ANALYSIS QUALIFYING EXAM SOLUTIONS

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These solutions should contain a majority of the questions asked from Spring 2010 to Fall 2020. Some of these solutions are inspired by the solutions from Adam Lott's solutions which can be found *here* and for Steven Truong's solutions which can be found *here*.

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Contents

1. Spring 2010

Problem 1. Let $1 \leq p < \infty$. Show that if a sequence of real-valued functions $\{f_n\}_{n\geq 1}$ converges in $L^p(\mathbb{R}^n)$, then it contains a subsequence that converges almost everywhere.

Also give an example of a sequence of functions converging to zero in $L^2(\mathbb{R})$ that does not converge almost everywhere.

Proof. Assume $f_n \to f$ in $L^p(\mathbb{R})$. Then there exists a subsequence $\{f_{n_k}\}$ such that

$$||f_{n_k} - f||^p_{L^p(\mathbb{R})} \leq \frac{1}{2^{-k}}$$

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Therefore,

$$\sum_{k=1}^{\infty}\int_{\mathbb{R}}|f_{n_k}-f|^p dx \leq 1$$

which by the monotone convergence theorem tells us that

$$\sum_{k=1}^{\infty}\int_{\mathbb{R}}|f_{n_k}-f_{n_{k+1}}|^p dx = \int_{\mathbb{R}}\sum_{k=1}^{\infty}|f_{n_k}-f|^p dx \leq 1$$

Therefore, it follows that for *a.e.* x that $\sum_{k=1}^{\infty}|f_{n_k}(x)-f(x)|^p<\infty$ so $f_{n_k}(x)-f(x)\to 0$ for a.e. x □

Problem 2. Let $\{p_i\}_{i=1}^n$ be distinct points in $\mathbb{C}$ and let $U:=\mathbb{C}\setminus\{p_1,..,p_n\}$. Let A be the vector space of real harmonic functions on U and let $B\subset A$ be the subspace of real parts of complex analytic functions on U. Find the dimension of the quotient vector space A/B, giver a basis of this quotient space, and prove that it is a basis.

Proof. Fix $\psi(z)\in A$ then define

$$g(z):=\partial_x\psi(z)-i\partial_y\psi(z)$$

then observe that this function satisfies the Cauchy-Riemann Equations since ψ is harmonic. And as $g(z)$ is real differentiable, we see that is must be holomorphic. Now notice that we have isolated singularities at $\{p_i\}_{i=1}^n$, so for each of these isolated singularities define c_j as the residue of g at p_i. Then notice

$$h(z):=g(z)-\sum_{j=1}^{n}\frac{c_j}{z-p_j}$$

is a holomorphic function that integrates to zero over any closed curve $\gamma\subset U$ thanks to the Residue Theorem. Therefore, $h(z)$ has a primitive which we denote by $u(z)$. Now observe if we define

$$\tilde{u}(z):=\psi(z)-\sum_{j=1}^{n}c_j\log|z-p_j|\Rightarrow\partial_x\tilde{u}(z)-i\partial_y\tilde{u}(z)=h(z)$$

And the Cauchy Riemann Equations tells us that

$$u'(z)=\frac{\partial u}{\partial x}=\partial_x\text{Re}(u(z))-i\partial_y\text{Re}(u(z))=h(z)$$

i.e. $\tilde{u}$ is the real part of $u(z)$ up to some positive constant. So it follows that $\tilde{u}$ is the real part of a holomorphic function. Therefore, we have shown that the set of functions

$$\{\log|z-p_j|\}_{j=1}^n$$

spans A/B. And they are also linearly independent. Indeed, if

$$\sum_{j=1}^{n}c_j\log|z-p_j|=0$$

So taking exponentials give

$$\prod_{j=1}^{n}|z-p_j|^{c_j}=1$$

Say $c_j\neq 0$ then taking $z=p_j$ gives $0=1$, so we must have $c_j=0$ for all j. Therefore, $\{\log|z-p_j|\}_{j=1}^n$ is a basis of $A\setminus B$, so this vector space has dimension n with the above basis.

□

Problem 3. For an $f : \mathbb{R} \to \mathbb{R}$, $f \in L^1(\mathbb{R})$ define the Hardy-Littlewood maximal functions as

$$(Mf)(x) := \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| dy$$

Prove that it has the following property: There is a constant A such that for any $\lambda > 0$,

$$m(\{x \in \mathbb{R} : (Mf)(x) > \lambda\}) \leq \frac{A}{\lambda} ||f||_{L^1}$$

Proof. Let $m(A)$ denote the Lebesgue Measure of the measurable subset A. We first need the following covering lemma:

Lemma: Given a finite collection of balls $\{B_i\}_{i=1}^N$ then there exists a collection of balls $\{B_{i_j}\}_{j=1}^m$ that are disjoint such that

$$m(\bigcup_{i=1}^N B_i) \leq 3 \sum_{j=1}^m m(B_{i_j})$$

Proof of Lemma: Notice if B and B' are balls such that they intersect with the radius of B greater than or equal to B' then B' is contained in the ball concentric with B with 3 times its radius. Now we proceed with a greedy algorithim. Choose B_{i_1} such that $m(B_{i_1}) \geq m(B_j)$ for any j then consider all the other balls that intersect B_{i_1}. Then those balls are contained in a concentric ball of B_{i_1} with 3 times the radius denoted by $\tilde{B_{i_1}}$. Now iterate this process of choosing the maximal remaining balls (maximal in the sense of volume) we eventialy stop to obtian a collection $\{\tilde{B_{i_1}}\}$ such that they cover $\bigcup_{i=1}^N B_i$ and are disjoint such that

$$m(\bigcup_{i=1}^N B_i) \leq m(\bigcup_{j=1}^m \tilde{B_{i_j}}) = \sum_{j=1}^m m(\tilde{B_{i_j}}) = 3 \sum_{j=1}^m m(B_{i_j})$$

and the covering lemma is proven.

Now fix a compact subset $\lambda > 0$ and fix a comapct subset $K \subset \{x : (Mf)(x) > \lambda\}$. Now by the translation continuity of the lebegsue integral we know for any $x \in K$ there exists an $\varepsilon_x > 0$ such that $B_{\varepsilon_x}(x) \subset \{x : (Mf)(x) > \lambda\}$. Therefore, by compactness we can find a finite subcover say B_{ε_i} for $i = 1, .., N$. This implies that by our covering lemma we can find a collection $\tilde{B}_{\varepsilon_{i_j}}$ such that they cover the original balls and the measure inequality holds. Therefore,

$$m(K) \leq \sum_{i=1}^N m(B_{\varepsilon_i}) \leq 3 \sum_{j=1}^n m(\tilde{B_{\varepsilon_{i_j}}}) \leq \frac{3}{\lambda} \int_{\bigcup \tilde{B_{\varepsilon_{i_j}}}} |f(x)| \leq \frac{3}{\lambda} ||f||_{L^1}$$

where the last inequality is due to the balls are disjoint. And the second last inequality is due to

$$\int_{\tilde{B}_{\varepsilon_{i_j}}} |f(x)| \geq \alpha m(\tilde{B}_{\varepsilon_{i_j}})$$

and all the sets are disjoint. Now we use the inner regularity of the Lebesgue measure as its a Radon Measure to conclude. □

□

Problem 4. Let $f(z)$ be a continuous function ont he closed unit disk $\overline{D}$ such that $f(z)$ is analytic on the open disk $\mathbb{D}$ and $f(0) \neq 0$.

(1) Prove that if $0 < r < 1$ and if $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta \geq \log |f(0)|$$

(2) Use (a) to prove that $m(\{\theta \in [0, 2\pi] : f(e^{i\theta} = 0\}) = 0$

Proof. Let $f(z) : \Omega \to \mathbb{C}$ be holomorphic then we claim $\log(|f(z)|)$ is subharmonic. That is it satisfies the mean value inequality: if $z \in \mathbb{C}$ and $\overline{B_\varepsilon(z)} \subset \Omega$ then

$$\log|f(z)| \le \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log|f(\varepsilon e^{i\theta} + z)| d\theta$$

Indeed, if $f(p) \neq 0$ then on a small ball around p it's non-zero so locally we can write $\log|f(z)|$ as the real part of a holomorphic function, so $\log|f(z)|$ is harmonic and the inequality is an equality. Now if $f(p) = 0$ then $\log|f(p)| = -\infty$ and the inequality is obvious. So this implies $\log|f(z)|$ is locally sub-harmonic which is equivalent to being sub-harmonic. Now apply this claim to $z = 0$ to get a). An alternative approach would be to derive Jensen's formula. □

Proof. Define the cut off function

$$g_n(z) := \max(\log|f(z)|, -n)$$

and notice that is is continuous and $g_n(z) \to \log|f(z)|$ as $n \to \infty$. Now observe by continuity and Fatou's lemma that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_n(e^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \liminf_{r\to 1} |g_n(re^{i\theta})| \le \liminf_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |g_n(re^{i\theta})|$$

Also observe that $g_n(z)$ is still sub-harmonic since it is the maximum of two sub-harmonic functions, so we have

$$\frac{1}{2\pi}\int_0^{2\pi} g_n(re^{i\theta}) d\theta \ge g_n(0)$$

and as $f \in C(\overline{\mathbb{D}})$ we have that f is bounded. So we have that if f^+, f^- denotes the positive and negative part of f then

$$\frac{1}{2\pi}\int_0^{2\pi} g_n^+(re^{i\theta}) - g_n^-(re^{i\theta}) d\theta \ge g_n(0) \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} g_n^-(re^{i\theta}) d\theta \le -g_n(0) + \frac{1}{2\pi}\int_0^{2\pi} g_n^+(re^{i\theta}) d\theta$$

Notice that f bounded implies $\log|f(z)|$ is bounded above, so we have

$$\frac{1}{2\pi}\int_0^{2\pi} g_n^-(re^{i\theta}) \le -g_n(0) + C = -f(0) + C$$

for n large since $f(0) \neq 0$. So it follows that for n large that $g_n(re^{i\theta}) \in L^1([0, 2\pi])$ with a L^1 bound independent of n or r when n is large i.e.

$$\frac{1}{2\pi}\int_0^{2\pi} |g_n(e^{i\theta})| d\theta \le C$$

so now again using Fatou's Lemam gives

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|f(e^{i\theta})|| d\theta = \frac{1}{2\pi}\int_0^{2\pi} \liminf_{n\to\infty} |g_n(e^{i\theta})| d\theta \le \liminf_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |g_n(e^{i\theta})| d\theta \le C$$

so $\log(f(e^{i\theta})) \in L^1([0, 2\pi)$ so $\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0\}$ has zero measure.

