepsilon(t) \subset \{y : \rho(x, y) \leq r\}$. So it is also open since every point is an interior point.

Problem 7. We need the following lemma: **Lemma 1** If A is a real normal matrix, then A is unitarily equivalent to the following matrix

$$B := R_1 \bigoplus R_2 \bigoplus .... \bigoplus R_m$$

where

$$R_i = \begin{bmatrix} \lambda_i \end{bmatrix} \text{ or } \begin{bmatrix} |\lambda_i| \cos(\theta_i) & -|\lambda_i| \sin(\theta_i) \\ |\lambda_i| \sin(\theta_i) & |\lambda_i| \cos(\theta_i) \end{bmatrix}$$

Since A is normal it is complex diagonalizable and since A is real all the complex eigenvalues come in conjugate pairs. Fix a complex eigenvalue say λ then $\lambda = |\lambda| e^{i\theta}$ for some $\theta \in [0, 2\pi]$. Then the eigenvalues of

$$\begin{bmatrix} |\lambda| \cos(\theta) & -|\lambda| \sin(\theta) \\ |\lambda| \sin(\theta) & |\lambda| \cos(\theta) \end{bmatrix}$$

is exactly $|\lambda|(\cos(\theta) + i \sin(\theta)) = \lambda$ and $|\lambda|(\cos(\theta) - i \sin(\theta)) = \overline{\lambda}$. This matrix is normal so it is unitarily similar to

$$\text{diag}(\lambda, \overline{\lambda})$$

By composing bases we get the lemma.

Now since M is orthogonal it is normal and eigenvalues have magnitude 1 so the lemma implies there exists complex unitary matrix such that

$$M = U^T (R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m) U$$

Then this implies

$$UM = (R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m) U$$

Then $U = A + iB$ where A and B are real. So we get for any $r \in \mathbb{C}$

$$(A + rB)M = (R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m)(A + rB)$$

Let $p(r) := \det(A + rB)$ then $p(i) \neq 0$ since U is invertible. Therefore, p is not the zero-polynomial so there exists only finitely many roots so there exists an $r \in \mathbb{R}$ such that $p(r) \neq 0$. In particular,

$$M = (A + rB)^{-1} (R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m)(A + rB)$$

Let $V := (A + rB)$ which is real. So

$$M = V^{-1} (R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m) V$$

Assume that M has $2k$ complex roots. WLOG assume that $R_1, .., R_k$ be the rotation matrix in lemma 1 while $R_{k+1}, .., R_m$ be the diagonal matrix. Consider

$$L_i := V^{-1}(I_1 \bigoplus ... \bigoplus I_{i-1} \bigoplus R_i \bigoplus I_{i+1} \bigoplus ... \bigoplus I_m)V$$

where I_j is the identity matrix with the size of R_j and L_i is real. Then we clearly have

$$\prod_{i=1}^{m} L_i = V^T(R_1 \bigoplus R_2 \bigoplus ... \bigoplus R_m)V = M$$

so if there exists even a single complex eigenvalue we have $m \leq n-1$. So the claim is proved for unitary with even a single complex eigenvalue since L_i is the identity on a $n-2$ dimension subspace. When the unitary matrix only has real eigenvalues relabel the R_i such that $R_i = [1]$ for $1 \leq i \leq k$ and $R_{k+i} = [-1]$ for $1 \leq i \leq n-k$. If $k = n$ then $M = I$ and there is nothing to prove. We just replace R_{k+i} and R_{k+i+1} with

$$W_i := \text{diag}(-1, -1, 1, 1.., 1)$$

Then if k is even we are done, but if k is odd the the we keep the last entry as $W_\ell := [-1]$. Then we do

$$\prod_{i=1}^{k} L_i \prod_{i=1}^{\ell} W_i = M$$

and $k + \ell \leq n-1$ as long as $k > 1$. If $k = 1$ we just do

$$W_\ell := \text{diag}(1, -1, 1, 1.., 1)$$

to get the result.

Problem 8. Assume

$$(I + A)x = 0$$

then

$$\begin{cases} x_1 + a_{11}x_1 + a_{21}x_2 = 0 \\ x_2 + a_{12}x_1 + a_{22}x_2 = 0 \end{cases}$$

so we have

$$\begin{aligned} ||x||^2 &= (a_{11}x_1 + a_{21}x_2)^2 + (a_{12}x_1 + a_{22}x_2)^2 \\ &\leq ||x||^2 (\sum_{i,j=1}^{2} a_{ij}^2) \leq \frac{1}{10}||x||^2 \end{aligned}$$

by Cauchy Schwarz, so we must have $||x|| = 0$ i.e. $x = 0$ so $I + A$ is invertible.

Problem 9. Let $A \in \mathbb{R}^{3\times 3}$ be defined via

$$A := [v_1, v_2, v_3]$$

then $det(A) \neq 0$ iff v_1, v_2, v_3 are linearly independent over $\mathbb{R}$. and

$$det(A) = x(2 - x^2)$$

so they are linearly independent over $\mathbb{R}$ iff $x \neq 0$ or $\pm\sqrt{2}$.

Note that if they are linearly independent over $\mathbb{R}$ they are linearly independent over $\mathbb{Q}$. So it suffices to check at $x = 0$ and $x = \pm\sqrt{2}$. It is easy to show that at $x = 0$ they are not linearly independent over . It is easy to verify the Kernals at $x = \sqrt{2}$ are spanned by

$$\begin{bmatrix} -1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

and at $x = -\sqrt{2}$ is spanned by

$$\begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

so they are linearly independent over .

Problem 10a. Fix $A \in Mat(3, \mathbb{C})$. By Schur Decomposition there exists a unitary matrix U and an upper triangular matrix T such that

$$A = U^*TU$$

Fix a sequence of numbers $\{h_1^{(k)}, ..., h_n^{(k)}\}$ such that $T_{ii} + h_i^{(k)} \neq T_{jj} + h_j^{(k)}$ for $i \neq j$ and $\sum_{i=1}^n |h_1^{(k)}| \leq \frac{1}{k}$ then define

$$A_k := U^*(T + diag(h_1^{(k)}, ..., h_n^{(k)})U$$

Then A_k has distinct eigenvalues and $A_k \to A$ as $k \to \infty$

Problem 10b. Let $A := diag(1, 2, 3)$ then if $A_n \to A$ then we have $det(A_n - \lambda I) \to det(A - \lambda I)$ but if A_n has only one Jordan Block then we must have

$$det(A_n - \lambda I) = (\lambda - \lambda_n)^3$$

for some λ_n but this can never converge to

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = det(A - \lambda I)$$

Problem 11.

Problem 12. As A is self adjoint and $Av_k = (2k-1)v_k$ we have $v_i \perp v_j$ for any $i \neq j$. The result then follows from bilinearity of the inner product.

14. Fall 2016

Problem 1. We claim similar matrix share the same eigenvalues. Indeed, if

$$SAS^{-1} = B \Rightarrow SA(x) = BS(x)$$

for all x. Let $Ax = \lambda x$ then

$$\lambda Sx = BSx$$

and as S is invertible we have $Sx \neq 0$ so Sx is an eigenvector of B with eigenvalue λ. This shows all the eigenvalues of A are eigenvalues of B and a similar argument shows that they share the same eigenvalue. We then note that the Jordan Canonical Form implies that all the eigenvalues of B^3 are the eigenvalues of B cubed. But as B is similar to B^3 for each eigenvalue λ we must have $\lambda^3 = \lambda \Rightarrow \lambda(\lambda^2 - 1) = 0$ so either $\lambda = 0$ or $\lambda = \pm 1$. But as B is invertible 0 cannot be an eigenvalue so the only eigenvalues are ± 1 which are roots of unity.

Problem 2. Note that A is a Jordan Block so we have

$$A^n = \begin{bmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{bmatrix}$$

So

$$\exp(A) = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{2^n}{n!} & \sum_{n=0}^{\infty} \frac{n2^{n-1}}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{2^n}{n!} \end{bmatrix}$$
$$= \begin{bmatrix} e^2 & e^2 \\ 0 & e^2 \end{bmatrix}$$

Let $B := \exp(A)$ then

$$||B||^2 = \sup_{||x||=1} ||Bx||^2 = \sup_{||x||=1,||y||=1} (Bx, By)$$
$$= \sup_{||x||=1,||y||=1} (B^T Bx, y)$$

And $B^T B$ is symmetric so it is real diagonalizable. So we get if we let λ be the largest eigenvalue of $B^T B$ that

$$||B|| = \sqrt{\lambda}$$

So one just computes the eigenvalues of $B^T B$ to get $||B||$.

Problem 3.

Problem 4. We use the following lemma: A matrix A has $rank(A) \geq r$ iff there exists an $r \times r$ submatrix of A such that the submatrix is invertible. This can be easily seen to be equivalent to having r linearly independent columns so we omit the details. Now let $r = rank(A)$ then there exists an $r \times r$ submatrix that is full rank. Then as $A_n \to A$ we must have the same entries in the $r \times r$ submatrix converge to the $r \times r$ submatrix of A. Then as the det is a polynomial map of the coefficients we must have for sufficiently large n the det of the $r \times r$ submatrix of A_n is non-zero due to continuity. So this implies for sufficiently large n we have $rank(A_n) \geq r$ i.e.

$$rank(A) \leq \liminf_{n\to\infty} rank(A_n)$$

Problem 5.

Problem 6a. Let

$$v := (\sqrt{2}, \pi, ..., 1)$$

then $\lambda\sqrt{2} \in \mathbb{Q}$ iff $\lambda = q\sqrt{2}$ for $q \in \mathbb{Q}$ but $q\sqrt{2}\pi \notin \mathbb{Q}$.

Problem 6b. Notice that $A - 3I$ is a rational matrix. We consider the companion matrix

$$\begin{bmatrix} A - 3I & 0 \end{bmatrix}$$

where the 0 represents an $n \times 1$ matrix. As 3 is an eigenvalue of A we have that $A - 3I$ has a non-zero kernal. So to find the Kernal we just preform Gaussian Elimination till $A - 3I$ is of the form

$$\begin{bmatrix} I_{r\times r} & 0_{n-r\times n-r} & v_1 \\ 0 & 0 & v_2 \end{bmatrix}$$

where $I_{r\times r}$ is the $r\times r$ identity matrix for $r = rank(A-3I)$ and we have $r \le n-1$. Then we have that v_1 and v_2 are in the Kernal of $A-3I$ but preforming Gaussian Elimination on rational entries leaves the entries rational i.e. $\begin{bmatrix} v_1 & 0 \end{bmatrix} \in \mathbb{Q}^n$ and this is also in the Kernal.

Problem 7.

Problem 8a. This is a standard diagonal argument.

Problem 8b. Let

$$f_n(x) := \begin{cases} 0 \text{ if } x \le n \\ 1 \text{ if } x \ge n+1 \end{cases}$$

with a line connection the two between $x = n$ and $n+1$. Then we have $f_n(m) = 0$ for any $m > n$ so $\lim_{n\to\infty} f_n(m) = 0$ but $\lim_{m\to\infty} f_n(m) = 1$ for all n. So the double limits do not agree.

Problem 9. If $f_n \to f$ uniformly then for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $||f_n - f_m||_{L^\infty} < \frac{\varepsilon}{3}$ for $n, m \ge N$. Then for any $n, m \ge N$ we have

$$\begin{aligned} |f_n(x) - f_n(y)| &\le |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &\le \frac{2\varepsilon}{3} + |f_N(x) - f_N(y)| \end{aligned}$$

As f_N is continuous there exists a $\delta_N > 0$ such that if $d(x,y) < \delta_N \Rightarrow d(f_N(x), f_N(y)) < \frac{\varepsilon}{3}$ so

$$|f_n(x) - f_n(y)| < \varepsilon$$

for x, y such that $d(x,y) < \delta_N$. Then $f_1, .., f_{N-1}$ are uniformly continuous let δ_i be chosen such that $d(x,y) < \delta_i \Rightarrow d(f_i(x), f_i(y)) < \varepsilon$ then let $\delta := \min\{\delta_1, ..., \delta_N\}$ then if $d(x,y) < \delta$ we have

$$d(f_i(x), f_i(y)) < \varepsilon$$

i.e. the family is equicontinuous.

Now if $f_n \to f$ pointwise and $\{f_n\}$ is equicontiuous. Then fix $\varepsilon > 0$ then there exists a $\delta > 0$ such that if $|f_n(x) - f_n(y)| < \varepsilon$ and $|f(x) - f(y)| < \varepsilon$ whenever $d(x,y) < \delta$. Now consider the open cover of $[0,1]$

$$[0,1] \subset \bigcup_{x\in[0,1]} B_\delta(x)$$

so there exists a finite subcover $B_\delta(x_i)$ for $i = 1, .., N$. Then as $f_n(x_i) \to f(x_i)$ there exists an M_i such that if $n \ge M_i$ then $|f_n(x_i) - f(x_i)| < \varepsilon$. Let $M := \max\{M_1, .., M_N\}$ then $|f_n(x_i) - f(x_i)| \le \varepsilon$ for any i when $n \ge M$. Then for any x it must live in some δ ball say x_i then

$$|f(x) - f_n(x)| \le |f(x) - f(x_i)| + |f(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \le 3\varepsilon$$

so

$$\sup ||f - f_n|| < 3\varepsilon$$

for any $n \ge M$ i.e. uniform covergence.