□

Problem 5. For $f \in L^2(\mathbb{R})$ and a sequence $\{x_n\} \subset \mathbb{R}$ which converges to zero, define

$$f_n(x) := f(x + x_n)$$

show that $\{f_n\}$ converges to f int he L^2 sense.

Let $W \subset \mathbb{R}$ be a Lebesgue measurable set of positive Lebesgue measure. Show that the set of differences

$$W - W = \{x - y : x, y \in W\}$$

contains an open neighborhood of the origin.

Proof. Note that compactly supported continuous functions are dense in $L^2(\mathbb{R})$. Therefore, for any $\varepsilon > 0$ there exists a $g \in C_c(\mathbb{R})$ such that $||f-g||_{L^2} \leq \varepsilon$. Say $\text{supp}(g) = K$ then by uniform continuity we have that for n sufficiently large we have $\sup_{x\in K} |g(x) - g(x+x_n)| \leq \varepsilon$. Therefore,

$$\int_{\mathbb{R}} |f(x) - f(x+x_n)|^2 dx \leq 4\left\{\int_{\mathbb{R}} |f(x)-g(x)|^2 + |g(x) - f(x+x_n)|^2\right\}$$

$$\leq 16\left\{\int_{\mathbb{R}} |f(x)-g(x)|^2 + |g(x)-g(x+x_n)|^2 + |g(x+x_n) - f(x+x_n)|^2\right\}$$

$$\leq 16\left(\varepsilon + 2\sqrt{\varepsilon}m(K) + \varepsilon\right) \to 0$$

For the second part, let $W_R := W \cap B_R(0)$ then notice

$$\chi_{W_R} \in L^2$$

and for R sufficiently large we have

$$0 < m(W_R) = \int_{\mathbb{R}} \chi_{W_R}^2 dx$$

now assume for the sake of contradiction that $W - W$ does not have a neighborhood of zero then there exists a sequence $x_n \to 0$ such that $x_n \notin W - W$. Therefore,

$$\int_{\mathbb{R}} \chi_W(x)\chi_W(x+x_n)dx = \int_W \chi_W(x+x_n)dx = 0$$

since if $w \in W$ and $w + x_n \in W$ then $x_n = (w+x_n) - (w)$ which would imply $x_n \in W - W$ so the integrand is zero. But we also have from from arguing like in $5a)$ that

$$0 = \int_W \chi_W(x+x_n)dx \to \int_W \chi_{W_R} > 0$$

which is our contradiction. ☐ ☐

Problem 6. Let μ be a finite, positive, regular Borel measure supported on a compact subset of $\mathbb{C}$ and define the Newtonian potential of μ to be

$$U_\mu(z) := \int_{\mathbb{C}} \left|\frac{1}{z-w}\right| d\mu(w)$$

(1) Prove that U_μ exists lebesgue a.e. and that

$$\int\int_K U_\mu(z)dxdy < \infty$$

for every compact $K \subset \mathbb{C}$.

(2) Prove that for almost every horizontal or vertical line $L \subset \mathbb{C}$, $\mu(L) = 0$ and $\int_K U_\mu(z)ds < \infty$ for every compact subset $K \subset L$ where ds denotes Lebesgue linear measure on L.

(3) Define the Cauchy Potential of μ to be

$$S_\mu(z) := \int_{\mathbb{C}} \frac{1}{z-w} d\mu(w)$$

which trivially exists whenever $U_\mu(z) < \infty$. Let R be a rectangle in $\mathbb{C}$ whose four sides are contained in lines L having the conditions of (b). Prove that

$$\frac{1}{2\pi i}\int_{\partial R} S_\mu(z)dz = \mu(R)$$

Proof. Define

$$U_\mu(z) := \int_{\mathbb{C}} \frac{1}{|z-w|} d\mu(w)$$

then if $K \subset \mathbb{C}$ is compact then

$$\int_K U_\mu(z)dxdy = \int_{\mathbb{C}}\int_K \frac{1}{|z-w|}dxdyd\mu(w) \text{ thanks to Tonelli}$$

$$= \int_{\mathbb{C}} \int_{K \cap B_1(w)} \frac{1}{|z-w|} dxdyd\mu(w) + \int_{\mathbb{C}} \int_{K \cap |z-w| \geq 1} \frac{1}{|z-w|} dxdyd\mu(w) = (I) + (II)$$

Then observe

$$\int_{B_1(w)} \frac{1}{|z-w|} dxdy \leq C$$

because $1/|x|$ is integrable om $B_1(0)$. So

$$(I) \leq \int_{\mathbb{C}} C d\mu(w) \leq K$$

since μ is of finite measure and

$$(II) \leq \int_{\mathbb{C}} \int_K 1 dxdyd\mu(w) = m(K)\mu(\mathbb{C}) = K_2 < \infty$$

Therefore, for any compact set K we have $U_\mu \in L^1(K, dxdy)$. Now this implies U_μ is finite a.e. i.e. exists a.e.in K and let $K = \overline{B_r(0)}$ and send $r \to \infty$ to deduce U_μ is finite a.e. on $\mathbb{C}$.

For the second part, assume for the sake of contradiction that there was the set of lines such that $\mu(L) > 0$ had positive (Lebesgue) measure. Then there must exist an $n \in \mathbb{N}$ such that there are infinitely many disjoint lines $\{L_m\}_{m \in \mathbb{N}}$ such that $\mu(L_m) \geq 1/n$. Notice that

$$\mu(\mathbb{C}) \geq \mu(\bigcup_{m \in \mathbb{N}} L_m) = \sum_{m \in \mathbb{N}} \mu(L_m) \geq \sum_{m \in \mathbb{N}} 1/n = \infty$$

which contradicts that μ is finite. Therefore, for almost every horizontal or vertical line we have $\mu(L) = 0$.

Now fix $R > 0$ and $n \in \mathbb{Z}$ then $[n, n+1] \times [-R, R]$ is a compact set of $\mathbb{C}$ so by part $a)$ we have with Fubini-Tonelli

$$\int_n^{n+1} \int_{-R}^{R} U_\mu(x+iy) dxdy < \infty$$

this implies for a.e. y that

$$\int_{-R}^{R} U_\mu(x+iy) dx < \infty$$

i.e. integrating along horizontal lines U_μ is finite for a.e. $y \in [n, n+1]$. For each $n \in \mathbb{Z}$ define

$$Y_R^n := \{y \in [n, n+1] : \int_{-R}^{R} U_\mu(x+iy) dx < \infty\}$$

define $Y_R := \bigcap_{n \in \mathbb{Z}} Y_R^n$ and notice $Y_n^c = \bigcup_{n \in \mathbb{Z}} (Y_R^n)^c$ and each $(Y_R^n)^c$ has measure zero, so Y_R is a set of full measure. And if $y \in Y_R$ we have

$$\int_{-R}^{R} U_\mu(x+iy) dx < \infty$$

i.e. if $L = \{(x, y) : x \in [-R, R]\}$ then

$$\int_L U_\mu(x+iy) dx < \infty$$

Now finally define $Y := \bigcap_{R \in \mathbb{Z}} Y_R$ then arguing as before we see that Y is a set of full measure and if $y \in Y$ then we have for any $R > 0$

$$\int_{-R}^{R} U_\mu(x+iy) dx < \infty$$

so it follows if we let L be the horizontal lines with y coordinates in Y then if K is compact then

$$\int_K U_\mu(x+iy) dx < \infty$$

as desired.

Finally for the third part. Observe that

$$\frac{1}{2\pi i}\int_{\partial R} S_\mu(z)dz = \frac{1}{2\pi i}\int_{\mathbb{C}}\int_{\partial R}\frac{1}{z-w}dzd\mu(w)$$

where Fubini is justified since $S_\mu(z) \in L^1$ and Cauchy's Theorem tells us that

$$\int_{\partial R}\frac{1}{z-w}dz = \begin{cases} 2\pi i \text{ if } w \in R \\ 0 \text{ else} \end{cases}$$

so

$$= \int_{\mathbb{C}} \chi_R(w)d\mu(w) = \mu(R)$$

□

Problem 7. Let H be a Hilbert space and let E be a closed convex subset of H. prove that there exists a unique element $x \in E$ such that

$$||x|| = \inf_{y\in E} ||y||$$

Proof. Let E be a closed subset of H. Then let

$$m := \inf_{y\in E} ||y||$$

and let x_n be a minimizing sequence that is $x_n \in E$ and

$$||x_n|| \to m$$

It suffices to show $\{x_n\}$ is Cauchy since H is complete. Indeed, observe that as E is convex that $(x_n + x_m)/2$ and $(x_n - x_m)/2 \in E$. Therefore, $m \le ||(x_n + x_m)/2|| \le ||x_n/2|| + ||x_m/2|| \to m$ so $||(x_n + x_m)/2|| \to m$. But then by the parallelogram law we have

$$||x_n||^2 + ||x_m|^2 - 2||x_n + x_m||^2 = 2||x_n - x_m||^2$$

and the left hand side approaches to 0 as $n, m \to \infty$. Therefore, the sequence is Cauchy and we are done. □

Problem 8. Let $F(z)$ be a non-constant meromorphic function on $\mathbb{C}$ such that for all $z \in \mathbb{C}$

$$F(z+1) = F(z) \text{ and } F(z+i) = F(z)$$

Let Q be a square with vertices $z, z+1, z+i, z+1+i$ such that F has no poles or zeros on ∂Q. Prove that inside Q the function F has the same number of zeros and poles (counting multiplicities).

Proof. By the argument principle

$$\int_{\gamma_{\partial Q}} \frac{F'(z)}{F(z)}dz = \text{number of zeros in } Q \ - \text{number of poles in } Q$$

counted with multiplicity. But as the above integral is zero due to the periodicity we are done. □

Problem 9. Let

$$A = \{x \in \ell^2 : \sum_{n\ge 1} n|x_n|^2 \le 1\}$$

(1) Show that A is compact in the ℓ^2 topology.
(2) Show that the mapping from A to $\mathbb{R}$ defined by

$$x \mapsto \int_0^{2\pi} |\sum_{n\ge 1} x_n e^{in\theta}| \frac{d\theta}{2\pi}$$

achieves its maximum on A.

Proof. Fix $\{x_n\} \subset A$ where we use the notation x_n refers to the sequence $x_n = \{x_{n,1}, x_{n,2}...\}$. Then notice for all n that $\{x_{n,1}\}_{n\in\mathbb{N}}$ is a bounded sequence. So there is a subsequence n_j^1 such that $\{x_{n_j^1,1}\}_{j\in\mathbb{N}}$ converges to a number denoted by x_1. Now note that again $\{x_{n_j^1,2}\}$ is a bounded subsequence so there exists a subsequence $n_j^2 \subset n_j^1$ with $\{x_{n_j^2,2}\}$ converging to a number x_2. Iterate this process for all n. Define the sequence $x = \{x_1, x_2, .., \}$ and consider the diagonal subsequence $\{x_{n^n}\}$. Then we claim $x_{n^n} \to x$ in ℓ^2. First note $x \in A$ since for any N fixed we have

$$1 \geq \lim_{j\to\infty} \sum_{n=1}^{N} n|x_{j^j,n}|^2 = \sum_{n=1}^{N} n|x_n|^2$$

due to pointwise convergence so taking limits gives

$$1 \geq \sum_{n=1}^{\infty} n|x_n|^2$$

Now observe

$$\begin{aligned}\sum_{n=1}^{\infty} |x_n - x_{j^j,n}|^2 &= \sum_{n=1}^{N} |x_n - x_{j^j,n}|^2 + \frac{N}{N} \sum_{n=N+1}^{\infty} |x_n - x_{j^j,n}|^2 \\ &\leq \sum_{n=1}^{N} |x_n - x_{j^j,n}|^2 + \frac{1}{N} \sum_{n=N+1}^{\infty} n|x_n - x_{j^j,n}|^2 \\ &\leq \sum_{n=1}^{N} |x_n - x_{j^j,n}|^2 + \frac{2}{N}\end{aligned}$$

now fix $\varepsilon > 0$ and choose N so that $2/N < \varepsilon/2$ and now make j so large such that the first sum is $< \varepsilon/2$ which can be done thanks to pointwise convergence. Therefore, $x_{j^j} \to x$ in ℓ^2 so A is compact.

For the second part observe that since $(e^{in\theta}, e^{im\theta}) = 2\pi\delta_{nm}$ that

$$||\sum_{n=1}^{\infty} a_n e^{in\theta}||_{L^2}^2 = (\sum_{n=1}^{\infty} a_n e^{in\theta}, \sum_{n=1}^{\infty} a_n e^{in\theta}) = 2\pi \sum_{n=1}^{\infty} |a_n|^2$$

so it follows that for

$$f(x) := \int_0^{2\pi} \left| \sum_{n\geq 1} x_n e^{in\theta} \right| \frac{d\theta}{2\pi}$$

that

$$|f(x) - f(y)| = \left| \int_0^{2\pi} \left| \sum_{n\geq 1} x_n e^{in\theta} \right| - \left| \sum_{n\geq 1} y_n e^{in\theta} \right| \frac{d\theta}{2\pi} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right| d\theta$$

$$\leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} \left(\sum_{n\geq 1} (x_n - y_n) e^{in\theta} \right)^2 \right)^{1/2} = \sqrt{2\pi} \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2} = \sqrt{2\pi}||x - y||_{\ell^2}$$

where the equality is due to $\{e^{in\theta}/2\pi\}$ being an orthonormal basis of $L^2([0, 2\pi])$ which shows that f is continuous so it attains its max over A since A is compact. □

Problem 10. Let $\Omega \subset \mathbb{C}$ be a connected open set, let $z_0 \in \Omega$, and let $\mathcal{U}$ be the set of positive harmonic functions U on Ω such that $U(z_0) = 1$. Prove for every compact set $K \subset \Omega$ there is a finite constant M (depending on Ω, z_0, and K) such that

$$\sup_{U\in\mathcal{U}} \sup_{z\in K} U(z) \leq M$$

You may use Harnack's inequality for the disk without proving it, provided you state it correctly.

Proof. Harnack's Inequality on a ball states that if $B_r(w)$ is a ball of radius r and center w and A is the set of positive harmonic functions on $B_r(w)$ then there exists a constant $C(r)$ depending only on r such that

$$\sup_{z\in B_r(w)} u(z) \leq C(r) \inf_{z\in B_r(w)} u(z)$$

for any $u \in A$. Now if $K \subset \Omega$ is compact then there exists a $\delta > 0$ such that $d(K,\Omega) > 2\delta$. Then the collection $\{B_\delta(z)\}_{z\in K}$ is an open cover of K so there exists a finite sub-cover $\{B_\delta(z_k)\}_{k=0}^N$ where we adjoined $B_{\delta/2}(z_0)$. Therefore, by Harnack's on each of these balls we have

$$\sup_{z\in B_\delta(z_i)} u(z) \leq C \inf_{z\in B_{\delta/2}(z_i)} u(z)$$

Now as Ω is connected in $\mathbb{C}$ it is path connected, so after a cyclic permutation on $\{z_i\}_{i=1}^N$ let γ be a piece wise line that connects z_i to z_{i+1} for $i = 0, .., n-1$ where $B_{\delta/2}(z_i) \cap B_{\delta/2}(z_{i+1}) \neq \emptyset$. It therefore, follows that

$$\sup_{z\in B_\delta(z_0)} u(z) \leq C$$

since $u(z_0) = 1$, which implies as $B_\delta(z_0) \cap B_\delta(z_1) \neq \emptyset$ that

$$\sup_{z\in B_\delta(z_1)} u(z) \leq C \sup_{B_\delta(z_0)} u(z) \leq C^2$$

then again as all the balls are not disjoint we can iterate to get

$$\sup_{z\in B_\delta(z_i)} \leq C^{i+1}$$

Therefore, it follows that

$$\sup_{z\in \bigcup_{i=0}^N B_\delta(z_i)} u(z) \leq \max\{C, C^{N+1}\} := A$$

and as K is contained in these balls we have

$$\sup_{z\in K} u(z) \leq A$$

as desired. □

Problem 11. Let $\phi : \mathbb{R} \to \mathbb{R}$ be in $C_c(\mathbb{R})$. Prove that there is a constant A such that

$$||f * \phi||_{L^q} \leq A||f||_{L^p} \text{ for all } 1 \leq p \leq q \leq \infty \text{ and } f \in L^p$$

Problem 11. We first prove Young's Convolution Inequality: If $p, r, q \geq 1$ are such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

with $f \in L^p$ and $g \in L^q$. Then we have

$$||f * g||_{L^r} \leq ||f||_{L^p}||g||_{L^q}$$

Indeed, observe that one has

$$|f * g| \leq \int |f(y)g(x-y)|dy = \int |f(y)|^{1-p/r}|g(x-y)|^{1-q/r}|f(y)|^{p/r}|g(x-y)|^{q/r}dy$$
$$= \int (|f(y)|^p|g(x-y)|^q)^{1/r} (|f(y)|^{(r-p)})^{1/r}(|g(x-y)|^{(r-q)})^{1/r}$$

Notice that

$$\frac{1}{r} + \frac{r-p}{pr} + \frac{r-q}{rq} = \frac{1}{r} + \frac{1}{p} - \frac{1}{r} + \frac{1}{q} - \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$$

so we can apply Holder's Inequality to obtain

$$|f * g| \leq (||f||_p^{p/r}||g||_q^{q/r})(||f||_p^{\frac{r-p}{r}})(||g||_q^{\frac{r-q}{r}})$$

So now we have from the previous computation

$$\int |f * g|^r dx \leq \int \left| \int f(x-y)g(y)dy \right|^r$$

$$\leq \int \left(\int |f(x-y)|^{p/r} |g(y)|^{q/r} |f(x-y)|^{1-p/r} |g(y)|^{1-q/r} dy \right)^r$$
$$\leq ||f||_p^{r-p} ||g||_q^{r-q} \int \int |f(x-y)|^p |g(y)|^q dx dy$$
$$\leq ||f||_p^{r-p} ||g||_q^{r-q} \int |g(y)|^q \int |f(x-y)|^p$$
$$= ||f||_p^r ||g||_q^r$$

so we have

$$||f * g||_{L^r} \leq ||f||_p ||g||_q$$

Now back to the problem, define $r \geq 1$ such that

$$\frac{1}{r} := 1 + \frac{1}{q} - \frac{1}{p}$$

then we can define

$$A := ||\varphi||_{L^r}$$

to get the desired result.

Problem 12. Let F be a function from the open unit disk $\mathbb{D}$ to $\mathbb{D}$ such that whenever z_1, z_2 and z_3 are distinct points of $\mathbb{D}$ there exists an analytic function f_{z_1,z_2,z_3} from $\mathbb{D}$ to $\mathbb{D}$ such that

$$F(z_j) = f_{z_1,z_2,z_3}(z_j) \text{ for } j = 1, 2, 3$$

Prove that F is analytic at every point of $\mathbb{D}$.

Proof. By Montel's Theorem the family $\{f_{z_1,z_3,z_3}\}_{z_1,z_2,z_3 \in \mathbb{D}}$ is uniformly Lipschitz. Therefore, the difference quotient

$$\frac{F(z_n) - F(z)}{z_n - z} = \frac{f_{z_n,z,w}(z_n) - f_{z_n,z,w}(z)}{z_n - z}$$

is uniformly bounded where $w \neq z_n$ or z. So along a sub-sequence the difference quotients converge. Now let $z_{i(n)}$ be a sub-sequence such that the difference quotient converges and $z_{j(n)}$ be another sub-sequence. Then the family

$$\{f_{z_{i(n)},z_{j(n)},z}\}_{n \in \mathbb{N}}$$

is pre-compact by Montel's Theorem. So for $\delta > 0$ so small such that $\overline{B_\delta(z)} \subset \mathbb{D}$ we have that along a subsequence n_k the functions $f_{i(n),j(n),z} := f_{z_{i(n)},z_{j(n)},z}$ converge uniformly to a holomorphic function f on $\overline{B_\delta(z)}$. This implies $f'_{i(n),j(n),z} \to f'$ uniformly on $\overline{B_\delta(z)}$ (denote the convergent sub-sequence as n_k. So as

$$\frac{F(z_{i(n_k)}) - F(z)}{z_{i(n_k)} - z} = \frac{f_{i(n_k),j(n_k),z}(z_{i(n_k)}) - f_{i(n_k),j(n_k),z}(z)}{z_{i(n_k)} - z} \to f'(z)$$

and similarly

$$\frac{F(z_{j(n_k)}) - F(z)}{z_{i(n_k)} - z} = \frac{f_{i(n_k),j(n_k),z}(z_{j(n_k)}) - f_{i(n_k),j(n_k),z}(z)}{z_{i(n_k)} - z} \to f'(z)$$

where the convergence is due to uniform convergence of this families derivative. Therefore, the limit on any two sub-sequence is the same, so we know the limit exists and is $f'(z)$. Therefore, F is holomorphic on $\mathbb{D}$.