Problem 10. We are asked to minimize for $g(x,y) := x^4 + y^4 - 2$

$$\min_{(x,y):g(x,y)=0}$$

Problem 11. Let

$$f_n(t) := \begin{cases} \sin(n) \text{ for } t \le \frac{1}{n} \\ \sin(\frac{1}{t}) \text{ for } t \ge \frac{1}{n} \end{cases}$$

then if there exists a sub-sequence that converges to a limit f then we must have $f(t) = \sin(\frac{1}{t})$ for $t \in (0,1]$. But this limit cannot be continuous since $\sin(1/t)$ cannot be extended to be continuous on $[0,1]$. So it is not compact, hence not complete.

Problem 12. As f is convex we have for any $x < z_t < y$ where $z_t := (1-\lambda)x + \lambda y$ for $\lambda \in [0,1]$ that

$$\frac{f(x) - f(z_t)}{x - z_t} \le \frac{f(x) - f(y)}{x - y} \le \frac{f(y) - f(z_t)}{y - z_t}$$

In particular sending $z_t \to y$ and $z_t \to x$ that

$$f'(x) \le \frac{f(y) - f(x)}{y - x} \le f'(y)$$

so we have

$$(y-x)f'(x)+f(x) \le f(y) \le (y-x)f'(y)+f(x)$$

i.e.

$$f(y) \ge f(x)+f'(x)(y-x)$$

whenever $y > x$ and if $x > y$ we have the inequality

$$\frac{f(x)-f(y)}{x-y} \le f'(x)$$

i.e.

$$f(x) \le (x-y)f'(x)+f(y) \Rightarrow f(x)+(y-x)f'(x) \le f(y)$$

whenever $x > y$. The inequality is trivial for $x = y$ so we have arrived at the conclusion.

For the reverse fix x, y and let $z := \lambda x + (1-\lambda)y$ then

$$f(y) \ge f(z) + \lambda f'(z)(x-y)$$

and

$$f(x) \ge f(z) + (1-\lambda)f'(z)(y-x)$$

so

$$(1-\lambda)f(y) + \lambda f(y) \ge f(z)$$

as desired.

15. Spring 2017

Problem 1. It is clear that $range(MM^T) \subset range(M)$ so it suffices to show $rank(MM^T) = rank(M)$. Now fix $x \in Ker(MM^T)$ thenf or any $y \in \mathbb{R}^n$ we have

$$(MM^T x, y) = 0 \Rightarrow (M^T x, M^T y) = 0$$

taking $y = x$ we get $M^T x = 0$. In particular, we have shown that $Ker(MM^T) \subset Ker(M^T)$. But we trivially have $Ker(M^T) \subset Ker(MM^T)$ so $Ker(M^T) = Ker(MM^T)$ i.e. $nullity(M^T) = nullity(MM^T)$. But using $nullity(M^T) = nullity(M)$ we get

$$rank(MM^T) = rank(M)$$

so we have $range(MM^T) = range(M)$ as desired.

Problem 2.

Problem 3a. As $M = M^T$ and $MM^T = I$ we get $M^2 = I$. In particular, as M is normal it is complex diagnolizable so it has a basis of eigenvectors. Let λ be an eigenvalue associated with the eigenvector x then we have $x = M^2 x = \lambda^2 x$ so $\lambda = \pm 1$. As M is positive def all the eigenvalues are positive i.e. $\lambda = 1$. Therefore, by the spectral theorem we have the existence of complex unitary matrix U such that

$$M = U^T I U = I$$

so $M = I$.

Problem 3b. No.

Problem 4.

Problem 5. Fix a polynomial $F(X)$ then $F(x) = \alpha \prod_{i=1}^{n}(x - x_i)$ for some x_i and α. Then we claim $F(T)x = 0$ for $x \neq 0$ iff F has an eigenvalue as a root. The direction eigenvalue as a root imply $F(T)x = 0$ for $x \neq 0$ is trivial by taking an eigenvector as x. For the other direction let $k \geq 0$ with $\prod_{i=1}^{0}(T - x_i I) := x$ be the largest integer such that $\prod_{i=1}^{k}(T - x_i I)x \neq 0$ then we have $\prod_{i=1}^{k}(T - x_i I)x$ is an eigenvector of $T - x_{k+1}I$. So this implies $F(T)$ is invertible iff F does not have any eigenvalues as roots i.e. iff the minimal polynomial and F do not share ant roots.

Problem 6b. Just do Grahm-Schmidt on

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 7. Let the operator $T : C([0,1]) \to C([0,1])$ be defined via

$$T(f) := 1 - \left(\int_0^x tf\right)^2$$

Indeed this maps to $C([0,1])$ since

$$0 \leq \left(\int_0^x tf\right)^2 \leq \left(\int_0^1 tf\right)^2 \leq \int_0^1 t^2 f^2 \leq 1$$

where the second last inequality is due to Jensen's Inequality. So in particular,

$$0 \leq T(f) \leq 1$$

and the continuity is clear. Then observe

$$T(f) - T(g) = \left(\int_0^x tg\right)^2 - \left(\int_0^x tf\right)^2 = (\int_0^x tg - tf)(\int_0^x tf + tg)$$

$$\int_0^x tg \leq \int_0^1 |tg| \leq \frac{1}{2}$$

where we used $||g||_{L^\infty} \leq 1$. In particular this implies

$$|T(f) - T(g)| \leq \int_0^x |tf - tg| \mathrm{d}t \leq \int_0^x ||f - g||_{L^\infty} t = \frac{1}{2}||f - g||_{L^\infty}$$

i.e. T is a contraction map and an operator on a complete metric space. So it follows from Banach Fixed Point Theorem that this admits a unique fixed point.

Problem 8. Note that by integrating by parts we have

$$\int_0^1 f(x)\mathrm{d}x = f(x)(x-\frac{1}{2})|_{x=0}^1 - \int_0^1 (x-\frac{1}{2})f'(x)$$
$$= \frac{f(1)+f(0)}{2} - \int_0^1 (x-\frac{1}{2})f'$$
$$= \frac{f(1)+f(0)}{2} + f'(x)(\frac{x^2}{2}-\frac{x}{2})|_{x=0}^1 + \int_0^1 (\frac{x^2}{2}-\frac{x}{2})f''(x)$$
$$= \frac{f(1)+f(0)}{2} + \int_0^1 (\frac{x^2}{2}-\frac{x}{2})f''(x)$$

Therefore,

$$\left|\frac{f(1)+f(0)}{2} - \int_0^1 f(x)\mathrm{d}x\right| = \left|\int_0^1 (\frac{x^2}{2}-\frac{x}{2})f''(x)\right| \leq \frac{1}{8}\int_0^1 |f''(x)|$$

where we got $1/8$ since $\max_{x\in[0,1]}|\frac{x^2}{2}-\frac{x}{2}| = \frac{1}{8}$

Problem 9a. Let $d(x,y) := \text{dist}(x,y)$ then notice that by the reverse triangle inequality it suffices to show for any $\varepsilon > 0$ and any $x \in X$ that there exists a $z \in Z_\varepsilon$ where Z_ε is coutable such that

$$d(x,z) < \varepsilon$$

As $C(X)$ is separable there exists a countable set of functions $\{f_n\}$ such that for any $g \in C(X)$ we have an n such that

$$||g(x) - f_n(x)||_{L^\infty} < \frac{\varepsilon}{2}$$

Fix an $x \in X$ ad let $g(z) := d(x,z)$ then $g(z) \in C(X)$ so there exists an n such that

$$||g(x) - f_n(x)|| = ||f_n(x)|| \leq ||g(z) - f_n(z)||_{L^\infty} < \frac{\varepsilon}{2}$$

Let $\{g_n\} \subset \{f_n\}$ be chosen such that for each n there exists an x_n such that

$$||d(x_n,z) - g_n(z)||_{L^\infty} < \frac{\varepsilon}{2}$$

then we have $g_n(x_n) < \frac{\varepsilon}{2}$. Then fix an arbitrary $x \in X$ then this implies there exists an n such that

$$||d(x,z) - g_n(z)|| < \frac{\varepsilon}{2}$$

In particular this implies

$$d(x,x_n) \leq ||d(x,x_n) - g_n(x_n)|| + ||g_n(x_n)|| < \varepsilon$$

If we let $Z_\epsilon := \{x_n\}$ then the claim is shown.

Problem 9b. In a we have shown for any $\varepsilon > 0$ there exists a countable set Z_ε such that for any $x \in X$ there exists a $z \in Z_\varepsilon$ such that

$$d(x,z) < \varepsilon$$

Let $\varepsilon_n := \frac{1}{n}$ and consider $Z := \bigcup_{n=1}^\infty Z_{\varepsilon_n}$ which is countable since it is a countable union of countable sets and for any $\varepsilon > 0$ and $x \in X$ there exists an $z \in Z$ such that

$$d(x,z) < \varepsilon$$

so X is separable.

Problem 10a. As K is compact it is closed. In particular, if K can be written as the union of two separated sets A and B then A and B are both closed. But as K is bounded so are A and B. In particular, A and B are compact and

$$K = A \cup B \text{ such that} A \cap B = \emptyset$$

Then we must have the existence of an $\varepsilon_0 > 0$ such that $d(A,B) > \varepsilon_0$. Let $a \in A$ and $b \in B$ then by assumption there exists a sequence $x_0, x_1, .., x_n$ such that $x_0 = a$ and $x_n = b$ with $||x_k - x_{k-1}|| < \frac{\varepsilon_0}{2}$. As $x_0 = a \in A$ and $x_n = b \in B$ this implies there exists an integer such that $x_k \in A$ and $x_{k+1} \in B$ and we have $||x_{k+1} - x_k|| < \frac{\varepsilon_0}{2}$ but this contradicts $d(A,B) > \varepsilon_0$. Therefore, K is connected.

Problem 10b. Take the topologist sine curve.

Problem 11. We claim that the family is uniformly bounded on $[0,1]$ and that it is equicontinuous on $[\frac{1}{k},1]$ for any $k\in\mathbb{N}$. Indeed, note that

$$||F_n||_{L^\infty}\le\int_0^1 1+\frac{n}{1+n^2x^2}\mathrm{d}x=1+\int_0^n\frac{1}{1+t^2}\mathrm{d}t=1+\arctan(n)\le 1+\frac{\pi}{2}$$

so the family is uniformly bounded. And we have by the fundamental theorem of calculus as f_n are continuous that

$$|F_n'(x)|=|f_n(x)|\le 1+\frac{n}{1+n^2x^2}$$

and on $[\frac{1}{k},1]$ we have

$$||F_n'(x)||_{L^\infty(\frac{1}{k},1)}\le 1+\frac{n}{1+\frac{n^2}{k^2}}$$

Noting that

$$\lim_{n\to\infty}\frac{n}{1+\frac{n^2}{k^2}}=0$$

we get that there exists a constant that depends only on k such that $C=C(k)$

$$||F_n'(x)||_{L^\infty(1/k,1)}\le 1+C(k)$$

So it is equicontinuous with lipschitz constant $1+C(k)$. So by Arzela-Ascoli there exist a uniformly convergent subsbequence on $[\frac{1}{2},1]$ denoted by $n_k^{(2)}$ to a function $f_1(x)$. We can similarly find a uniformly convergent subsequence on $[\frac{1}{3},1]$ where $n_k^{(3)}\subset n_k^{(2)}$ to a function $f_2(x)$. Note that by uniqueness of limit we have $f_1(x)=f_2(x)$ for $x\in[\frac{1}{2},1]$, so we may as well call this limit f.Do this for all k. Then we define $n_k:=n_k^{(k)}$ i.e. the diagonal subsequence. Then note $F_n(0)=0$ for all n. Then define $f(0):=0$ then $F_{n_k}(x)\to f(x)$ for all $x\in[0,1]$.

Problem 12. Let

$$F(y;t):=y^4+ty^2+t^2y$$

then note that as $|y|\to+\infty$ that we have $F(y;t)\to\infty$. Therefore, there exists a compact set $K(t)$ such that for $x\notin K(t)$ we have $F(x)>1$ but $F(0;t)=0$. Therefore, as $F(y;t)$ is continuous for any fixed t there exists a min of $F(y;t)$ over $K(t)$. This minimum is the global minimum since $F(y;t)>F(0,0)$ for $x\notin K(t)$. So a global minimum exists on a compact subset. We can take a slightly larger compact subset to ensure that the global minimum is an interior point then we must have $\partial_y F(y,t)=0$ i.e.

$$4y^3+2yt+t^2=0$$

so there is at most 3 candidates for the global min which we denote by $\{y_1(t),y_2(t),y_3(t)\}$. But notice that

$$\partial^2_{yy}F(y;t)=12y^2+2t$$

so if $t>0$ then F is uniformly convex so the minimum is unique. So assume $t\le 0$. Then we have on $A_1(t):=(-\infty,-\frac{\sqrt{-t}}{6}]$, $A_2(t):=(-\frac{\sqrt{-t}}{6},\frac{\sqrt{-t}}{6})$, and $A_3(t):=[\frac{\sqrt{-t}}{6},\infty)$ then the global mi cannot occur in $A_3(t)$ since if $y\ge 0$ then $F(-y;t)<F(y,t)$. And if the min occurs in $A_2(t)$ then $F(y;t)|_{A_2(t)}$ is concave so y_i must be a max since critical points of concave functions are global maxs. Therefore, the global min must occur in $A_3(t)$. But on $A_3(t)$ we have$\partial^2_{yy}F(y;t)<0$ for $x-\sqrt{-t}6$ so there is at most one zero. Therefore, there is a unique global minimum $y_i(t)$.

16. Fall 2017

Problem 1b. Do grahm-schmidt on $\{1, x, x^2, x^3\}$.

Problem 2a. Indeed we have

$$det(AB - \lambda I) = det(AA^{-1}(AB - \lambda I)) = det(A^{-1}(AB - \lambda I)A)$$
$$= det(BA - \lambda I)$$

so AB and BA do have the same characteristic polynomial when A is invertible.

Problem 2b. Note that the set of complex invertible matrix are dense and that $det(A) : \mathbb{C}^{n\times n} \to \mathbb{C}$ is a continuous map since it is a polynomial of the entries of A. Indeed, given a matrix A we can consider it as an operator on $\mathbb{C}$ then Schur's Theorem tells us that there exists unitary matrix U and an upper triangular matrix T such that

$$A = U^*TU$$

then consider

$$A_k := U^*(T + \frac{1}{n}I)U$$

then $A_k \to A$ in entry wise. And there exists an $N \in \mathbb{N}$ such that if $k \geq N$ then $(A_k)_{ii} \neq 0$ for any $1 \leq i \leq n$ i.e. 0 is not an eigenvalue of A_k. Therefore, given an $A \in \mathbb{C}^{n\times n}$ then we have a sequence $A_k \to A$ such that A_k is invertible then

$$det(A_kB - \lambda I) = det(BA_k - \lambda I)$$

for all k. Then taking limits along with the continuity of det gives

$$det(AB - \lambda I) = det(BA - \lambda I)$$

Problem 3. The matrix is diagonlizable so we get

$$\begin{cases} x_1 = \alpha e^{5t} + \beta e^{4t} \\ x_2 = \alpha e^{5t} + 2\beta e^{4t} \end{cases}$$

for $\alpha, \beta \in \mathbb{R}$.

Problem 4a. Fix an arbitrary finite collection of $\{e_i\}_{i\in\mathcal{F}}$ then if

$$L := \sum_{i\in\mathcal{F}} \alpha_i e_i^{\#} = 0$$

then for any $i \in \mathcal{F}$ we get

$$L(e_i) = 0 \Rightarrow \alpha_i = 0$$

since $e_j^{\#}(e_i) = \delta_{ij}$

Problem 4b. We claim this is a basis iff V is finite dimensional. Indeed if V is finite dimensional, then given an $T \in V^*$ i.e. $T : V \to \mathbb{R}$ let $\{v_1, .., v_n\}$ be a basis of V then for any $x \in V$ there exists unique constant $\alpha_1, .., \alpha_n$ such that

$$T(x) = T(\sum_{i=1}^{n} \alpha_i v_i) = \sum_{i=1}^{n} \alpha_i T(v_i) = \sum_{i=1}^{n} T(v_i) e_i^{\#}(x)$$

since $e_i^{\#}(e_j) = \alpha_i$ so $\{e_i^{\#}\}$ spans V^* and is also linearly independent, so it is a basis of V when V is finite dimensional.

Now assume for the sake of contradiction that $\{e_i^{\#}\}$ is a basis and V is infinite dimensional. Let $\{e_i\}_{i\in I}$ be a basis of V where I is an infinite counting set. Then consider the operator $T : V \to \mathbb{R}$ defined via

$$Id(x) := x$$

Then fix any finite collection of $\{e_i\}_{i\in\mathcal{F}}$ then fix $j \notin \mathcal{F}$ then

$$L(x) := \sum_{i\in\mathcal{F}} \alpha_i e_i$$

has

$$L(e_j) = 0$$

since $j \notin \mathcal{F}$. Therefore, we cannot represent the Id operator with finite linear combinations of $\{e_i^{\#}\}$ so it is not a basis.

Problem 5a. Yes. We have $0 \in X$ then if $f, g \in X$ then $af + g \in X$ since there exists $i_1, .., i_N$ and $j_1, ..j_M$ such that $\{af(e_{i_1}), ..., af(e_{i_N})\}$ and $\{g(e_{j_1}), ..., g(e_{j_M})\}$ span the image of f and g so the image of $af + g$ is a subset of the span of $span\{af(e_{i_1}), ..., af(e_{i_N})\} + span\{g(e_{j_1}), ..., g(e_{j_M})\}$ which has at most dimension $N + M$ so $af + g \in X$

Problem 5b. No for $I \in Y$ and $-I \in Y$ but $I - I = 0 \notin Y$

Problem 5c. We claim $X \cap Y = \emptyset$. Fix a basis $\{e_i\}_{i \in I}$ of V. Then if $f \in X$ then there exists $e_{i_1}, .., e_{i_N}$ such that $\{f(e_{i_1}), .., f(e_{i_N})\}$ is a basis of $im(f)$. In particular this implies for $k \notin \{i_1, .., i_N\}$ that there exists constant $\alpha_1, .., \alpha_N, \alpha_k$ such that

$$\sum_{i=1}^{N} \alpha_j f(e_{i_j}) + \alpha_k f(e_{i_k}) = 0$$

In particular, this implies $\{\alpha_1 e_{i_1}, ..., \alpha_N e_{i_N}, \alpha_k e_{i_k}\} \in ker(f)$ for all $k \notin \{i_1, .., i_N\}$ and there are infinitely many such k and the set $\{\alpha_1 e_{i_1}, ..., \alpha_N e_{i_N}, \alpha_k e_{i_k}\}$ and $\{\tilde{\alpha_1} e_{i_1}, ..., \tilde{\alpha_N} e_{i_N}, \alpha_m e_{i_m}\}$ are linearly independent whenever $i_m \neq i_k$ and both are not in $i_1, .., i_N$. So the kernel of f is infinite dimensional which implies $X \cap Y = \emptyset$.

Problem 6a. This is true by Spectral Theorem.

Problem 6b. This is false. We need it to be Hermitian or Normal to be able to apply the Spectral Theorem. Take

$$A = \begin{bmatrix} 1+i & 1 \\ 1 & 1-i \end{bmatrix}$$

then characteristic polynomial is $(x-1)^2$ but $A - I \neq 0$ so it only has one eigenvector. Therefore, it is not diagnolizable.

Problem 6c. Consider the matrix

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

then over $\mathbb{R}$ its characteristic polynomial is $x^2 - 3x - 2 = x^2 - 2$ over $\mathbb{Z}/3\mathbb{Z}$ which does not admit any roots over $\mathbb{Z}/3\mathbb{Z}$ so it has no eigenvalues so it is not diagnolizable.

Problem 7. As a_n is decreasing we have the following inequality

$$\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} 2^n a_{2^n} \leq 2 \sum_{n=1}^{\infty} a_n$$

In particular, $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges iff $\sum_{n=1}^{\infty} a_n$ converges. Therefore, we must have $2^n a_{2^n} \to 0$. Now fix an $k \in \mathbb{N}$ then there exists an n such that $k \in [2^n, 2^{n+1}]$ then we have from the decreasing condition that

$$k a_k \leq k a_{2^n} \leq 2^{n+1} a_{2^n} = 2(2^n a_{2^n}) \to 0$$

so we have $k a_k \to 0$ since $2^n a_{2n} \to 0$.

Problem 8a. Assume L is discontinuous at x_0. Then there exists an $\varepsilon_0 > 0$ such that there exists a sequence $x_n \to x_0$ and

$$|L(x_n) - L(x_0)| > \varepsilon_0$$

In particular as $L(x_n) = \lim_{x \to x_n} f(x)$ this implies that there exists an N such that if $d(x_0, y) < \frac{1}{N}$ then $|L(x_0) - L(y)| < \frac{\varepsilon_0}{2}$. As $y_n \to x_0$ we can find an N_1 such that $d(y_n, x_0) < \frac{1}{2N}$. Then as $L(y_n) = \lim_{z \to y_n} f(z)$ this means we can find an N_2 such that if $d(z, y_n) < \frac{1}{N_2}$ then $d(f(z), L(y_n)) < \frac{\varepsilon_0}{2}$. Choose z such that $d(z, y_n) < \min\{\frac{1}{2N_2}, \frac{1}{2N}\}$ then $d(z, y_n) \leq \frac{1}{N_2}$ and $d(z, x_0) \leq d(z, y_n) + d(y_n, x_0) < \frac{1}{N}$ so we have

$$|L(x_n) - L(x_0)| \leq |L(x_n) - f(z)| + |f(z) - L(x_0)| < \frac{\varepsilon_0}{2} + \frac{\varepsilon_0}{2} = \varepsilon_0$$

which is a contradiction.

Problem 8b. Let $A := \{x \in [a,b] : f(x) \neq L(x)\} = \bigcup_{n=1}^{\infty}\{x \in [a,b] : |f(x) - L(x)| \geq \frac{1}{n}\} := \bigcup_{n=1}^{\infty} A_n$ We claim that each A_n is countable. Indeed, if not there exists an n such that A_n is uncountable thus there exists a sequence $\{x_k\} \subset A_n$ with infinitely many distinct terms. Thus as $[a,b]$ is compact there exists a subsequence such that x_k converges to a limit x. We still denote this subsequence as x_k. Then we have

$$\begin{cases} L(x_k) \to L(x) \\ f(x_k) \to L(x) \end{cases}$$

since $x_k \to x$ but then this implies by uniform continuity of L that there exists a $\delta > 0$ if $d(x_k, x) < \delta$ then

$$\begin{cases} |L(x_k) - L(x)| < \frac{1}{3n} \\ |f(x_k) - L(x)| < \frac{1}{3n} \end{cases}$$

choose large enough k such that $d(x_k, x) < \delta$ then

$$|f(x_k) - L(x_k)| \leq |f(x_k) - L(x)| + |L(x) - L(x_k)| \leq \frac{2}{3n}$$

but

$$|f(x_k) - L(x_k)| > \frac{1}{n}$$

since $x_k \in A_n$ which is a contradiction. Therefore, there exists only countably many terms in each A_n. Therefore, as A is a countable union of countable sets it is countable.

Problem 8c. Note that b implies that $f(x)$ is continuous except for a countable set. Now we claim that if f is continuous except for on a countable set that it is Riemann Integrable. Indeed, fix $\varepsilon > 0$ and let $\omega(f, I_i)$ denote the oscillation of f on the interval I_i. Enumerate the subset of discontinuity where $\omega(f, q_n) \geq \alpha := D$ where $\omega(f, q_n) := \lim_{r\to 0} \omega(f, B_r(q_n))$ and consider

$$I_n := B_{\frac{\varepsilon}{2^n}}(q_n)$$

and for each $x \notin D$ Then we have for any $x \notin D$ the existence of an $\varepsilon_x > 0$ such that $\omega(f, B_{e_x}(x)) < \alpha$

$$[a,b] \subset \bigcup_{n=1}^{\infty} I_n \cup \bigcup_{x \in [a,b]-D} B_{\varepsilon_x}(x)$$

So compactness ensures there exists a finite subcover say

$$[a,b] \subset \bigcup_{j=1}^{N} I_{i_j} \cup \bigcup_{i=1}^{M} B_{\varepsilon_i}(x_i)$$

Let $B_{\varepsilon_i}(x_i) := J_i$. Let P be any partition that contains the points

$$\bigcup_{j=1}^{N}\{q_{i_j} - \frac{\varepsilon}{2^{i_j}}, q_{i_j} + \frac{\varepsilon}{2^{i_j}}\} \cup \bigcup_{i=1}^{M}\{x_i - \varepsilon_i, x_i + \varepsilon_i\}$$

Then if $P = \{a = t_0 < t_1 < ... < t_K = b\}$ and let $\Delta t_i := t_i - t_{i-1}$ and $T_i := [t_{i-1}, t_i]$ then note that for any i we have either

$$T_i \subset \{q_{i_j} - \frac{\varepsilon}{2^{i_j}}, q_{i_j} + \frac{\varepsilon}{2^{i_j}}\} \text{ or } \{x_i - \varepsilon_i, x_i + \varepsilon_i\}$$

for some i_j or i. Then

$$\sum_{j=1}^{K} \Delta t_i \omega(f, T_i) \leq \sum_{j=1}^{n} \frac{\varepsilon}{2^{i_j}} \omega(f, I_i) + \sum_{i=1}^{M} 2\varepsilon_i \omega(f, J_i)$$

Let $M := ||f||_{L^\infty} < \infty$ then we have

$$\leq 2\varepsilon M + \sum_{i=1}^{M} 2\varepsilon_i \alpha$$

$$\leq 2\varepsilon M + 2(b-a)\alpha$$

choose $\alpha = \varepsilon$ then we have

$$\leq \varepsilon(2M + (b-a))$$

i.e. the lower and upper sums are within $C\varepsilon$ for a constant $C > 0$ so f is Riemann Integrable.