□

Problem 13. Let X and Y be two Banach Spaces. Let $A : X \to Y$ be a compact operator. Suppose X is reflexive and X^* is separable. Show that A is compact iff for every bounded sequence $\{x_n\}$ there exists a sub-sequence $\{x_{n_j}\}$ and a vector $\phi \in X$ such that $x_{n_j} = \phi + r_{n_j}$ and $Ar_{n_j} \to 0$ in Y

Proof. Let A be compact and $\{x_n\}$ bounded. By Banach Alagou (since X is reflexive and X^* is separable so we can upgrade weak* compactness to weak* subsequential compactness) the closed ball $\overline{B_M(0)} \subset X$ is weakly compact for $M > 0$. And as X^* is separable, we know that $\overline{B_M(0)}$ is weakly subsequentially compact, so there is a subsequence $\{x_{n_j}\}$ such that $x_{n_j} \rightharpoonup \phi \in \overline{B_M(0)}$. Define $r_{n_j} := (x_{n_j} - \phi)$ then $x_{n_j} = \phi + r_{n_j}$. As A is compact we can by looking at a furhter subsequence if necessary assume that Ax_{n_j} converges. So

$$\lim_{j\to\infty} A(x_{n_j}) = \lim_{j\to\infty} A(\phi + r_{n_j}) = A(\phi) + \lim_{j\to\infty} A(r_{n_j})$$

Thus it satisfies to show

$$\lim_{j\to\infty} A(x_{n_j}) = A(\phi)$$

Indeed, let $f \in Y^*$ and define $y := \lim_{j\to\infty} A(x_{n_j})$

$$f(y) = \lim_{j\to\infty} f(A(x_{n_j})) = \lim_{j\to\infty} (A^t f)(x_{n_j}) = A^t f(\phi) = f(A(\phi))$$

which implies $y = A\phi$ so this direction is proven.

The other direction implies $A(x_{n_j}) \to A(\phi)$ by linearity so A is compact.

□

2. Fall 2010

Problem 1. For this problem, consider just the Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$ together with the Lebesgue emasure.

(1) State Fatou's Lemma (no proof is required).
(2) State and prove the Dominated Convergence Theorem.
(3) Given an example where $f_n(x) \to 0$ a.e., but $\int f_n(x)dx \to 1$

Problem 1a. Fatou's Lemma states that if $f_n : [0,1] \to [0,\infty)$ are a sequence of Lebesgue measurable functions then we have

$$\int_0^1 \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \int_0^1 f_n(x)dx$$

Problem 1b. DCT states that if $f_n \in L^1([0,1])$ are such that $f_n \to f$ pointwise a.e. and if there exists a $g \in L^1([0,1])$ such that $|f_n| \leq g$ then we have

$$f_n \to f \text{ in } L^1([0,1])$$

To see the proof first observe that $2g - |f - f_n| \geq 0$ so we have by Fatou's Lemma

$$\int_0^1 2g(x)dx = \int_0^1 \liminf_{n\to\infty} \{2g - |f - f_n|\}\, dx \leq \liminf_{n\to\infty} \int_0^1 2g - |f - f_n| dx$$

so we have

$$0 \leq \liminf_{n\to\infty} \int_0^1 -|f - f_n| dx = -\limsup_{n\to\infty} \int_0^1 |f - f_n| \text{ i.e. } 0 \geq \limsup_{n\to\infty} \int_0^1 |f - f_n|$$

so $f_n \to f$ in $L^1([0,1])$ $\square$.

Problem 1c. Take $f(x) = n\chi_{[0,1/n]}(x)$ $\square$

Problem 2. Prove the following form of Jensen's inequality: If $f : [0,1] \to \mathbb{R}$ is continuous then

$$\int_0^1 e^{f(x)}dx \geq \exp\{\int_0^1 f(x)dx\}$$

Proof. As e^x is convex we have that for any $y, z \in \mathbb{R}$

$$e^y \geq e^z + e^z(y - z)$$

Taking $z = \int_0^1 f(x)$ and $y = f(x)$ we have

$$e^{f(x)} \geq e^{\int_0^1 f(x)dx} + e^{\int_0^1 f(x)dx}(f(x) - \int_0^1 f(x)dx)$$

so integrating this gives

$$\int_0^1 e^{f(x)}dx \geq \exp\left\{\int_0^1 f(x)dx\right\}$$

as desired $\square$.

$\square$

Problem 3. Consider the following sequence of functions:

$$f_n : [0,1] \to \mathbb{R} \text{ by } f_n(x) = \exp(\sin(2\pi nx))$$

(1) Prove that f_n converges weakly in $L^1([0,1])$
(2) Prove that f_n converges weakly-* in $L^\infty([0,1])$ viewed as the dual of $L^1([0,1])$

Proof. Observe that

$$\int_0^1 \exp(\sin(2\pi nx))dx = \frac{1}{n}\int_0^n \exp(\sin(2\pi y))dy = \frac{1}{n}n\int_0^1 \exp(\sin(2\pi y))dy = \int_0^1 f_1(x)dx$$

So it follows that $L : (L^1([0,1]))^* \to \mathbb{R}$ defined via

$$L(f) := \int_0^1 f(x)dx$$

satisfies $L(f_n) \to C := \int_0^1 f_1(x)dx$ so we make the guess that $f_n \rightharpoonup C$. Let $L \in (L^1([0,1]))^*$ then by Risez-Representation Theorem there exists a $g \in L^\infty([0,1])$ such that

$$L(f) = \int_0^1 gf dx$$

for all $f \in L^1([0,1])$. Note that $L^\infty([0,1]) \subset L^1([0,1])$, so it suffices to show the problem $L^1([0,1])$ functions. We will first prove the problem when $g \in L^\infty([0,1]) \cap C([0,1])$ and use density to conclude (in L^1 norm). Indeed, observe

$$L(f_n) = \int_0^1 g(x)\exp(\sin(2\pi nx))dx = \frac{1}{n}\int_0^n g(\frac{x}{n})\exp(\sin(2\pi x))dx = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{(k+1)} g(\frac{x}{n})\exp(\sin(2\pi x))dx$$

so

$$L(f_n) - \frac{C}{n}\sum_{k=0}^{n-1} g(\frac{k}{n}) = \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1}\left(g(\frac{x}{n}) - g(\frac{k}{n})\right)\exp(\sin(2\pi x))dx$$

Therefore, by uniform continuity if $\varepsilon > 0$ there exists a $\delta > 0$ such that if $|x-y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon$. Choose N so large such that if $n \geq N$ then $1/n < \delta$ then

$$|L(f_n) - \frac{C}{n}\sum_{k=0}^{n-1} g(\frac{k}{n})| \leq \frac{1}{n}\sum_{k=0}^{n-1}\int_k^{k+1} \varepsilon\exp(\sin(2\pi x))dx = C\varepsilon$$

so as

$$\frac{C}{n}\sum_{k=0}^{n-1} g(\frac{k}{n}) \to C\int_0^1 g(x)dx$$

it follows that $L(f_n) \to C\int_0^1 g(x)dx = L(C)$. Now the general case follows from density. Indeed, if $f_n \in C([0,1])$ such that $f_n \to f$ in $L^1([0,1])$ then we have

$$\int_0^1 |(f_m - f)\exp(\sin(2\pi nx))| \lesssim ||f_m - f||_{L^\infty}$$

so

$$\int_0^1 |f\exp(\sin(2\pi nx)) - Cf|dx \leq \int_0^1 |(f-f_m)\exp(\sin(2\pi nx))| + \int_0^1 |f_m\exp(\sin(2\pi nx)) - Cf_m|dx + \int_0^1 |Cf - Cf_m| \to 0$$

so if $f \in L^1([0,1])$ then $L(f_n) \to \int_0^1 Cf(x)dx$ i.e. $(\sin(2\pi nx)) \rightharpoonup C$ in L^1.

Note we proved the second part in the proof of the first part. □

Problem 4. Let T be a linear transformation on $C_c(\mathbb{R})$ such that

$$||Tf||_{L^\infty} \leq ||f||_{L^\infty} \text{ and } m(\{x \in \mathbb{R} : |Tf(x)| > \lambda\}) \leq \frac{||f||_{L^1}}{\lambda}$$

Prove that for all $f \in C_c(\mathbb{R})$

$$\int_{\mathbb{R}} |Tf(x)|^2 \lesssim \int_{\mathbb{R}} |f(x)|^2$$

Problem 4. Fix $f \in C_c^0(\mathbb{R})$ and assume that $f \geq 0$ then by the Layer Cake Decomposition we have

$$\int_{\mathbb{R}} .|Tf(x)|^2 dx = 2\int_{\mathbb{R}} s|x : |Tf(x)| \geq s|ds$$

Now we decompose $f(x) = g(x,s) + h(x,s)$ where $g(x,s) := \min\{f(x), s/2\}$ and $h(x,s) = 0$ if $f(x) < s/2$ and $h(x,s) = f(x) - s/2$ for $f(x) \geq s/2$. Then observe that since $|Tf| \leq |Tg| + |Th|$ so if $|Tf| > s$ then we must have $|Tg| > s/2$ or $|Th| > s/2$ so

$$\{x : |Tf(x)| > s\} \subset \{x : |Tg(x)| > s/2\} \cup \{x : |Th(x)| > s/2\}$$

But as $||Tg||_{L^\infty} \leq ||g||_{L^\infty} = s/2$ we conclude the first set is a null set. Therefore,

$$\int_{\mathbb{R}} |Tf(x)|^2 dx \leq 2\int_{\mathbb{R}} s|x : |Th(x)| > s/2|ds \leq 4\int_{\mathbb{R}} ||h(x,s)||_{L^1(\mathbb{R})} ds$$
$$= 4\int_{\mathbb{R}}\int_{\{x:f(x)\geq s/2\}} (f(x) - s/2)dxds \leq 4\int_{\mathbb{R}}\int_{\{x:f(x)\geq s/2\}} f(x)dxds$$
$$= 4\int_{\mathbb{R}}\int_0^{2f(x)} f(x)dsdx = 8\int_{\mathbb{R}} |f(x)|^2 dx$$

For a general f decompose it into its positive and imaginary part □

Problem 5. Let $\mathbb{R} \setminus \mathbb{Z}$ denote the torus (whose elements we will write as cosets) and fix an irrational number $\alpha > 0$

(1) Show that
$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha + \mathbb{Z}) = \int_0^1 f(x + \mathbb{Z})dx$$
for all $f \in C(\mathbb{R} \setminus \mathbb{Z})$

(2) Show that the conclusion is also true when f is the characteristic function of a closed interval.