Problem 9. Fix $x \in X$ and $n, m \in \mathbb{N}$ then assume $m > n$ so there is a k such that $n + k = m$ then

$$\rho(f^n(x), f^{n+k}(x)) \leq \sum_{i=0}^{k-1} \rho(f^{n+i}(x), f^{n+1+i}(x)) \leq \rho(x, f(x))[c_n + c_{n+1} + ... + c_{n+k}]$$

and as

$$\sum_{n=1}^{\infty} c_n < \infty$$

and $c_n \geq 0$ we get that $\{f^n(x)\} := \{x_n\}$ is a Cauchy Sequence, so completeness implies there exists a limit x^*. But by continuity (since f is Lipschitz with constant c_1) we have

$$\lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f(x_n) = f(x)$$

and

$$\lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f^n(x) = x$$

so $f(x) = x$. Uniqueness follows from if there exists two fixed points then $c_n \geq 1$ for all n so we do not have $\sum c_n < \infty$.

Problem 10. As $[a, b]$ is compact and $f \in C([a, b])$ stone Weiestrass implies that there exists a sequence of polynomial $p_n \to f$ uniformly but for any p_n we have

$$\int_a^b f(x)p_n(x) = 0$$

so by uniform convergence we have

$$0 = \lim_{n\to\infty} \int_a^b f(x)p_n(x) = \int_a^b \lim_{n\to\infty} f(x)p_n(x) = \int_a^b f^2(x)$$

i.e. $f = 0$ everywhere due to continuity.

Problem 11. Note that $x \mapsto \log(x)$ is concave on $(0, \infty)$. We can assume $a, b \neq 0$ for otherwise the inequality is trivial. So we have

$$\log(\frac{a^p}{p} + \frac{b^q}{q}) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q)$$

$$\log(\frac{a^p}{p} + \frac{b^q}{q}) \geq \log(a) + \log(b)$$

So exponentiation of both sides give

$$\frac{a^p}{p} + \frac{b^q}{q} \geq ab$$

as desired

Problem 12. See Fall 2014 number 10 and 11.

17. Spring 2018

Problem 1. Let $V := C^\infty([0,2])$ and let $L : V \to V$ be defined via for $f \in C^\infty([0,2])$

$$L(f) = f'$$

Then

$$L(e^{kt}) = ke^{kt}$$

i.e. e^{kt} is an eigenvector with eigenvalue k. Then we claim $\{e^{kt}\}_{k=1}^n$ is linearly independent. Indeed this is trivially true for $k = 1$ so assume it holds when $k = n - 1$. Then if

$$\sum_{k=1}^{n} \alpha_k e^{kt} = 0$$

for all $t \in [0,2]$ then applying L gives

$$\sum_{k=1}^{n} k\alpha_k e^{kt} = 0$$

and multiplying the first quantity by n gives

$$\sum_{k=1}^{n} n\alpha_k e^{kt} = 0$$

Subtracting these quantities give

$$\sum_{k=1}^{n-1} (n-k)\alpha_k e^{kt}$$

which by induction implies $\alpha_k(n-k) = 0$ for all $1 \leq k \leq n-1$ but as $n \neq k$ we get $\alpha_1, .., a_{n-1} = 0$ which then implies $\alpha_k = 0$.

Problem 2. As $A^2 = A$ we have for any $x \in \mathbb{R}^5$ that

$$x = Ax + (x - Ax)$$

and $x - Ax \in ker(A)$ since $A^2 = A$ i.e.

$$\mathbb{R}^5 = range(A) \bigoplus kernal(A)$$

Now we claim that we have $A|_{range(A)} = Id$. Indeed, fix $x \in \mathbb{R}^5$ then we have $x = u + v$ for $u \in range(A)$ and $v \in ker(A)$ so there exists a w such that $u = Aw$ then we get $Ax = Au = A^2w = Aw = u$ i.e. if $x \in range(A)$ we have $Ax = x$. Therefore, if

$$I - (A + B) \text{ is invertible}$$

then for any $x \in range(A)$ such that $x \neq 0$ we have

$$-B(x) \neq 0$$

In particular, $\mathbb{R}^5 = range(A) \bigoplus kernal(A)$ implies that $ker(B) \subset ker(A)$. And we can repeat the same argument to get $ker(B) = ker(A)$. Therefore, rank nullity implies $rank(A) = rank(B)$.

Problem 3. Take

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

then $A^2 = 0$ and $A^n = 0$ for all $n \geq 2$. Therefore,

$$e^A = I + A + 0 = I + A + \frac{A^2}{2}$$

but $A \neq 0$.

Problem 4. All of these matrix have eigenvalues 1 so they have a real Jordan Canonical Form. So they are similar iff they have the same Jordan Canonical Form. Note we get by computation that A, B, C, D, E all have a minimal polynomial $(x-1)^2$ so we have that they all have the following Jordan Form

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

but F Jordan Form is

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

so A, B, C, D and E are similar to one another while F is similar to itself.

Problem 5a. Note that A is positive definite iff

$$\sum_{i,j=1}^{2} \xi_i A_{ij} x_j > 0$$

for all $\xi = [\xi_1, \xi_2]^T \neq 0$. Then if A and B are positive definite we have

$$\sum_{i,j=1}^{2} \xi_i (A_{ij} + B_{ij}) x_j = \sum_{i,j=1}^{2} \xi_i A_{ij} x_j + \sum_{i,j=1}^{2} \xi_i B_{ij} x_j > 0$$

for all $\xi \neq 0$ since A and B are positive definite. Therefore, $A + B$ is positive definite.

Problem 5b.

Problem 6.

Problem 7. By Dirichlet's test since $\frac{1}{n} \to 0$ monotonically it suffices to show for any p there exists an $M = M(p)$ such that for any N

$$\left| \sum_{i=1}^{N} \sin(\pi n/p) \right| \leq M$$

By Euler's Identity we have

$$\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin(\theta)$$

So

$$\sum_{i=1}^{N} \sin(\pi n/p) = \frac{1}{2i} \sum_{i=1}^{N} e^{i\frac{\pi n}{p}} - e^{-i\frac{\pi n}{p}}$$

$$= \frac{1}{2i} \left(\frac{1 - e^{i\pi(n+1)/p}}{1 - e^{i\frac{\pi}{p}}} - \frac{1 - e^{-i\pi(n+1)/p}}{1 - e^{-i\frac{\pi}{p}}} \right)$$

since the denominator is never zero we have

$$\left| \frac{1}{2i} \left(\frac{1 - e^{i\pi(n+1)/p}}{1 - e^{i\frac{\pi}{p}}} - \frac{1 - e^{-i\pi(n+1)/p}}{1 - e^{-i\frac{\pi}{p}}} \right) \right| \leq \frac{1}{2} \left(\frac{2}{|1 - e^{i\pi/p}|} + \frac{2}{|1 - e^{i\pi/p}|} \right) := M(p) < \infty$$

so the sum converges for any p.

Problem 8. Let us follow the hint. We first claim $x_n \to 0$. Indeed, we will use induction to show $0 \leq x_n \leq 1$ then Taylor's Theorem with remainder implies $x_{n+1} := \sin(x_n) \leq x_n$. The base case is given then if $0 \leq x_n \leq 1$ then we have $0 \leq \sin(x_n) = x_{n+1} \leq 1$. Therefore, $\{x_n\}$ converges since it is a bounded monotonic sequence. Say the limit is x. Now it converges to 0 since the continuity of sin gives us

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sin(x_n) = \sin(x)$$

$$x = \lim_{n\to\infty} x_{n+1} = \sin(x)$$

so we have $x = 0$ since Taylor's theorem with remainder implies the unique fixed point has to be at $x = 0$. Now we proceed with the hint: we claim that

$$\lim_{x\to 0} \frac{1}{\sin^2(x)} - \frac{1}{x^2}$$

exists. Indeed for any fixed x Taylor's Theorem with remainder implies $\sin(x) = x - \frac{x^3}{6}\cos(\xi(x))$ for $\xi(x) \in (0, x)$ so

$$\frac{1}{\sin^2(x)} - \frac{1}{x^2} = \frac{1}{x^2} = \frac{\frac{1}{3}x^4 \cos(\xi(x)) - \cos^2(\xi)\frac{x^6}{36}}{x^4(1 - \frac{x^2}{3}\cos(\xi(x)) + \cos^2(\xi(x))\frac{x^2}{36})} \to \frac{1}{3}$$

In particular, this means
$$\frac{1}{x_{k+1}^2} - \frac{1}{x_k^2} \to \frac{1}{3}$$
Now we claim if $a_{n+1} - a_n \to L$ then $\frac{a_n}{n} \to L$ Indeed,
$$\begin{aligned}\frac{a_n}{n} - L &= \frac{\sum_{i=1}^{n-1}(a_{i+1} - a_i) + a_1}{n} - L \\ &= \frac{\sum_{i=1}^{n-1}(a_{i+1} - a_i) + a_1 - nL}{n} \\ &= \frac{\sum_{i=1}^{n-1}(a_{i+1} - a_i - L) + (a_1 - L)}{n}\end{aligned}$$
So we have for any $\varepsilon > 0$ an $N \in \mathbb{N}$ such that if $n \geq N$ then $|a_{n+1} - a_n - L| < \varepsilon$ and
$$\begin{aligned}\left|\frac{a_n}{n} - L\right| &\leq \frac{1}{n}\sum_{i=N}^{n}|a_{i+1} - a_i - L| + \frac{1}{n}\left(\left(\sum_{i=1}^{N}|a_{i+1} - a_i - L|\right) + |a_1 - L|\right) \\ &\leq \frac{\varepsilon(n-N)}{n} + \frac{M(N+1)}{n}\end{aligned}$$
where $M := \max\{\max_{i=1,..,M}|a_{i+1} - a_i - L|, |a_1 - L|\}$ Taking n to be sufficiently large we get
$$\leq 2\varepsilon$$
so we have $\frac{a_n}{n} \to L$. Therefore, we have
$$\frac{1}{nx_n^2} \to \frac{1}{3}$$
i.e.
$$\frac{nx_n^2}{1} \to 3$$
i.e.
$$\sqrt{n}x_n \to \sqrt{3}$$
as desired.

Problem 9. Fix an interval $[a, b]$ and $\varepsilon > 0$ then we have for any partition $P := \{x_0 = a < ... < x_n = b\}$ with uniform step size $\Delta x < \varepsilon$ with intervals $I_i := [x_{i-1}, x_i]$ and $\omega(f, I_i) := \sup_{x,y\in I_i}|f(x) - f(y)|$ that
$$\sum_{i=1}^{n}\Delta x\omega(f, I_i) = \varepsilon\sum_{i=1}^{n}f(x_i) - f(x_{i-1}) = \varepsilon(f(b) - f(a)) \leq 2M\varepsilon$$
where $M := ||f||_{L^\infty[a,b]}$ so f is Riemann Integrable.

Problem 10. Fix $x \in U$ then let $A := f^{-1}(f(x)) \cap U$ i.e. the preimage of f on $f(x)$. Then this is closed in U since f is continuous since it is C^1 (since the partials are continuous on U). We also claim it is open. Indeed fix $x \in A$ then $x \in U$ so there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. Then for any $y \in B_\varepsilon(x)$ we have $tx + (1-t)y \in B_\varepsilon(x)$ for $0 \leq t \leq 1$ since balls are convex. In particular, we get
$$g(t) := f(tx + (1-t)y)$$
is C^1 and
$$g(1) - g(0) = g'(\xi)$$
for some $\xi \in (0, 1)$. But $g'(t) = \nabla f(tx + (1-t)y) \cdot (x - y) = 0$ so $g(1) = g(0)$ i.e. $f(y) = f(x)$. Therefore, A is open. But as U is connected we must have $A = U$ i.e. for all $y \in U$ we have $y \in f^{-1}(f(x))$ i.e. $f(y) = f(x)$ for all $y \in U$ i.e. f is constant.

Problem 11. As X is compact and f is continuous we have
$$\overline{f(X)} = f(X)$$
now fix $x \in X$ then consider $x_n := f(x_{n-1})$ with $x_0 := x$. Then as X is compact there exists a subsequence such that $x_{n_k} \to y$. Then for any $\varepsilon > 0$ we have for large enough n_k that
$$\begin{aligned}\varepsilon > \rho(x_{n_k}, x_{n_{k+1}}) &= \rho(f^{n_k-1}(x), f^{n_{k+1}-1}(x)) \\ &= \rho(x, f^{n_{k+1}-n_k}(x))\end{aligned}$$

i.e. for $y := f^{n_{k+1}-n_k-1}(x)$ we have

$$\rho(x, f(y)) < \varepsilon$$

so $x \in \overline{f(X)} = f(X)$. In particular, we get $X \subset \overline{f(X)} = f(X)$ but the other subset is trivial so $X = f(X)$ i.e. f is surjective.

Problem 12. Let $\varepsilon > 0$ then by equicontinuity there exists a $\delta > 0$ such that if $d(x,y) < \delta$ then for any $f \in \mathcal{F}$ we have $d(f(x), f(y)) < \varepsilon$ so choose $x \in X$ and let $y \in X$ such that $d(x,y) < \delta$ then there exists an $f_1 \in \mathcal{F}$ such that

$$g(x) \leq f_1(x) + \varepsilon \leq f_1(y) + 2\varepsilon \leq g(y) + 2\varepsilon$$

i.e.

$$g(x) - g(y) < 2\varepsilon$$

and we also have the existence of an f_2 such that

$$g(y) < f_2(y) + \varepsilon \leq f_2(x) + 2\varepsilon \leq g(x) + 2\varepsilon$$

i.e.

$$|g(x) - g(y)| \leq 2\varepsilon$$

whenever $d(x,y) < \delta$ so g is uniformly continuous.

18. Fall 2018

Problem 1. Assume that

$$\sum_{n=1}^{\infty} \frac{a_n}{2a_n+1}$$

converges then we must have $\frac{a_n}{2a_n+1} \to 0$ i.e. $a_n \to 0$. So there exists an $n \geq N$ such that

$$2a_n + 1 \leq 3$$

so for any fixed $\varepsilon > 0$ there is an M such that for any $n, m \geq M$

$$\varepsilon \geq \sum_{j=n}^{m} \frac{a_j}{2a_j+1} \geq \sum_{j=n}^{m} \frac{a_j}{3}$$

for Therefore, as a_j is non-negative we must have $\sum a_j$ converges which is a contradiction.

Problem 2. Assume

$$A \cup B = X \cup Y \quad X \cap \overline{Y} = \overline{X} \cap Y = \emptyset$$

as A is connected we have $A \subset X$ or $A \subset Y$. WLOG assume $A \subset X$ then $Y \subset B$. Then

$$\mathbb{R}^n = X \cup Y \cup C$$

and

$$(X \cup C) \cap \overline{Y} = (X \cap \overline{Y}) \cup (C \cap \overline{Y}) \subset \emptyset \cup (C \cap \overline{B}) = \emptyset$$

and

$$\overline{(X \cup C)} \cap Y \subset (\overline{X} \cap Y) \cup (\overline{C} \cap Y) \subset \emptyset \cup (\overline{B} \cap Y) = \emptyset$$

but $\mathbb{R}^n$ is connected so this is a contradiction.

Problem 3. Let f and g be Riemann Integrable such that there is an $\alpha > 0$ with

$$|g(x) - g(y)| \geq \alpha|x - y|$$

Then g is injective since if $g(x) = g(y)$ then

$$0 = |g(x) - g(y)| \geq \alpha|x - y|$$

so we can define its inverse on $im(G)$. Then its inverse is Lipschitz since

$$|x - y| = |g(g^{-1}(x)) - g(g^{-1}(y))| \geq \alpha|g^{-1}(x) - g^{-1}(y)|$$

Now we just need to show that $f \circ g$ set of discontinuity has measure zero. In particular, if we let $E := \{x : f \text{ is discontinuous}\}$ then we want to show $g^{-1}(E \cap im(G))$ has measure zero. But as g^{-1} is Lipschitz this set indeed has measure zero. Indeed we have for any $\varepsilon > 0$ there are open intervals (a_n, b_n) such that $E \cap im(G) \subset \bigcup_{n=1}^{\infty}(a_n, b_n)$ with $\sum b_n - a_n \leq \varepsilon$ because $E \cap im(G)$ has zero measure. Then $g^{-1}(E \cap im(G)) \subset \bigcup_{n=1}^{\infty} g^{-1}(a_n, b_n) = \bigcup_{n=1}^{\infty}(c_n, d_n)$ and $\sum d_n - c_n = \sum g^{-1}(b_n^*) - g^{-1}(a_n^*) \leq \sum \frac{1}{\alpha}(b_n^* - a_n^*) \leq \frac{1}{\alpha} \sum b_n - a_n = \frac{\varepsilon}{\alpha}$ letting $\varepsilon \to 0$ shows $g^{-1}(E \cap im(G))$ has zero measure. Then as g set of discontinuity has measure zero, we conclude that $f \circ g$ has a set of discontinuity has measure zero. So $f \circ g$ is Riemann integrable.

Problem 4. We claim that f is concave i.e. we have the following secant line inequality if $x < z_t < y$ then

$$\frac{f(x) - f(z_t)}{x - z_t} \geq \frac{f(x) - f(y)}{x - y} \geq \frac{f(y) - f(z_t)}{y - z_t}$$

Indeed, as f is differentiable on (x, y) MVT implies there exists a $\xi_1 \in (x, z_t)$ such that $\frac{f(x)-f(z_t)}{x-z_t} = f'(\xi_1)$ ad a $\xi_2 \in (z_t, y)$ such that $\frac{f(y)-f(z_t)}{y-z_t} = f'(\xi_2)$ so in particular, as $\xi_1 \leq \xi_2$ we have

$$\frac{f(x) - f(z_t)}{x - z_t} \geq \frac{f(y) - f(z_t)}{y - z_t}$$

As $y \in (0, 1)$ there exists an $\varepsilon > 0$ such that $y + \varepsilon_0 \in (0, 1)$ so

$$\frac{f(x) - f(z_t)}{f - z_t} \geq \frac{f(y + \varepsilon) - f(z_t)}{y + \varepsilon - z_t} := L(\varepsilon)$$

Note that L is well defined for $0 \leq \leq_0$ and is continuous so we have

$$\frac{f(x)-f(z_t)}{f-z_t} \geq \lim_{\varepsilon \to 0} L(\varepsilon) = \frac{f(y)-f(z_t)}{y-z_t}$$

and a similar argument gives

$$\frac{f(x)-f(z_t)}{x-z_t} \geq \frac{f(x)-f(y)}{x-y} \geq \frac{f(y)-f(z_t)}{y-z_t}$$

Letting $x_1 < x_2 \in (0,1)$ we have $0 < x_1 < x_2$ so the secant line inequality with $x = 0, z_t = x_1, y = x_2$ gives

$$\frac{f(x_1)}{x_2} \geq \frac{f(x_2)}{x_2}$$

since $f(0) = 0$ i.e.

$$g(x) := \frac{f(x)}{x}$$

is a decreasing function on $(0,1)$.

Problem 5a. Fix $x, y \in \partial B$ then as

$$h(z) := g(z) + |x - z|$$

is a continuous map on ∂B which is compact, we get the existence of a minimum i.e. a x^* such that

$$h(x^*) = \inf_{z \in \partial B}[g(z) + |x-z|]$$

i.e.

$$f(x) = g(x^*) + |x - x^*|$$

and then we have that

$$f(y) \leq g(x^*) + |y - x^*|$$

since $f(y) \leq g(z) + |y - z|$ for any $z \in \partial B$. Then

$$f(y) - f(x) \leq g(x^*) + |y - x^*| - g(x^*) - |x - x^*| = |y - x^*| - |x - x^*| \leq |x - y|$$

We can repeat a similar argument to get

$$f(x) - f(y) \leq |x - y|$$

which implies

$$|f(x) - f(y)| \leq |x - y|$$

so f is 1-Lipschitz.

Problem 5b. By Arzela-Ascoli it suffices to show $M(g)$ is equicontinuous, closed, and uniformly bounded. It is closed since if $f_n \subset M(g)$ converge to f uniformly, then we have $f|_{\partial B} = g$ since uniform implies pointwise and f would be $1-$Lipschitz since

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|$$

and for any $\varepsilon > 0$ we can find an N such that if $n \geq N$ the $||f_n - f||_{L^\infty} < \frac{\varepsilon}{2}$ so we get

$$|f(x) - f(y)| \leq \varepsilon + |f_n(x) - f_n(y)| \leq \varepsilon + |x - y|$$

as ε is arbitrary we conclude that f is 1-Lipschitz i.e. $M(g)$ is a closed subset of $C(\overline{B})$.

Now we claim that for any $f \in M(g)$ that f is $1-$Lipschitz on $\overline{B}$. We now that f on $\text{int}(B)$ is $1-$Lipschitz and f on ∂B is $1-$Lipschitz so it suffices to show if $x \in$int(B) and $y \in \partial B$ then

$$|f(x) - f(y)| \leq |x - y|$$

Indeed fix $t \in (0,1)$ and define

$$\ell(t) := f(ty + (1-t)x)$$

then $\ell(0) = f(x)$ and $\ell(1) = f(y)$ and $ty + (1-t)x \in$int(B) since $||tx + (1-t)y|| \leq |t|||x|| + (1-t)||y|| = |t|||x|| + (1-t) < 1$ for $t \neq 1$. Then we have

$$|f(x) - \ell(t)| \leq t|x - y| \leq |x - y|$$

letting $t \to 1$ and using continuity of $\ell(t)$ we get

$$|f(x) - f(y)| \leq |x - y|$$

Let $M := \sup_{x \in \partial B} |g(x)|$ then for any $f \in M(g)$ and any $y \in \overline{B}$ we have for any $x \in \partial B$

$$|f(y)| \leq |f(y) - f(x)| + |f(x)| \leq |y - x| + M \leq 1 + M$$

so by Arzela-Ascoli we conclude $M(g)$ is a compact subset of $C(\overline{B})$

Problem 6. We will show that F is C^1. Fix $\varepsilon > 0$ then consider

$$\frac{F(x+h) - F(x)}{h} - \int_0^\infty \frac{e^{-tx}}{t^{1/2}} \mathrm{d}t$$

Notice these integrals have finite volume since

$$\begin{aligned}\int_0^\infty \frac{1 - e^{-tx}}{t^{3/2}} &\leq \int_0^1 \frac{1 - e^{-tx}}{t^{3/2}} + \int_1^\infty \frac{1 - e^{-tx}}{t^{3/2}} \\ &\leq \int_0^1 \frac{tx}{t^{3/2}} + \int_1^\infty \frac{1}{t^{3/2}} \\ &= 2x + 2\end{aligned}$$

where we used that e^{-tx} is convex to get that its tangent line lies below e^{-tx}. And we have

$$\begin{aligned}\int_0^\infty \frac{e^{-tx}}{t^{1/2}} \leq \int_0^1 \frac{e^{-tx}}{t^{1/2}} + \int_1^\infty \frac{e^{-tx}}{t^{1/2}} &\leq \int_0^1 \frac{1}{t^{1/2}} + \int_1^\infty e^{-tx} \\ &= 2 + \frac{e^{-x}}{x}\end{aligned}$$

which is finite for any $x \in (0, \infty)$. Then we have

$$\begin{aligned}\frac{F(x+h) - F(x)}{h} &= \frac{1}{h} \int_0^\infty \frac{e^{-tx} - e^{-t(x+h)}}{t^{3/2}} \\ &= \frac{1}{h} \int_0^\infty \frac{hte^{-tx} + \frac{h^2}{2} t^2 e^{-t\xi(x)}}{t^{3/2}} \\ &= \int_0^\infty \frac{te^{-tx}}{t^{3/2}} + O(h)\end{aligned}$$

since the $O(h^2)$ term is integrable. So we get

$$F'(x) = \int_0^\infty \frac{e^{-tx}}{t^{1/2}} \mathrm{d}t$$

which is continuous since

$$F'(x+h) - F'(x) = \int_0^\infty \frac{hte^{-tx} + O(h^2)}{t^{1/2}}$$

where the $O(h^2)$ term is integrable. Then as

$$\int_0^\infty h\sqrt{t} e^{-tx} \leq h \int_0^\infty e^{-tx} = hC(x) \to 0$$

so it is continuous. It is also injective since $F' > 0$ and its inverse is well defined on $range(F)$ and is C^1 thanks to

$$(F^{-1})'(F(x)) = \frac{1}{F'(x)}$$

and $F'(x) \neq 0$. It is easy to see $\lim_{x \to 0} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = +\infty$ so F is actually a bijection.

Problem 7. We claim that

$$\mathbb{R}^n = range(T) \bigoplus Ker(T)$$

Indeed, given $x \in \mathbb{R}^n$ we have $x = Tx + (x - Tx)$ and $x - Tx \in Ker(T)$. So it suffices that $range(T) \cap Ker(T) = \{0\}$. If $x \in range(T) \cap Ker(T)$ there is a y such that $Ty = x$ and $Tx = 0$ implies $T^2 y = 0$ but $T^2 y = Ty$ so $0 = Ty = x$. So the claim is proved.

Now we claim that $T|_{range(T)} = Id$. Indeed, if $x \in range(T)$ then there is a y such that $T(y) = x$ then $T(x) = T^2(y) = T(y) = x$. Now fix a basis of $range(T)$ and extend it to a basis of $ker(T)$ i.e. $\{v_1, .., v_m, w_1, .., w_{n-m}\}$ where $v_i \in range(T)$ and $w_i \in Ker(T)$ then T on this basis is

$$[Tv_1, Tv_2, .., Tv_m, Tw_1, .., Tw_{n-m}] = [v_1, v_2, .., v_m, 0, .., 0]$$

i.e. if we let $\beta := \{v_1, .., v_m, w_1, .., w_{n-m}\}$ then

$$[T]_\beta = \begin{bmatrix} Id & 0 \\ 0 & 0 \end{bmatrix}$$

where Id is an $m \times m$ block of the identity matrix where $m = rank(T)$. This is the desired bases.