Proof. By Stone-Weiestrass trigonometric polynomials are dense on $\mathbb{R}/\mathbb{Z} \simeq [0,1]$. By linearity it suffices to show that $e^{2\pi ikx}$ for $k \in \mathbb{Z}$ satisfies the desired conclusion to have it hold for all trigonometric polynomials. Then

$$\int_0^1 e^{2\pi ikx} dx = \int_0^1 \cos(2\pi kx) + i\sin(2\pi kx)dx = 0$$

and we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp(2\pi ikn\alpha) = \frac{1}{N}\sum_{n=0}^{N-1} (\exp(2\pi ik\alpha))^n = \frac{1}{N}\left(\frac{1 - \exp(2\pi ikn\alpha)}{1 - \exp(2\pi ik\alpha)}\right)$$

now because α is irrational we have the denominator is never 0, so as the numerator is bounded by 2 we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \exp(2\pi ikn\alpha) \to 0$$

so we have the desired conclusion for all trigonometric polynomials. Therefore, it follows for all continuous functions by density. Since the sum operator is uniformly bounded for any N by the L^∞ norm and the integral is continuous w.r.t. uniform convergence.

For the second part, if $f(x) := \chi_{[a,b]}(x)$ then it is obvious that there exists a sequence of functions $0 \leq u_n(x) \leq f(x) \leq v_n(x) \leq 1$ with $u_n(x)$ and $v_n(x)$ pointwise converging to $f(x)$ where u_n, v_n are continuous. Then we have

$$\int_0^1 u_n(x)dx = \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} u_n(n\alpha+\mathbb{Z}) \leq \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) \leq \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} v_n(n\alpha+\mathbb{Z}) = \int_0^1 v_n(x)dx$$

Then as all these functions are dominated by 1 DCT tells us that

$$\lim_{n\to\infty}\int_0^1 u_n(x) = \lim_{n\to\infty}\int_0^1 v_n(x) = \int_0^1 f(x)$$

so it follows that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f(n\alpha+\mathbb{Z}) = \int_0^1 f(x)$$

□

Problem 6. Consider the Hilbert space

$$\mathcal{H} := \{f : \overline{\mathbb{D}} \to \mathbb{C} : f(z) = \sum_{k=0}^{\infty}\hat{f}(k)z^k \text{ with } ||f||^2 = \sum_{k=0}^{\infty}(1+|k|^2)|\hat{f}(k)|^2 < \infty\}$$

(1) Prove that the linear functional $L : f \mapsto f(1)$ is bounded.
(2) Find the element $g \in \mathcal{H}$ representing L.
(3) Sow that $f \mapsto \mathrm{Re}L(f)$ achieves its maximal value on the set

$$\mathcal{B} := \{f \in \mathcal{H} : ||f|| \leq 1 \text{ and } f(0) = 0\}$$

that this maximum occurs at a unique point, and determine its maximal value.

Problem 6a. Observe that

$$f(1) = \sum_{k=0}^{\infty}\hat{f}(k)$$

and

$$|f(1)| \leq \sum_{k=0}^{\infty}|\hat{f}(k)|\frac{\sqrt{1+|k|^2}}{\sqrt{1+|k|^2}} \leq \left(\sum_{k=0}^{\infty}|\hat{f}(k)|^2(1+|k|^2)\right)^{1/2}\left(\sum_{k=0}^{\infty}\frac{1}{1+|k|^2}\right)^{1/2} \lesssim ||f||$$

where the second inequality is due to Holder and the last one is due to the sum on the right is a convergent sum.

Problem 6b. Observe that the inner product is

$$(f,g) = \sum_{k=0}^{\infty}(1+|k|^2)\hat{f}(k)\overline{\hat{g}(k)}$$

so if $\overline{\hat{g}(k)} = \frac{1}{1+|k|^2}$ then

$$(f,g) = f(1)$$

Therefore, define

$$g(z) := \sum_{k=0}^{\infty}\frac{z^k}{1+|k|^2}$$

which is a well defiend function on $\overline{D}$ since $\sum\frac{1}{1+|k|^2}$ converges absolutely and $||g||^2 = \sum\frac{1}{1+|k|^2} < \infty$ so g is the desired element representing L.

Problem 6c. Note on $\mathcal{B}$ that we have

$$h(z) := \sum_{k=1}^{\infty}\frac{z^k}{1+|k|^2}$$

is such that

$$(f,g) = f(1)$$

for $f \in \mathcal{B}$ by an identical argument as above since $f(0) = 0$. Now by Cauchy-Schwarz we have

$$|\mathrm{Re}L(f)| = |\mathrm{Re}f(1)| = |\mathrm{Re}(h,f)| \leq ||h||$$

since $||f|| \leq 1$. Now taking $f := h/||h||$ shows that this maximal value occurs. This maximal is also unique since equality in Cauchy-Schwarz inequality happens iff $f = \lambda h$ for some λ and by the constraint $||\lambda h|| = 1$ implies $f = h/||h||$.

Problem 7. Suppose $f : \mathbb{C} \to \mathbb{C}$ is continuous on $\mathbb{C}$ and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. Prove that f is entire.

Proof. This is a standard application of Morrera's Theorem/Schwarz Reflection Principle. □

Problem 8. Let $A(\mathbb{D})$ be the $\mathbb{C}$-vector space of all holomorphic functions on $\mathbb{D}$ and suppose that $L : A(\mathbb{D}) \to C$ is a multiplicative linear functional, that is

$$L(af + bg) = aL(f) + bL(g) \text{ and } L(fg) = L(f)L(g)$$

for all $a, b \in \mathbb{C}$ and all $f, g \in A(\mathbb{D})$. If L is not identically zero, show that there is a $z_0 \in \mathbb{D}$ so that $L(f) = f(z_0)$ for all $f \in A(\mathbb{D})$.

Proof. Observe that for any $f \in A(\mathbb{D})$ that

$$L(f) = L(f \cdot 1) = L(f)L(1) \Rightarrow L(1) = 1$$

So linearlity implies for any constant C that we have $L(C) = 1$. Define

$$z_0 := L(z)$$

then observe

$$L(z^n) = L(\prod_{i=1}^{n} z) = (z_0)^n$$

so linearity implies for any polynomial P we have

$$L(P(z)) = P(z_0)$$

Now we show $z_0 \in \mathbb{D}$. Assume $z_0 \notin \mathbb{D}$ then $\frac{1}{z - z_0} \in A(\mathbb{D})$ so

$$1 = L(1) = L(\frac{z - z_0}{z - z_0}) = L(z - z_0)L(\frac{1}{z - z_0})$$

but as $L(z - z_0) = 0$ we arrive at a contradiction. Therefore, $z_0 \in \mathbb{D}$, so we have for any $f \in A(\mathbb{D})$ that

$$L(f - f(z_0)) = L((z - z_0)g(z)) = 0$$

where $g(z) \in A(\mathbb{D})$. So in particular linearity implies $L(f) = L(f(z_0)) = f(z_0)$ for any $f \in A(\mathbb{D})$. □

Problem 9. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be holomorphic in $\mathbb{D}$. Show that if

$$\sum_{n=2}^{\infty} n|a_n| \leq |a_1|$$

with $a_1 \neq 0$ then f is injective.

Proof. If $f'(z_0) \neq 0$ for all $z_0 \in \mathbb{D}$ then the inverse function theorem implies $f(z)$ is locally injective on $\mathbb{D}$ and the open mapping theorem implies $f(z)$ is injective on $\mathbb{D}$. So we compute

$$|f'(z)| = \left|\sum_{n=1}^{\infty} na_n z^{n-1}\right| = \left|a_1 + \sum_{n=2}^{\infty} na_n z^{n-1}\right| \geq |a_1| - |\sum_{n=2}^{\infty} na_n z^{n-1}|$$

Now for any $z \in \mathbb{D}$ we have $|z| < 1$ so we have by the triangle inequality

$$|\sum_{n=2}^{\infty} na_n z^{n-1}| \leq \sum_{n=2}^{\infty} n|a_n||z| < \sum_{n=2}^{\infty} n|a_n|$$

from which it follows that

$$|f'(z)| > |a_1| - \sum_{n=2}^{\infty} n|a_n| \geq 0$$

so it follows that $f'(z) \neq 0$ on $\mathbb{D}$, so f is injective.

□

Problem 10. Prove that the punctured disk $\{z \in \mathbb{C} : 0 < |z| < 1\}$ and the annulus given by $\{z \in \mathbb{C} : 1 < |z| < 2\}$ are not conformally equivalent.

Proof. Assume for the sake of contradiction that there exists a conformal map φ from the punctured unit disk to the annulus $A_{1,2} := \{z \in \mathbb{C} : 1 < |z| < 2\}$. Then notice this implies φ is bounded, so by Riemann Extension Theorem φ extends to a holomorphic map on $\mathbb{D}$. Note by the mean value property we deduce that $1 < |\varphi(0)| < 2$ so φ is still a surjective map onto $A_{1,2}$. Now observe it is still injective since if $\varphi(0) = \varphi(w)$ for some $w \neq 0$ the open mapping principle tells us by taking small balls aroinnd 0 and w that φ is not injective on the punctured unit disk which is a contradiction. Therefore, we have found a homeomorphism from the unit disk to the annulus $A_{1,2}$, which is a contradiction since homeomorphisms preserve simply connectedness. □ □

Problem 11. Let $\Omega \subset \mathbb{C}$ be a non-empty open connected set. If $f : \Omega \to \mathbb{C}$ is harmonic such that f^2 is also harmonic, show that either f or $\overline{f}$ is holomorphic on Ω.