Problem 8. As X is symmetric we know from the spectral theorem that there exists a unitary matrix U such that

$$X = UDU^T$$

where $D = \text{diag}(\lambda_1, ..., \lambda_n)$ and $\lambda_i \in \mathbb{R}$. Then we have for $z\ in$ with $\text{im}(z) > 0$ that

$$X - zI = U(D - zI)U^T$$

so

$$G := (X - zI)^{-1} = U\tilde{D}U^T$$

where $\tilde{D} = \text{diag}(\frac{1}{\lambda_1 - z}, ..., \frac{1}{\lambda_n - z})$ note that $\lambda_i - z \neq 0$ since $\lambda_i \in \mathbb{R}$ and $\text{Im}(z) > 0$. Then notice that

$$\sum_{j=1}^{n} |G_{ij}|^2 = (G^* G)_{ii} = \sum_{j=1}^{n} \frac{1}{\lambda_j^2 + |z|^2} u_{ij}^2$$

and

$$G_{ii} = \sum_{i=1}^{n} \frac{1}{\lambda_i - z} u_{ij}^2$$

Then note that

$$\frac{\text{Im}(G_{ii})}{\text{Im}(z)} = \sum_{j=1}^{n} \frac{1}{\lambda_i^2 + |z|^2} u_{ij}^2$$

as desired

Problem 9. We claim $ker(f) + ker(g) = \mathbb{R}^n$. Indeed, it suffices to show $\dim(ker(f) + ker(g)) = n$. Indeed as $f, g \in V^*$ are linearly independent we have $f, g \neq 0$ so $dim(ker(f)) = dim(ker(g)) = n-1$ since $Im(f) = \mathbb{R}$. But as f and$g$ are linearly independent we know that $ker(f) \neq ker(g)$ for if $ker(f) = ker(g)$ then

$$f = cg$$

which implies they are not linearly independent. Therefore,

$$dim(ker(f) + ker(g)) = dim(ker(f)) + dim(ker(g)) - dim(ker(f) \cap ker(g))$$

$$\geq n - 1 + n - 1 - (n - 2) = n$$

so $ker(f) + ker(g) = \mathbb{R}^n$ so this means for any $v \in \mathbb{R}^n$ there exists a $v_2 \in ker(f)$ and $v_1 \in ker(g)$ such that $v = v_1 + v_2$. Therefore, by linearity,

$$\begin{cases} f(v) = f(v_1) + f(v_2) = f(v_1) \\ g(v) = g(v_1) + g(v_2) = g(v_2) \end{cases}$$

as desired.

Problem 10. Note we diagonlize the matrix into

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

so we have

$$A^n = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2^n} & 0 \\ 0 & 0 & \frac{1}{3^n} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 - \frac{1}{2^n} & \frac{1}{2^n} & 0 \\ 1 - \frac{1}{2^{n-1}} + \frac{1}{3^n} & \frac{1}{2^{n-1}} - \frac{2}{3^n} & \frac{1}{3^n} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Problem 11. Note that if we let W be the set of 3×3 symmetric matrix we have

$$\mathbb{R}^{3\times 3} = V \bigoplus W$$

since

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$

and $V \cap W = \{0\}$ In particular,

$$\dim(V) = 3$$

since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

are 6 linearly independent matrix in W and

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

are 3 linearly independent matrix in V so this forms a basis of V since $\mathbb{R}^{3\times 3} = V \bigoplus W$. Note that it is an inner product since

$$< A + \lambda B, C >= \frac{1}{2}\text{Tr}((A + \lambda B)C^T) = \frac{1}{2}\text{Tr}(AC^T) + \frac{\lambda}{2}\text{Tr}(BC^T)$$

thanks to the linearity of trace and

$$< A, B >= \frac{1}{2}\text{Tr}(AB^T) = \frac{1}{2}\text{Tr}(BA^T) =< B, A)$$

since $trace(A^T) = trace(A)$. Then note that

$$< A, A >= \frac{1}{2}\text{Tr}(AA^T) = \frac{1}{2}\sum_{i,j=1}^{n} |a_{ij}|^2$$

so $< A, A >\geq 0$ with equality to zero iff $A = 0$. So it is an inner product. To find an orthonormal basis do Grahm-Schmit on the basis vectors of W mentioned above.

Problem 12. Fix a basis $\{v_1, ..., v_n\}$ of V. Then let $W_i := \{v_1, .., v_{i-1}, v_{i+1}, ..., v_n\}$ so as $T(v_j) \in W^i$ for all $i \neq j$ we get that for $i \neq j$

$$T(v_j) = \sum_{k \neq i} \alpha_k v_k$$

Fix a $\ell \neq i$ or j then we get

$$T(v_j) = \sum_{k \neq \ell} \alpha_k v_k$$

which implies that $\alpha_\ell = \alpha_i = 0$ We can repeat this argument for all $k \neq j$ to get

$$T(v_j) = \alpha_j v_j$$

So T in this basis is $T = \text{diag}(\alpha_1, .., \alpha_n)$ so it suffices to show $\alpha_i = \alpha_j$ for all i, j. Fix $i \neq j$ and then we have for $E_{ij} := span\{v_i + v_j\}$ and let $W_{ij} := \{e_k : k \neq i, j\}$ then let $M := E_{ij} + W_{ij}$ which is $n-1$ dimensional. Therefore, we have as $T(M) \subset M$ that

$$T(v_i + v_j) = \alpha(v_i + v_j) + \sum_{k \neq i,k} \beta_k v_k$$

but

$$T(v_i + v_j) = \alpha_i v_i + \alpha_j v_j$$

so we must have $\beta_k = 0$ for all $k \neq i, k$ and $\alpha = \alpha_i = \alpha_j$. Iterating this with i fixed at 1 and letting $2 \leq j \leq n$ shows $\alpha = \alpha_i$ for all $1 \leq i \leq n$ i.e. $Tv_i = \alpha v_i$ for all i so T is a constant multiple of the identity.

19. Spring 2019

Problem 1. To show it is complete it suffices to show it is a closed subsbet of the complete metric space $(C([0,1]), ||\cdot||_{L^\infty})$. Indeed, fix $\varepsilon > 0$ then if $X \ni f_n \to f$ then there exists an N such that if $n \geq N$ then $||f_n(x) - f(x)||_{L^\infty} < \frac{\varepsilon}{2}$ so we have

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \leq \varepsilon + |f_n(x) - f_n(y)|$$
$$\leq \varepsilon + |x - y|$$

Letting $\varepsilon \to 0$ gives

$$|f(x) - f(y)| \leq |x - y|$$

so $f \in X$ so it is a closed subset of a complete metric space, so X is complete.

We will show that X is path connected which implies it is connected. Fix $f, g \in X$ and define

$$\gamma(t) := (1-t)f + tg$$

then for any fixed $t \in [0,1]$ we have $\gamma(t)(x) \in X$ since

$$|\gamma(t)(x) - \gamma(t)(y)| \leq (1-t)|f(x) - f(y)| + t|g(x) - g(y)|$$
$$\leq |x - y|$$

Now we claim γ is continuous. Indeed, if given an $\varepsilon > 0$ then if $|t_1 - t_2| < \varepsilon$ then

$$||\gamma(t_1) - \gamma(t_2)||_{L^\infty} \leq |t_1 - t_2|(||f||_{L^\infty} + ||g||_{L^\infty}) < \varepsilon((||f||_{L^\infty} + ||g||_{L^\infty})$$

so γ is continuous. Therefore, X is path connected.

Problem 2. Note that

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

So in particular we have

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$$

We can solve for a_n and compute $\lim_{n\to\infty}$ directly by diagonalizing this matrix.

Problem 3. First observe that since $f(x) \geq \delta$ that $\frac{1}{f(x)}$ is finite so

$$\frac{1}{f(x)} - \frac{1}{f(y)} = \frac{f(y) - f(x)}{f(x)f(y)}$$

so

$$\left|\frac{1}{f(x)} - \frac{1}{f(y)}\right| \leq \frac{|f(y) - f(x)|}{\delta^2}$$

Now as f is Riemann Integrable for any $\varepsilon > 0$ there exists a partition $P = \{a = x_0 < ... < x_n = b\}$ with $\Delta x_i := x_i - x_{i-1}$, $I_i := [x_{i-1}, x_i]$, and $\omega(f, I_i) := \sup_{x,y\in I_i} |f(x) - f(y)|$ such that

$$\sum_{i=1}^n \Delta x_i \omega(f, I_i) \leq \delta^2 \varepsilon$$

then as

$$\sum_{i=1}^n \Delta x_i \omega(\frac{1}{f}, I_i) \leq \frac{1}{\delta^2} \sum_{i=1}^n \Delta x_i \omega(f, I_i) \leq \varepsilon$$

so the lower and upper Riemann sums are within ε and as ε is arbitrary we conclude that $\frac{1}{f}$ is Riemann Integrable.

Problem 4. Let

$$\ell(x) := g(x)(f(b) - f(a)) - f(x)(g(b) - g(a))$$

Then note that

$$\ell(a) = \ell(b) = f(b)g(a) - g(b)f(a)$$

so Rolle's Theorem implies there exists a ξ such that

$$\ell'(\xi) = 0$$

i.e.

$$g'(\xi)(f(b) - f(a)) = f'(\xi)(g(b) - g(a))$$

as desired where $\xi \in (a, b)$.

Problem 5. Let X^* denote the completion of X. Embed (X, d) to (X^*, d^*) via the identity map. Then we next claim theres an isometric embedding of (X^*, d^*) to $(C(X^*), ||\cdot||_{L^\infty}))$ which is a Banach Space. Indeed, fix an $x \in X^*$ and define

$$\Phi_{x^*}(y) : d^*(x^*, y)$$

then $\Phi_{x^*} : X^* \to C(X^*)$ and

$$||\Phi_{x^*}(y) - \Phi_{y^*}(y)||_{L^\infty} = \sup_{y \in X^*} |d(x^*, y) - d(y^*, y)| \leq d(x^*, y^*)$$

and taking $y = y^*$ gives

$$\sup_{y \in X^*} |d(x^*, y) - d(y^*, y)| \geq d(x^*, y^*)$$

i.e.

$$||\Phi_{x^*}(y) - \Phi_{y^*}(y)||_{L^\infty} = d^*(x^*, y^*)$$

so the map Φ is an isometric embedding of X^* to $C(X^*)$. Let $X := C(X^*)$ with norm $||\cdot||_{L^\infty}$ then for any fixed x the map $\psi_x : X \to C(X)$ defined via $\psi_x(y) := d(x, y)$ is an isometric embedding into $C(X) \subset C(X^*)$ which is a Banach Space where we used that $d(x, y) = d^*(x, y)$ when $x, y \in X$.

Problem 6. Note that

$$\int_0^\infty \frac{x}{n^2} e^{-x/n} \mathrm{d}x = \int_0^\infty u e^{-u} \mathrm{d}u = -\int_0^\infty e^{-u} \mathrm{d}u = 1$$

and we note that $\lim_{x \to \infty} x e^{-\frac{x}{n}} = 0$ since exponential decay is much faster than linear growth, so the maximum over $\overline{R^+}$ must be attained om a compact set. In particular, we will have $\partial_x x e^{-\frac{x}{n}} = 0$ at the max or $x = 0$. The critical point x^* satisfies

$$e^{-\frac{x^*}{n}} - \frac{x^*}{n} e^{-\frac{x^*}{n}} = 0$$

so

$$1 - \frac{x^*}{n} = 0 \Rightarrow x^* = n$$

so

$$f_n(x^*) = \frac{1}{en}$$

but $f_n(0) = 0$ so the max must occur at x^* so we have

$$||f_n||_{L^\infty} = \frac{1}{ne} \to 0$$

so f_n uniformly converges to 0.

Problem 7. We find the characteristic polynomial to find out it is

$$\chi(x) = -1 - 2x^2 + 13x - x^3$$

then Cayley-Hamilton gives us

$$0 = \chi(A) = -Id - 2A^2 + 13A - A^3 \Rightarrow Id = A(-2A + 13Id - A^3)$$

so $A^{-1} = (13Id - A^3 - 2A)$.

Problem 8. We claim that this subspace is the space of trace zero matrix and its dimension is $n^2 - 1$. Indeed, trace zero matrix have dimension $n^2 - 1$ since if we define the matrix $E^{(ij)}$ to satisfy

$$E^{(ij)}_{k\ell} = \begin{cases} 1 \text{ if } k = i \text{ and } j = \ell \\ 0 \text{ else} \end{cases}$$

Then $E^{(ij)}$ for $i \neq j$ is in the space of trace zero matrix, so there are at least $n^2 - n$ of them. Then we define the $n - 1$ matrix $E^{(ii)}$ for $2 \leq i \leq n$ with

$$E^{(ij)}_{k\ell} = \begin{cases} 1 \text{ if } k = i \text{ and } j = i \\ -1 \text{ if } k = i-1 \text{ and } \ell = i-1 \\ 0 \text{ else} \end{cases}$$

then there are $n-1$ matrix in the space of trace zero matrix. Then these matrix are linearly independent, so there at least $n^2 - 1$ independent matrix in the space of trace zero matrix. But as Id has non-zero trace the dimension cannot be greater than $n^2 - 1$, so the dimension of trace zero matrix is $n^2 - 1$. We will prove these two definitions are equivalent over $\mathbb{R}^2$ so we have the dimension is 3. Let U denote the subspace of trace zero matrix. Then we clearly have $W \subset U$ since if $C \in W$ then $C = AB - BA$ for some A, B then $tr(AB - BA) = tr(AB) - tr(BA) = 0$.