Problem 11. Define the Wirtinger Derivatives

$$\partial_z := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\partial_{\overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and the Cauchy Riemann equations imply f is holomorphic iff $\partial_{\overline{z}} f = 0$ and $\overline{f}$ is holomorphic iff $\partial_z \overline{f} = 0$. A standard computation yields

$$= 4\frac{\partial^2 f}{\partial z \partial \overline{z}}$$

Now observe since $\Delta f = \Delta f^2 = 0$ we obtain

$$2\frac{\partial f}{\partial \overline{z}}\frac{\partial f}{\partial z} = 0$$

so as f is C^2 in the real sense we conclude either $\partial_z f = 0$ or $\partial_{\overline{z}} f = 0$ in Ω i.e. f or $\overline{f}$ is holomorphic. □

Problem 12. Let $\mathcal{F}$ be the family of holomorphic functions on $\mathbb{D}$ with

$$\int_{\mathbb{D}} |f(x+iy)|^2 dA(x,y) < 1$$

prove that for each compact subset $K \subset \mathbb{D}$ there is an $A > 0$ such that $|f(z)| < A$ for all $z \in K$ and $f \in \mathcal{F}$

Proof. Let $r := \operatorname{dist}(K, \Omega)/2$ where K is a compact subset of Ω. Then for any $z \in K$ observe that $B_r(z) \subset \Omega$ so the mean value property tells us that

$$u(z) = \frac{1}{\pi r^2}\int_{B_r(z)} u(z)dz$$

so we have

$$|u(z)| \leq \frac{1}{\pi r^2}\int_{B_r(z)} |u(z)|dz$$
$$\leq \frac{1}{\pi r^2}\left(\sqrt{\pi} r ||u||_{L^2(B_r(z))}\right) \leq \frac{1}{\sqrt{\pi} r}||u||_{L^2(B_1(0))} = \frac{1}{\sqrt{\pi} r}$$

and this bound is independent of $z \in K$ so we have the desired conclusion. □

3. Spring 2011

Problem 1. Define what it means to say that $f_n \rightharpoonup f$ weakly in $L^2([0,1])$.
Suppose $f_n \in L^2([0,1])$ converges weakly to $f \in L^2([0,1])$ and define the primitive functions

$$F_n(x) := \int_0^x f_n(t)dt \text{ and } F(x) := \int_0^x f(t)dt$$

Show that $F_n, F \in C([0,1])$ and that $F_n \to F$ uniformly on $[0,1]$.

Problem 1a. Let $X := L^2([0,1])$ then $X \ni f_n \rightharpoonup f \in X$ means for any $L \in X^*$ where X^* is the topological dual of X then $L(f_n) \to L(f)$. Fix $L \in X$ then by Risez Representation Theorem we have that there exists $g \in X$ such that $\forall f \in X$

$$L(f) = \int_0^1 f(x)g(x)dx$$

so it is equivalent that $f_n \rightharpoonup f$ to mean for any $g \in X$ that

$$\int_0^1 f_n(x)g(x)dx \to \int_0^1 f(x)g(x)dx$$

Problem 1b. Since $f_n \rightharpoonup f$ it follows that for any $L \in X^*$ that $\sup_n |L(f_n)| < \infty$ since it is a convergent sequence, so by the uniform boundness principle we have $\sup_n ||f_n||_{L^2([0,1])} < \infty$. So as

$$|F_n(x)| \leq \int_0^1 |f_n(t)|dt \leq ||f_n||_{L^2([0,1]}$$

due to Cauchy-Schwarz we have that the family is uniformly bounded. Now observrve it is equicontinuous since if $0 \leq x \leq y \leq 1$ we have

$$|F_n(x) - F_n(y)| \leq \int_x^y |f_n(t)|dt = \int_0^1 |f_n(t)|\chi_{[x,y]}dt \leq ||f_n||_{L^2}\sqrt{|x-y|} \lesssim \sqrt{|x-y|}$$

where the final inequality is due to $||f_n||_{L^2}$ being uniformly bounded, so the family is equicontinuous. So now observe that as $\chi_{[0,x]}(t) \in X^* = L^2([0,1])$ we have

$$F_n(x) = \int_0^x f_n(t)dt = \int_0^1 f_n(t)\chi_{[0,x]}(t)dt \to \int_0^x f(t) = F(x)$$

so $F_n \to F$ pointwise. Then for any subsequence we have from Arzela-Ascoli a further subsequence which uniformly converges to $F(x)$, which implies the whole sequence uniformly converges to $F(x)$ □.

Problem 2. Let $f \in L^3(\mathbb{R})$ and

$$\phi(x) := \begin{cases} \sin(\pi x) : |x| \leq 1 \\ 0 \text{ else} \end{cases}$$

Show that

$$f_n(x) := n\int_{\mathbb{R}} f(x-y)\phi(ny)dy \to 0$$

Lebesgue almost everywhere.

Proof. Note that if we define $\phi_n(x) := \phi(nx)$ we have that

$$\int_{\mathbb{R}} \phi_n(x) = \int_{-1/n}^{1/n} \sin(n\pi x)dx = 0$$

so we have

$$|f_n(x)| \leq n\int_{\mathbb{R}} |f(x-y) - f(x)||\phi(ny)|dy \leq n\int_{-1/n}^{1/n} |f(x-y) - f(x)|dy$$

and Holder's Inequality implies $f \in L^1_{loc}(\mathbb{R})$ so we conclude by the Lebesgue Differentiation Theorem that

$$\limsup_{n\to\infty}\left\{n\int_{-1/n}^{1/n}|f(x-y)-f(x)|dy\right\} = 0 \text{ a.e.}$$

so $f_n(x) \to 0$ a.e. □

Problem 3. Let μ be a Borel probability measure on $\mathbb{R}$ and define $f(t) := \int e^{itx}d\mu(x)$. Suppose also that

$$\lim_{t\to 0}\frac{f(0)-f(t)}{t^2} = 0$$

Show that μ is supported at $\{0\}$.

Proof. First observe that

$$\frac{f(0)-f(t)}{t^2} = \int_{\mathbb{R}}\frac{1-e^{itx}}{t^2}d\mu(x)$$

and by using Taylor Expansion, we see that

$$\lim_{t\to 0}\operatorname{Re}\left(\frac{1-e^{itx}}{t^2}\right) = \frac{x^2}{2}$$

and we have from Fatou's Lemma that (since $\operatorname{Re}(1-e^{itx}) \geq 0$ that

$$0 = \liminf_{t\to 0}\int_{\mathbb{R}}\frac{\operatorname{Re}(1-e^{itx})}{t^2}d\mu(x) \geq \int_{\mathbb{R}}\liminf_{t\to 0}\frac{\operatorname{Re}(1-e^{itx})}{t^2}d\mu(x) = \int_{\mathbb{R}}\frac{x^2}{2}d\mu(x)$$

So now suppose for the sake of contradiction that μ is not supported on $\{0\}$, this means we can find a measurable set with $0 \notin E$ such that $\mu(E) > 0$. As μ is a Borel probability measure on the metric space $\mathbb{R}$ we conclude that μ is regular, so we can find a compact set $K \subset E$ such that $\mu(K) > 0$. As K is compact there exists a minimum $m \in K$ such that $|m| > 0$ (since $0 \notin K$). Therefore,

$$0 = \int_{\mathbb{R}}\frac{x^2}{2}d\mu(x) \geq \int_K\frac{x^2}{2}d\mu(x) \geq \frac{m^2}{2}\mu(K) > 0$$

which is our desired contradiction. □

Problem 4. Let $f_n : [0,1] \to [0,\infty)$ be Borel functions with

$$\sup_n\int_0^1 f_n(x)\log(2+f_n(x))dx < \infty$$

Suppose $f_n \to f$ Lebesgue almost everywhere. Show that $f \in L^1$ and $f_n \to f$ in the L^1 sense.

Proof. Assume that there exists a C independent of n such that

$$\sup_{n\in\mathbb{N}}\int_0^1 f_n(x)\log(2+f_n(x))dx \leq C < \infty$$

and there exists an f such that $f_n \to f$ and $f_n : \mathbb{R} \to \mathbb{R}^+$. First observe that as $\log(x)$ is an increasing function that $\log(2+f_n(x)) \leq \log(2)$ so it follows that

$$\sup_{n\in\mathbb{N}}\int_0^1 f_n(x)\log(2)dx \leq \sup_{n\in\mathbb{N}}\int_0^1 f_n(x)\log(2+f_n(x))dx \leq C$$

so each $f_n \in L^1$ with a uniform bound of C. Then we have from Fatou's Lemma that

$$\int_0^1 f(x)\log(2)dx \leq \int_0^1 f(x)\log(2+f(x))dx \leq \liminf_{n\to\infty}\int_0^1 f_n(x)\log(2+f_n(x))dx \leq C$$

and we have that $f \in L^1$. Now fix $\varepsilon > 0$ and assume E is measurable with $|E| < \delta$ then for $A_n^M := \{x : |f_n(x)| \le M\}$

$$\int_E f_n(x)dx = \int_{E\cap A_n^M} f_n(x)dx + \int_{E\cap (A_n^M)^c} f_n(x)\frac{\log(2+f_n(x))}{\log(2+f_n(x))}dx = (I) + (II)$$

now observe

$$(II) \le \left(\int_0^1 f_n(x)\log(2+f_n(x))\right)/\log(2+M) \le \frac{C}{\log(2+M)}$$

and

$$(I) \le M\delta$$

so taking $\delta = \frac{\varepsilon}{2M}$ we conclude that if M is very large then

$$\int_E f_n(x)dx \le \varepsilon$$

so $\{f_n\}$ is uniformly integrable. Now choose a $\delta > 0$ so small such that if $|E| < \delta \Rightarrow \int_E f_n(x)dx \le \varepsilon/2$. Now by Egorov's theorem there exists a compact set $K \subset [0,1]$ such that $|[0,1]\setminus K| \le \delta$ and $f_n \to f$ on K. So if n is large then $\sup_K |f_n(x) - f(x)| \le \varepsilon/2$ so

$$\int_0^1 |f_n(x)-f(x)|dx = \int_K |f_n(x)-f(x)|dx + \int_{[0,1]\setminus K} |f_n(x)-f(x)|dx$$
$$\le \varepsilon/2 + \varepsilon/2 = \varepsilon$$

so $f_n \to f$ in L^1 $\square$.

$\square$

Problem 5. Show that $\ell^\infty(\mathbb{Z})$ contains continnuum many functions $x_\alpha : \mathbb{Z} \to \mathbb{R}$ obeying

$$||x_\alpha||_{\ell^\infty} = 1 \text{ and } ||x_\alpha - x_\beta||_{\ell^\infty} \ge 1$$

Deduce (assuming the axiom of choice) that the Banach Space $\ell^\infty(\mathbb{Z})$ is not separable.

Deduce that $\ell^1(\mathbb{Z})$ is not reflexive.

Proof. Consider the set of binary strings i.e. $x_\alpha = (x_{\alpha,1}, x_{\alpha,2}, ..)$ where each $x_i = 0$ or 1. This is uncountably many distinct elements in $\ell^\infty(\mathbb{Z})$. Fix any two distinct binary strings x_α and x_β that are not identically zero then

$$||x_\alpha||_{\ell^\infty} = 1$$

And since $x_\alpha \ne x_\beta$ there is a j such that $|x_{\alpha,j} - x_{\beta,j}| = 1$. Therefore,

$$||x_\alpha - x_\beta||_{\ell^\infty} \ge 1$$

Let $\{e_j\}_{j\in\mathbb{N}}$ be a countable subset of $\ell^\infty(\mathbb{Z})$. Arguing for the sake of a contradiction, if $\{e_j\}_{j\in\mathbb{N}}$ is dense, then for each j we can find an $\alpha(j)$ such that $x_{\alpha(j)}$ is a binary string and

$$||e_j - x_{\alpha(j)}||_{\ell^\infty} \le 1/2$$

Then this implies if $\beta \ne \alpha(j)$ then the reverse triangle inequality implies

$$||e_j - x_\beta|| \ge ||x_{\alpha(j)} - x_\beta|| - ||e_j - x_{\alpha(j)}|| \ge 1 - 1/2 = 1/2$$

Therefore, as $\{x_{\alpha(j)}\}_{j\in\mathbb{N}}$ is countable and $\{x_\alpha\}$ is uncountable, we may find a binary string $x_\beta \notin \{x_{\alpha(j)}\}_{j\in\mathbb{N}}$. Therefore, for all $j \in \mathbb{N}$ then

$$||e_j - x_\beta|| \ge 1/2$$

so e_j cannot be dense. So we have arrived at a contradiction, so $\ell^\infty(\mathbb{Z})$ is not separable.

Note that the dual of $\ell^1(\mathbb{Z})$ is $\ell^\infty(\mathbb{Z})$, so if $\ell^1(\mathbb{Z})$ was reflexive then $\ell^1(\mathbb{Z}) \cong (\ell^\infty(\mathbb{Z}))^*$. Now we claim that if X^* is separable then this implies X is separable (where X is a normed linear space). This lemma will then gives us $\ell^1(\mathbb{Z})$ is isomorphic to a non-separable space, so this implies $\ell^1(\mathbb{Z})$ is not separable, but $\ell^1(\mathbb{Z})$ is separable, which is our contradiction. So it suffices to prove the lemma.

Indeed, let $\{f_n\} \subset X^*$ be a dense countable set. Then for each f_n if we define

$$||f_n|| := \sup_{x \in X : ||x|| = 1} |f_n(x)|$$

then there is some $x_n \in X$ with $||x_n|| = 1$ such that

$$f_n(x_n) \geq 1/2||f_n||$$

thanks to linearity. Now let S be the set of finite rational combinations of x_n i.e. $x \in S$ if $x = \sum_{n=1}^N q_n x_n$ for some $q_n \in \mathbb{Q}$ and $N \in \mathbb{N}$. We claim $\overline{S} = X$, so assume for the sake of contradiction it is not. So fix $x \in X \setminus \overline{S}$ with $||x|| = 1$ then by Hanh-Banach there is some $f \in X^*$ such that $f(x) \neq 0$ and $f|_{\overline{S}} = 0$. But observe from the triangle inequality that

$$|f_n(x_n)| \leq |f(x_n) - f_n(x_n)| + |f(x_n)| = |f(x_n) - f_n(x_n)|$$

and by our choice of x_n we have

$$1/2||f_n|| \leq |f_n(x_n)| \leq |f(x_n) - f_n(x_n)|$$

i.e.

$$||f_n|| \leq 2||f - f_n||$$

and as f_n is dense we can find a subsequence n_k such that $||f_{n_k} - f|| \to 0$. Thus in particular, $||f_{n_k}|| \to 0$,from which it follows that $||f|| = 0$ i.e. f is the zero operator, which is our desired contradiction.

□

Problem 6. Suppose μ and ν are finite positive (regular) Borel measures on $\mathbb{R}^n$. Prove the existence of the Lebesgue decomposition: There is a unique pair of positive Borel measures μ_a and μ_s so that

$$\mu = \mu_a + \mu_s, \mu_a \ll \nu, \text{ and } \mu_s \perp \nu$$

Proof. As μ is finite we have for any Borel set E that $\mu(E) < \infty$. Therefore, consider

$$E := \sup_{f \in L^1(d\nu), f \geq 0 \ \nu \ a.e.} \{\int_{\mathbb{R}^n} f(x) d\nu(x) : \int_E f(x) d\nu(x) \leq \mu(E) \text{ for all } E \text{ Borel } \}$$

Let f_n be a maximizing sequence i.e.

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} f_n(x) d\nu(x) = E$$

where $f_n \in L^1(d\nu)$. Observe that if $g_m := \max\{f_1, .., f_m\}$ satisfy the above constraints too. Indeed, $g_m \geq 0$ trivially and if we define E_j to be the set where $g_m = f_j$ we have

$$\int_E g_m d\nu(x) = \sum_{j=1}^m \int_{E \cap E_j} f_j d\nu(x) \leq \sum_{j=1}^m \nu(E \cap E_j) = \nu(E)$$

where for the last equality we used E_j partitions E. So it follows that $f_n \leq g_n$ for all n and the monotone convergence theorem shows that

$$E = \int_{\mathbb{R}^n} \sup_{m \in \mathbb{N}} \{f_m(x)\} d\nu(x) \leq \mu(\mathbb{R}^n)$$

so if we define $f := \sup_{m \in \mathbb{N}} f_m(x)$ then it obtains this maximum. Now define

$$\mu_a(E) := \int_E f(x) d\nu(x)$$

and note $\mu_a \ll \nu$ $\mu_s := \mu - \mu_a$. Suppose for the sake of contradiction that μ_s is not perpendicular to ν. Then there is an $\varepsilon > 0$ and a borel set E with $\nu(E) > 0$ such that

$$(\mu_s - \varepsilon\nu) \geq 0 \text{ on } E$$

i.e. for any Borel set F we have

$$\mu_s(F \cap E) - \varepsilon\nu(F \cap S) \geq 0$$
$$\mu(F \cap E) \geq \varepsilon\nu(F \cap E) + \int_{F \cap E} f(x)d\nu(x)$$

this means if $g(x) := f(x) + \varepsilon\chi_E(x)$ then we have found a strictly bigger maximizer, which is a contradiction to the definition of $f(x)$. Therefore, $\mu_s \perp \nu$. So we have shown such a decomposition exists. Now if there were two such decomposition's denoted $\mu_{a1}, \mu_{s1}, \mu_{a2}, \mu_{s2}$ then we have

$$\mu_{s1} - \mu_{s2} = \mu_{a_2} - \mu_{a_1}$$

the left hand side is singular to ν and the right hand side is absolutely continuous to ν, so $\mu_{s1} = \mu_{s2}$ and $\mu_{a1} = \mu_{a2}$, so this decomposition is unique.

□

Problem 7. Prove Gorsat's Theorem: if $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable then for every triangle $\Delta \subset \mathbb{C}$

$$\int_{\partial\Delta} f(z)dz = 0$$

where the line integral is over the three sides of the triangle.

Proof. □

Problem 8. (1) Define upper-semicontinuous for functions $f : \mathbb{C} \to [-\infty, \infty)$.
(2) Define what it means for such an upper-semicontinuous to be subharmonic.
(3) Prove or refute each of the following
- The pointwise supremum of a bounded family of subharmonic function is subharmonic.
- The pointwise infimum of a family of subharmonic functions is subharmonic.

(4) Let $A(z)$ be a 2×2 matrix-valued holomorphic function (i.e. the entires are holomorphic). Show that

$$z \mapsto \log(||A(z)||) \text{ is subharmonic}$$

where $||A(z)||$ is the operator norm on $\mathbb{C}^2$.

Problem 8a. An upper-semicontinuous function $f : \mathbb{C} \to [-\infty, \infty)$ is a function such that for any $\alpha \in \mathbb{R}$

$$\{z \in \mathbb{C} : f(z) < \alpha\} \text{ is open}$$

Problem 8b. We say a upper-semicontinuous function $u : \mathbb{C} \to [-\infty, \infty)$ if for any $r > 0$ small enough we have

$$f(z) \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z + re^{i\theta})d\theta$$

i.e. the local sub-mean value property.

Problem 8c. The first claim is true as long as the pointwise supremum is upper semi-continuous. Indeed, observe if $\mathcal{F}$ is a family of subharmonic functions then for any $z \in \mathbb{C}$ and $f \in \mathcal{F}$ we have for $r > 0$ small enough that we see for $g(z) := \sup_{f \in \mathcal{F}} f(z)$

$$f(z) \leq \frac{1}{2\pi} \int_0^{2\pi} f(z + re^{i\theta})d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} g(z + re^{i\theta})d\theta$$

since $f \leq g$. Now taking the supremum over f for the left hand side implies

$$g(z) \leq \frac{1}{2\pi} \int_0^{2\pi} g(z + re^{i\theta})d\theta$$

so $g(z)$ satisfies the local mean value inequality, so if it is upper semi-continuous then it is subharmonic.

The second claim is false. Indeed, consider $f(x+iy) := x$ and $g(x+iy) = -x$ then f and g are harmonic so they are subharmonic. And $\min\{f(x+iy), g(x+iy)\} = -|x|$ and $-|x|$ is concave, so it is superharmonic. In particular $-|x|$ is not subharmonic.

Problem 8d. Note that

$$||A(z)|| = \sup_{w\in\mathbb{C}^2:||w||=1} ||A(z)w||_2 = \sup_{w,\xi\in C^2:||w||=||\xi||=1} |\langle A(z)w,\xi\rangle|$$

where $||\cdot||_2$ refers to the Euclidan norm on $\mathbb{C}^2$. So we have

$$\log(||A(z)||) = \sup_{w,\xi\in C^2:||w||=||\xi||=1} \log|\langle A(z)w,\xi\rangle|$$

and for each fixed w,ξ we have $(A(z)w,\xi)$ is holomorphic in z so $\log|(A(z)w,\xi)|$ is subharmonic, so $\log(||A(z)||)$ is subharmonic since it is the sup of a family of subharmonic functions.

Problem 9. Let $E \subset [0,1]$ be the Cantor Set. Embedding $[0,1]$ naturally into $\mathbb{C}$, we may regrad $E \subset \mathbb{C}$. Suppose $f : \mathbb{C}\setminus E \to \mathbb{C}$ is holomorphic and (uniformly) bounded. Show that f is constant.