Now it suffices to show that on the basis $E^{(22)}, E^{(12)}, E^{(21)}$ that this property is true. Fix any diagonal matrix $D = (1, 0)$ then for any matrix B we have

$$DB - BD = \begin{bmatrix} 0 & b_{12} \\ b_{21} & 0 \end{bmatrix}$$

so for $E^{(12)}$ and $E^{(21)}$ this property clearly holds. And

$$E^{(22)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so we have $U \subset W$ so $dim(W) = 3$.

Problem 9. We write D in its matrix form with respect to the standard bases $\{1, x, x^2, .., x^10\}$

$$D = \text{super diag}(1, 2, 3, .., 10)$$

Then

$$\exp(D) = I + \sum_{n=1}^{\infty} \frac{D^n}{n!}$$

$$= I + \sum_{n=1}^{10} \frac{D^n}{n!}$$

since D is nilpotent. Then as $\sum_{n=1}^{\infty} \frac{D^n}{n!}$ is nilpotent all of the eigenvalues must be one. This follows from if A and B are nilpotent such that $AB = BA$ then AB is nilpotent thanks to the binomial theorem. And we have

$$rank(\sum_{n=1}^{\infty} \frac{D^n}{n!}) = 10$$

so its kernal is one dimensional, so the only eigenvector are constants i.e.

$$\begin{bmatrix} c \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

Problem 10. As A is diagonalizable there exists a U such that and a diagonal matrix U such that

$$A = U^{-1} D U$$

$$\begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix} \begin{bmatrix} D & I \\ 0 & D \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} = \begin{bmatrix} A & I \\ 0 & A \end{bmatrix}$$

So if $\begin{bmatrix} A & I \\ 0 & A \end{bmatrix}$ is diagonalizable so would $\begin{bmatrix} D & I \\ 0 & D \end{bmatrix}$:= B. But observe that the characteristic polynomial B is just the characteristic polynomial of A squared. But note for $\lambda_i := D_{ii}$ we have

$$(B - \lambda_1 I)x = 0 \Longleftrightarrow \begin{bmatrix} x_{n+1} \\ \lambda_2 x_2 - x_{n+2} \\ \dots \\ \lambda_n x_n - x_{2n} \\ x_{n+1} \\ \lambda_2 x_{n+2} \\ \dots \\ \lambda_n x_{n+2} \end{bmatrix} = 0$$

so for all $\lambda_k \neq 0$ we get $x_{n+k} = 0$ so if $\lambda_j = 0$ we also get $\lambda_j x_j - x_{n+j} = 0 \Rightarrow x_{n+j} = 0$ so $x_{n+k} = 0$ for all $k \geq 1$. Therefore, we x must be of the form $x = (x_1, .., x_n, 0, .., 0)^T$. But

$$(B - \lambda I)x = Ax - \lambda x$$

so it must be an eigenvalue of A i.e. for each eigenspace the multiplicity of the eigenvectors is the same as D, so B cannot be diagonalized since it only has n eigenvectors and not $2n$.

Problem 11. As $rank(A) = rank(A^2)$ we must have $nullity(A^2) = nullity(A)$ i.e. the generalized eigenspace of 0 for A is the same as the eigespace. So in Jordan Canonical Form A must have no non-trivial blocks of 0. Then as we have only finitely many eigenvalues, we see for small enough λ that $A + \lambda I$ is invertible. So by JCF we have

$$A = U^{-1}(J_1 \bigoplus J_2 \bigoplus \dots \bigoplus J_k)U$$

for Jordan Blocks J_k and any Jordan block with a diagonal zero must be 1×1. In particular, we can write

$$A = U^{-1} \begin{bmatrix} \tilde{J} & 0 \\ 0 & 0 \end{bmatrix} U$$

where $\tilde{J}$ is an invertible matrix so we have

$$(A + \lambda I)^{-1} A = U^{-1} \begin{bmatrix} (\tilde{J} + \lambda I)^{-1} \tilde{J} & 0 \\ 0 & 0 \end{bmatrix} U$$

so

$$\lim_{\lambda \to 0} (A + \lambda I)^{-1} A = U^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U$$

So one direction is proved. Now if $\lim_{\lambda \to 0} (A + \lambda I)^{-1} A$ exists then it must have no non trivial size zero Jordan Blocks. Indeed,

$$A = U^{-1}(J_1 \bigoplus \dots \bigoplus J_k)U$$

then

$$(A + \lambda I)^{-1} = U^{-1}((J_1 + \lambda I)^{-1} \bigoplus \dots \bigoplus (J_k + \lambda I)^{-1})U$$

and say J_k is a Jordan Block with zero diagonals and is of size $k \times k$ for $k > 1$ then the diagonal terms of $(J_k + \lambda I)^{-1} J_k$ super diagonal have terms of the form C/λ for some constant C so we have it blows up, so the limit does not exist. So at most the Jordan Blocks of zero are of size 1×1. This means that the generalized eigenspace of zero is equal to the eigenspace of zero, so we have $ker(A) = ker(A^2)$ so we have $rank(A) = rank(A^2)$.

Problem 12. Assume otherwise then there is an x such that $Ax = 0$ with $||x|| = 1$

$$a_{ii} x_i = - \sum_{j=1, j \neq i}^{n} a_{ij} x_j$$

so taking norms squared gives

$$\sum_{i=1}^{n} (a_{ii} x_i)^2 = \sum_{i=1}^{n} \left(\sum_{j=1, j \neq i}^{n} a_{ij} x_j \right)^2$$

Applying Cauchy-Schwarz gives

$$\leq \sum_{i=1}^{n}(\sum_{j=1,j\neq i}^{n}(a_{ij})^2)||x|| \\ = \sum_{i\neq j} a_{ij}^2 < 1$$

but

$$\sum_{i=1}^{n}(a_{ii}x_i)^2 \geq \sum_{i=1}^{n}|x_i|^2 = 1$$

so we have the contradiction that $1 < 1$ so we must have A is invertible.

20. Fall 2019

Problem 1. If A_λ is invertible then as $A^{-1}(e_1) \neq 0$ we get

$$A_\lambda(A^{-1}e_1) \neq 0$$
$$= e_1 + \lambda(e_1, A^{-1}(e_1))e_1 \neq 0$$

i.e. $1 + \lambda(e_1, A^{-1}(e_1) \neq 0$. For the converse fix x such that $A_\lambda x = 0$ then

$$A^{-1}(Ax + \lambda(e_1, x)e_1) = 0$$

so

$$x + \lambda(e_1, x)A^{-1}(e_1) = 0$$

so if $x = (x_1, .., x_n)$ we get that

$$x_1 + \lambda x_1 A^{-1}_{11} = x_1(1 + \lambda(e_1, A^{-1}(e_1))) = 0$$

so we must have $x_1 = 0$ but this implies $x = 0$ since if $x_1 = 0$ then we have $A_\lambda(x) = Ax$ and $Ax = 0$ iff $x = 0$.

Problem 2. Notice that by staring at the matrix we get that the eigenvalues and eigenvectors of $A^2 + A$ are

$$\lambda = \{6, 6, 0, 0\} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

so we have

$$A^2 + A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

where for the last step we used that our eigenvectors form an orthogonal matrix. In particular, let

$$A := \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

then A is symmetric since

$$A^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 1 & 0 & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}^T$$

Problem 4.

Problem 5. View A as a complex operator then we have that

$$A = U^{-1}(J_1 \bigoplus ... \bigoplus J_m)U$$

where J_i is a Jordan Block. In particular

$$A^k = U^{-1}(J_1^k \bigoplus ... \bigoplus J_m^k)U$$

Then note that J_i^k is either λ_i^k or $\binom{k}{\ell}\lambda_i^{k-\ell}$ where λ_i are the eigenvalues of A. Then if $|\lambda_i| < 1$ we have $\lambda_i^k \to 0$ and $\binom{k}{\ell}\lambda_i^{k-\ell} \to 0$ since $\binom{k}{\ell}$ is polynomial growth while λ_i^k is exponential decay. So we have each entry of J_i^k goes to 0. In particular, this means all the entries of A^k converges to zero. Then as we have

$$||A||_{\text{op}}^2 = \sup_{||x||=1} (Ax, Ax) = \sup_{||x||=1} (A^T Ax, x) = \max_{1 \le i \le n} |\sigma_i|$$

where σ_i are the eigenvalues of A^TA (spectral theorem guarantees the existence of a basis of eigenvectors in $\mathbb{R}$). But as A^TA is positive definite, we have that $\max_{1\leq i\leq n}|\sigma_i| \leq \text{Tr}(A^TA) = ||A||_2^2$ so we have

$$||A||_{\text{op}} \leq ||A||_2 = \left(\sum_{ij}|a_{ij}|^2\right)^{\frac{1}{2}}$$

for any matrix. As A^k entry wise goes to zero there exists for any $\varepsilon > 0$ an N such that if $k \geq N$ then $|a_{ij}^{(k)}|^2 < \frac{\varepsilon^2}{n^2}$ where $a_{ij}^{(k)}$ is the ijth entry of A^k, so

$$||A^k||_{\text{op}} \leq \left(\sum_{ij}|a_{ij}^{(k)}|^2\right)^2 \leq \varepsilon$$

for $k \geq N$ so we have

$$||A^k||_{\text{op}} \to 0$$

For the converse fix let v be an eigenvector associated to the eigenvalue λ where $||v|| = 1$ then

$$|\lambda^k v| = |A^k v| \leq ||A^k||_{\text{op}} \to 0$$

so we must have $|\lambda| < 1$

Problem 6a. If B is invertible define $T : \mathcal{M}^n \to \mathcal{M}^n$ via

$$T_B(A) := (B^T)^{-1}AB^{-1}$$

since $rank(B) = rank(B^T)$ so B^T is also invertible. Then

$$T_B(L_B(A)) = (B^T)^{-1}B^TABB^{-1} = A$$
$$L_B(T_B(A)) = B^T(B^T)^{-1}AB^{-1}B = A$$

so L_B is invertible with inverse T_B.

Now assume L_B is invertible, but that B does not have full rank. Then $range(B) \neq \mathbb{R}^n$. Let

$$A := \begin{cases} 0 \text{ on } range(B) \\ Id \text{ on } range(B)^\perp \end{cases}$$

then A is not the zero operator but

$$L_B(A) = 0$$

which implies L_B is not invertible, so this is a contradiction.

Problem 6b and 6c. Assume $rank(B) = k$ then we define the new linear map

$$T_B : \mathcal{M}^n \to \mathcal{M}^n \text{ where } T_B(A) := (B^T)E_1AE_2B$$

where E_i are invertible matrix. In particular, as $rank(B) = rank(B^T)$ we have due to Jordan Elimination the existence of elementary matrix such that

$$(B^T)E_1 = \text{diag}(1, ..., 1, 0, .., 0) \quad E_2B = \text{diag}(1, ..., 1, 0, .., 0)$$

where both have k ones. Then this map has the same range as L_B since $L_B(E_1AE_2) = T_B(A)$ and $T_B(E_1^{-1}AE_2^{-1}) = L_B(A)$. This lets us deduce that for a general matrix A that $T_B(A)$ has $n^2 - k^2$ zeros. So the kernal(T_B) has dimension $n^2 - k^2$, so its range must have dimension $k^2 = rank(B)^2$.

Problem 7. Consider the operator $L : [0, 1] \to [0, 1]$ defined via

$$L(x) = \cos(x)$$

Note that this is well defined since $\cos([0, 1]) \subset [0, 1]$. And as $[0, 1]$ is a closed subset of $\mathbb{R}$ it is complete. Then note that for $x < y \in [0, 1]$

$$L(x) - L(y) = \cos(x) - \cos(y) = (x - y)\sin(\xi(x, y))$$

by MVT where $\xi(x, y) \in [x, y]$ but as sin is an increasing function we have

$$|L(x) - L(y)| \leq |\sin(1)||x - y|$$

and $|\sin(1)| < 1$ so we can apply Banach Fixed Point Theorem to obtain the existence and uniqueness of a fixed point on $[0,1]$ of the operator L i.e. there exists a unique solution to

$$x = \cos(x)$$

on $[0,1]$.

Problem 8. Note for any $h \in (0,1]$ that

$$\sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} \leq \sum_{n\in Z} \frac{h}{n^2h^2} = \frac{1}{h}\sum_{n\in\mathbb{Z}} \frac{1}{n^2} = \frac{2}{h}\sum_{n\in\mathbb{N}} \frac{1}{n^2} \leq C(h)$$

so the sum is well defined. Note that by symmetry we have

$$\sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} = 2\sum_{n\in\mathbb{N}} \frac{h}{1+n^2h^2}$$

Define

$$f(x;h) := \frac{h}{1+x^2h^2}$$

then for $x \in [0,\infty)$ we have $f(x;h)$ is a decreasing function. In particular, we have the following inequality

$$\sum_{n=0}^{\infty} \frac{h}{1+n^2h^2} \leq h + \int_0^\infty \frac{h}{1+x^2h^2}\mathrm{d}x$$
$$= h + \frac{\pi}{2} \leq 1 + \frac{\pi}{2}$$

so we have

$$0 \leq \sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} \leq 2+\pi$$

i.e.

$$\sup_{h\in(0,1]} \sum_{n\in\mathbb{Z}} \frac{h}{1+n^2h^2} < +\infty$$

Problem 9a. Let the map $F : \mathbb{R}^3 \to \mathbb{R}$ be defined via

$$F(x,y,z) = (2+x+y)e^z - z^2 - e^x - e^y$$

then $F \in C^\infty(\mathbb{R}^3)$ since the partials are smooth and

$$F(0,0,0) = 0$$

with

$$\nabla F(0,0,0) = (0,0,2)^T$$

so as the 1×1 submatrix corresponding to $\frac{\partial F}{\partial z}(0,0,0)$ is non-singular with $F \in C^1(\mathbb{R}^3)$, we can apply implicit function theorem to find an open subset $U \subset \mathbb{R}^2$ with $(0,0) \in U$ where since $\frac{\partial F}{\partial z} \neq 0$ at $(0,0,0)$ we can use continuity to make $\frac{\partial F}{\partial z} \neq 0$ in U and a function φ such that

$$F(x,y,\varphi(x,y)) = 0 \text{ for } (x,y) \in U$$

and $\varphi(0,0) = 0$. Then for the regularity of φ we have by the implicit function theorem that

$$\nabla\varphi = \left(\frac{\partial F}{\partial x}\left(\frac{\partial F}{\partial z}\right)^{-1}, \frac{\partial F}{\partial y}\left(\frac{\partial F}{\partial z}\right)^{-1}\right)^T$$

where $\partial_z F \neq 0$ in U. Then as F is smooth and $\frac{\partial F}{\partial z} \neq 0$ we can use product rule/quotient rule to see that $\varphi \in C^\infty(U)$

Problem 9b. Note that $\nabla\varphi(0,0) = (0,0)^T$ so it is a critical point. We compute the Hessian at $(0,0)$ to get

$$D^2\varphi(0,0) = \mathrm{diag}(1/2, 1/2)$$

so $D^2\varphi$ is positive definite, so it is at a min.

Problem 10.

Problem 11a. Fix $\{f_n\} \subset X$ that is a Cauchy Sequence. Then fix an $\varepsilon > 0$ then there is an N such that if $n \geq N$ then

$$||f_n(x) - f_m(x)||_{L^\infty} < \frac{\varepsilon}{2}$$

so in particular we have for any $x \in [0,1]$ that

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$$

by completeness of $[0,1]$ we determine that there exists an $f(x) \in [0,1]$ such that

$$f_n(x) \to f(x)$$

Then we notice that as $f_n(x) \to f(x)$ there is an N_x such that if $n \geq N_x$ then $|f_{N_x}(x) - f(x)| \leq \frac{\varepsilon}{2}$

$$|f(x) - f_n(x)| \leq |f(x) - f_{N_x}(x)| + |f_{N_x}(x) - f_n(x)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

since we may as well assume that $N_x \geq N$. So we have

$$||f(x) - f_n(x)||_{L^\infty} < \varepsilon$$

so we have $f_n(x) \to f(x)$ uniformly. So it suffices to show $f(x)$ is decreasing but this follows since by taking n large enough we have for $x \leq y$

$$f(x) \leq f_n(x) + \frac{\varepsilon}{2} \leq f_n(y) + \frac{\varepsilon}{2} \leq f(y) + \varepsilon$$

so letting $\varepsilon \to 0$ shows when $x \leq y$ we have

$$f(x) \leq f(y)$$

as desired. So it is complete

Problem 11b. Take $\{x^n\} \subset X$ since x^n is an increasing function and on $[0,1]$ we have $0 \leq x^n \leq 1$. But

$$x^n \to \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ if } x = 1 \end{cases} := f(x)$$

so if there exists a subsequence x^{n_k} that converged it must converge to $f(x)$. But as it is a uniformly convergent subsequence the limit must be continuous since $x^n \in C([0,1])$. This shows no subsequence uniformly converges so it is not sequentially compact.

Problem 12. We clearly have that if $f : \ell^\infty \to \mathbb{R}$ is continuous then $f|_K$ is continuous for any compact set K. So it suffices to show the other direction. Indeed, let $x_n \to x$ then define $K := \{x_n\} \cup \{x\}$ then we claim this is a compact subset of ℓ^∞. Indeed, take $\{y_n\} \subset K$ if it only has finitely many terms then we are done, so assume it is infinite. Then as $x_n \to x$ for a fixed $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that if $n \geq N$ then $d(x_n, x) < \varepsilon$. Then as $\{y_n\}$ is an infinite subset of K there must be a k such that if $n \geq k$ then $\{y_n\}_{n \geq k} \subset \{x_n\}_{n \geq N}$ so in particular x is a limit point of $\{y_n\}$ so K is compact. But as $f|_K$ is compact we have

$$f|_K(x_n) \to f|_K(x)$$

i.e. $f(x_n) \to f(x)$ so f is continuous.

21. Spring 2020

Problem 1. Note that

$$Tr(AB - BA) = Tr(AB) - Tr(BA) = 0$$

so we cannot have

$$AB - BA = Id$$

because it would imply $tr(Id) = 0$ which is false.

Problem 2. We claim if for a matrix C and D that if C is similar to D then they have the same eigenvalues. Indeed,

$$\det(C - \lambda I) = \det(SS^{-1}(C - \lambda I))$$
$$= \det(S^{-1}(C - \lambda I)S) = \det(D - \lambda I)$$

so they have the same characteristic polynomial. Indeed, then as B is similar to B^5 implies that if x is an eigen vector with eigenvalue λ then

$$Bx = \lambda x \quad B^5 x = \lambda^5 x$$

so for every eigenvalue λ we must have $\lambda = \lambda^5$. But as B is invertible we have $\lambda \neq 0$ so

$$\lambda^{25} = \lambda^5 = \lambda \Rightarrow \lambda^{24} = 1$$

Problem 3.

Problem 4a. Notice this implies for all $x, y \in \mathbb{C}$ that

$$(x + y, A(x + y)) = 0$$

i.e. since $(x, Ax) = y, Ay) = 0$

$$(y, Ax) + (x, Ay) = 0$$

Taking $xtoix$ also gives

$$(y, A(ix)) + (ix, Ay) = 0$$

$$\begin{cases} -i(y, Ax) + i(x, Ay) = 0 \\ (y, Ax) + (x, Ay) = 0 \end{cases}$$

implies

$$\begin{cases} -i(y, Ax) + i(x, Ay) = 0 \\ i(y, Ax) + i(x, Ay) = 0 \end{cases}$$

i.e.

$$2i(x, Ay) = 0$$

i.e.

$$(x, Ay) = 0$$

for all $x, y \in \mathbb{C}$ so

$$Ay = 0$$

for all y so A is the zero operator.

Problem 4b. Take

$$A = \begin{bmatrix} \cos(\frac{\pi}{2}) & -sin(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) & cos(\frac{\pi}{2}) \end{bmatrix}$$

then

$$(Av, v) = 0$$

since it is a rotation of v by 90 degrees.

Problem 5.

Problem 6. Assume that $T \in M_{m\times m}(\mathbb{R})$ eigenvalues satisfy $|\lambda_i| < 1$. By viewing T as an operator over $\mathbb{C}$ we can find a basis such that
$$T = U^{-1}JU$$
where
$$J = J_1 \bigoplus J_2 \bigoplus ... \bigoplus J_m$$
where J_i is a Jordan Block with diagonal entries being the eigenvalues of T. Note that
$$T^n = U^{-1}J^nU$$
and
$$J = J_1^n \bigoplus J_2^n \bigoplus ... \bigoplus J_m^n$$
So it suffices to show for an arbitrary Jordan Block J_i that $|(J_i)_{kk}| < 1$ implies $J_i^n = 0$ where J_i is of size mm. This follows from that
$$J^n = \begin{bmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & ... & \binom{n}{m-1}\lambda^{n-m+1} \\ 0 & \lambda^n & \binom{n}{1}\lambda^{n-1} & ... & \binom{n}{m-2}\lambda^{n-m+2} \\ 0 & 0 & \lambda^n & .... & \end{bmatrix}$$
i.e. each entry is either 0 or λ^n or $\binom{n}{k}\lambda^{n-k}$. But as $|\lambda| < 1$ we know that $\lambda^n \to 0$ and as exponentials decay much faster than polynomials grow L'hopital gives $\binom{n}{k}\lambda^{n-k}0$ as $n \to \infty$. Therefore, $J^n \to 0$. So we have $T^n \to 0$.

Now if $T^n \to 0$ fix an eigenvector x with eigenvalue λ then we have
$$T^n x = \lambda^n x \to 0$$
so $|\lambda| < 1$.

Problem 7. Both parts follow from IVT.

Problem 8. Density of polynomials in $C([a, b])$ implies this.

Problem 9. See Fall 2016 number 11.

Problem 10a. This is Banach's Fixed Point. Fix an $x \in X$ then let $x_n := f(x_{n-1})$ with $x_0 := x$ then if $n \geq m$
$$\begin{aligned} d(x_{n+1}, x_{m+1}) = d(f^n(x), f^m(x)) \leq \lambda^m d(f^{n-m}(x), x) \\ \leq \lambda^m \left(d(f^{n-m}(x), f^{n-m-1}(x)) + d(f^{n-m-1}(x), f^{n-m-2}(x)) + ...d(f(x), x)\right) \\ \leq d(f(x), x) \sum_{k=0}^{n-m} \lambda^{m+i} \to 0 \end{aligned}$$
since it is a convergent sum due to $\lambda < 1$ so $\{x_n\}$ is Cauchy. Then completeness implies there exists a limit say z then
$$\begin{cases} \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(z) \\ \lim_{n\to\infty} x_n = z \end{cases}$$
i.e. $f(z) = z$ where we used continuity of f. This is unique since if $f(z_1) = z_1$ and $f(z_2)$ and $z_1 \neq z_2$
$$d(z_1, z_2) = d(f(z_1), f(z_2)) \leq \lambda d(z_1, z_2) < d(z_1, z_2)$$
which is a contradiction so it is unique.

Problem 10b. Uniqueness follows from if $x \neq y$ and $f(x) = f(y)$ then we get
$$d(x, y) \leq \frac{d(x, y)^2}{1 + d(x, y)}$$
i.e.
$$1 \leq \frac{1}{1 + d(x, y)}$$
but as $d(x, y) < +\infty$ we have
$$\frac{1}{1 + d(x, y)} < 1$$

which is a contradiction so there is uniqueness.

To see existence fix $x \in X$ and define $x_0 := x$ with $x_{n+1} := f(x_n)$. Then we have

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}) \left(\frac{d(x_n, x_{n-1})}{1 + d(x_n, x_{n-1})} \right) \leq d(x_n, x_{n-1}) \leq d(x_0, x_1)$$

Note that the function

$$g(x) := \frac{x}{1+x}$$

is an increasing function so we have from $d(x_{n+1}, x_n) \leq d(x_0, x_1)$ that

$$\frac{d(x_n, x_{n-1})}{1 + d(x_n, x_{n-1})} \leq \frac{d(x_0, x_1)}{1 + d(x_0, x_1)} := \lambda < 1$$

so for our fixed x if we let $A := \{f^n(x) : n \in \mathbb{N}\}$ where f^n means the nth iterate of f then we have $f|_A : A \to A$. Now fix $u, v \in A$ then assume that $u = f^{n+1}(x)$ and $v = f^{n+m+1}(x)$. Then

$$\begin{aligned} d(u, v) = d(f^{n+1}(x), f^{n+m+1}(x)) \leq d(x, f^m(x)) \\ \leq d(x, f(x)) + d(f(x) f^2(x)) + ... + d(f^{m-1}(x), f^m(x)) \\ \leq d(x, f(x))(1 + \lambda + ... + \lambda^m) \\ \leq \sum_{i=1}^{\infty} \lambda^i < M \end{aligned}$$

i.e. there exists an $M > 0$ such that

$$d(f^n(x), f^m(x)) < M$$

for any n, m so we have

$$\frac{d(x, y)}{d(x, y) + 1} \leq \frac{M}{1 + M} := \alpha < 1$$

on A so we have $f|_A$ satisfies

$$d(f|_A(x), f|_A(y)) \leq \alpha d(x, y)$$

so it is a contraction mapping on A. Therefore, by the proof in 10a $\{f_n(x)\}$ is a Cauchy Sequence. By completeness of X we conclude a limit in X and an identical argument as in 10a concludes that the limit is a fixed point.

Problem 11.

Problem 12a. See Fall 2016 number 12.

Problem 12b. If $f'' \geq 0$ then by Taylor's Theorem we have

$$f(y) = f(x) + f'(x)(y - x) + f''(\xi(y)) \frac{(x - y)^2}{2}$$

for some $\xi(y) \in [\min\{x, y\}, \max\{x, y\}]$ but as $f'' \geq 0$ we have

$$f(y) \geq f(x) + f'(y)(y - x)$$

then part a) implies the desired result.