Proof. Note that E has measure zero. That is if $\varepsilon > 0$ there is a collection of balls $B(r_n, x_n) \subset \mathbb{C}$ such that

$$E \subset \bigcup_{n\in\mathbb{N}} B(x_n, r_n) \text{ and } \sum 2r_n < \varepsilon$$

Now let $R \subset \mathbb{C}$ be a rectangle. Then we will show $\int_{\partial R} f(z)dz = 0$. This is trivially true if $R \subset \mathbb{C}\setminus E$, so assume $R\cap E \neq \emptyset$. Now as $R\cap E$ is compact, we can find an $N \in \mathbb{N}$ such that $R\cap E \subset \bigcup_{i=1}^N B(r_n, x_n)$. Now observe that $R = (R\cap E)\cup(R\cap E^c)$ and on the second set, f is holomorphic so it integrates to zero over any closed curve in $(R\cap E^c)$, so the only remaining parts of the integral is where the balls are at. Now let γ_i be a closed curve parametrization of the connected components of $\partial(\bigcup_{n=1}^N B(r_n, x_n))$ where $i = 1, ..., m$ then

$$\int_{\partial R} f(z)dz = \sum_{i=1}^m \int_{\gamma_i} f(z)dz$$

so we have

$$\left|\int_{\partial R} f(z)dz\right| \le \sum_{i=1}^N M\ell(\gamma_i) \le \sum_{i=1}^N M\ell(\partial B(r_n, x_n))$$

where M is the upper bound of f and ℓ means length.

$$\le \sum_{n=1}^N M\ell 2\pi r_n = O(\varepsilon)$$

sending $\varepsilon \to 0$ implies that $\int_{\partial R} f(z)dz = 0$, so it follows that f extends to a bounded entire function by Morrera's Theorem, so it is constant.

□

Problem 10. Let $\Omega = \{z \in \mathbb{D} : \mathrm{Im}(z) > 0\}$. Evaluate

$$\sup\{\mathrm{Re} f'(i/2) : f : \Omega \to \mathbb{D} \text{ is holomorphic}\}$$

Proof. Consider the conformal map

$$\psi(z) := \frac{z+i}{z-i}$$

where ψ conformally maps Ω to $\mathbb{D}$. Then its inverse

$$\psi^{-1}(z) = i\frac{z+1}{1-z}$$

is a conformal map of $\mathbb{D}$ to Ω such that $-1/3$ gets mapped to $i/2$. Also consider the automorphisms of the disk for $\alpha \in \mathbb{D}$

$$\phi_\alpha(z) := \frac{\alpha - z}{1 - \overline{\alpha} z}$$

then ϕ maps 0 to α and α to 0. Therefore,

$$g(z) := \phi_{f(i/2)} \circ f \circ \psi^{-1} \circ \phi_{-1/3}$$

is a conformal map from $\mathbb{D}$ to $\mathbb{D}$ such that $g(0) = 0$. So by Schwarz Lemma we have

$$1 \geq |g'(0)| = |\phi'_{f(i/2)}(f(i/2))f'(i/2)[(\psi^{-1})'(-1/3)]\phi'_{-1/3}(0)|$$

and by computation

$$\phi'_\alpha(\alpha) = 1/(1 - |\alpha|^2) \text{ and } \phi'_\alpha(0) = 1 - |\alpha|^2 \text{ and } (\psi^{-1})'(z) = \frac{2i}{(1-z)^2}$$

i.e.

$$1 \geq \frac{1}{1 - |f(i/2)|^2}|f'(i/2)|$$

so we obtain the bound

$$1 \geq |f'(i/2)|$$

and by our computation above we see that this bound is obtained for any $f : \Omega \to \mathbb{D}$ such that $f(i/2) = 0$. For instance take

$$h(z) := \phi_{\psi(i/2)} \circ \psi(z)$$

and this function obtains the desired bound.

□

Problem 11. Consider the function defiend for $s \in (1, \infty)$ by

$$f(s) := \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

Show that f has an analytic continuation to $\{s \in \mathbb{C} : \text{Re}s > 0, s \neq 1\}$ with a simple pole at $s = 1$. Compute the residue at $s = 1$.

Proof. Note that

$$e^x - 1 = x + O(x^2) \text{ as } x \to 0$$

so the integrand is of $O(x^{s-2})$ near the origin, while far away it is of $O(e^{-x})$ so if $\text{Re}(s) > 1$ the integral is well defined. But notice by integration by parts with $u = x/(e^x - 1), dv = x^{s-2}$ that in this region we have

$$f(s) = -\frac{1}{s-1}\int_0^\infty x^{s-1}\left(\frac{e^x(1-x) - 1}{(e^x - 1)^2}\right) dx$$

And note that

$$\lim_{x \to 0}\left(\frac{e^x(1-x) - 1}{(e^x - 1)^2}\right) = -1/2$$

so the integrand is of $O(x^{s-1})$ near $x = 0$ and $O(e^{-x})$ as $x \to \infty$. Therefore, the integral converges for $\text{Re}(s) > 0$ with $s \neq 1$, so this is a meromorphic extension of f to $\{s \in \mathbb{C} : \text{Re}s > 0, s \neq 1\}$. It is also clear from the form of $f(s)$ that the pole is simple, so the residue at $s = 1$ is

$$-\int_0^\infty \frac{e^x(1-x) - 1}{(e^x - 1)^2} dx = 1$$

□

Problem 12. Let $\Omega := \mathbb{C} \setminus (-\infty, 0]$ and let $\log(z)$ be the branch of the complex logarithm on Ω that is real on the positive real axis (and analytic throughout Ω). Show that for $0 < t < \infty$, the number of solutions $z \in \Omega$ to

$$\log(z) = \frac{t}{z}$$

is finite and independent of t.

Proof. Notie that if $z = re^{i\theta}$ where $\theta \in (-\pi, \pi]$ then by our choice of log we have $\log(re^{i\theta}) = \log(r) + i\theta$ so if

$$\log(r) + i\theta = \frac{t}{r}(\cos(\theta) - i\sin(\theta))$$

we obtain

$$\theta = -\frac{t}{r}\sin(\theta)$$

and as $t/r > 0$ we see that θ and $\sin(\theta)$ have different signs, which means $\theta = 0$ because $\theta \in (-\pi, \pi]$. Therefore, we must have

$$\log(r) = \frac{t}{r}$$

i.e.

$$t = f(r) := r\log(r) > 0 \text{ on } r > 1$$

And observe that

$$\frac{df}{dr} = \log(r) + 1 > 0 \text{ on } r > 1$$

so we have that $f(r)$ is injective on $(0, \infty)$ so there is at most one solution of $\log(r) = t/r$. But it is clear

$$\lim_{r\to\infty} r\log(r) = \infty$$

so $f(r) : (1, \infty) \to (0, \infty)$ is surjective, so there is one solution for every t.

□

4. Fall 2011

Problem 1. Prove Egorov's Theorem

Proof. □

Problem 2a. Let $d\sigma$ denote the surface measure on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

(1) For $\xi \in \mathbb{R}^3$ compute
$$\int_{\mathbb{S}^2} e^{ix\cdot\xi} d\sigma(x)$$

(2) Using this or otherwise show that the mapping
$$f \mapsto \int_{\mathbb{S}^2}\int_{\mathbb{S}^2} f(x+y)d\sigma(x)d\sigma(y)$$
extends uniquely from $f \in C_c^\infty(\mathbb{R}^3)$ to a bounded linear functional on $L^2(\mathbb{R}^3)$.

Problem 2a. We claim that the integral only depends on $|\xi|$. Indeed for any orthogonal matrix we have that $Ax \cdot \xi = x \cdot A^{-1}\xi$ and $A^{-1} = A^t$. Then we have from integration by sub
$$\int_{S^2} e^{ix\cdot\xi} d\sigma(x) = \int_{S^2} e^{iA^{-1}x\cdot\xi} d\sigma(x) = \int_{S^2} e^{ix\cdot A\xi}$$
so wlog assume $\xi = (0, 0, |\xi|)$ then
$$\int_{S^2} e^{ix\cdot\xi} d\sigma(x) = \int_{S^2} \cos(z|\xi|) + i\sin(z|\xi|)d\sigma(x)$$
$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} [\cos(\cos(\phi)|\xi|) + i\sin(\cos(\phi)|\xi|)]\sin(\phi)d\phi d\theta$$
$$= \int_{\theta=0}^{2\pi}\int_{u=-1}^{1} \cos(u|\xi|) + i\sin(u|\xi|)dud\theta$$
$$= \frac{2\pi}{|\xi|}\int_{-|\xi|}^{|\xi|} \cos(w) + i\sin(w)dw$$
$$= \frac{4\pi}{|\xi|}\sin(|\xi|)$$

Problem 2b. Consider
$$L(f) := \int_{S^2}\int_{S^2} f(x+y)d\sigma(x)d\sigma(y)$$
then for $f \in C_c^\infty(\mathbb{R}^3)$ we have
$$L(f) = \int_{S^2}\int_{S^2}\int_{\mathbb{R}^d} \hat{f}(\xi)e^{-2\pi i(x+y)\cdot\xi} d\xi d\sigma(x)d\sigma(y)$$
and in particular by Fubini since $f \in C_c^\infty(\mathbb{R}^3)$ we have
$$L(f) = \int_{\mathbb{R}^d} \hat{f}(\xi)\int_{S^2} e^{-2\pi i x\cdot\xi}\int_{S^2} e^{-2\pi i y\cdot\xi} d\sigma(y)d\sigma(x)d\xi$$
$$= \int_{\mathbb{R}^d} \hat{f}(\xi)\frac{64\pi^2}{|\xi|^2}\sin(|\xi|)^2$$
so in particular,
$$|L(f)| \le C||\hat{f}||_{L^2}||\sin(|\xi|)^2/|\xi|^2||_{L^2} \le K||\hat{f}||_{L^2} = K||f||_{L^2}$$
so L extends to a continuous operator for L^2 functions.

Problem 3. Let $1 < p, q < \infty$ such that $1/p + 1/q = 1$. Fix $f \in L^p(\mathbb{R}^3)$ and $g \in L^q(\mathbb{R}^3)$.

(1) Show that
$$[f * g](x) := \int_{\mathbb{R}^3} f(x-y)g(y)dy$$
defines a continuous function on $\mathbb{R}^3$.

(2) Moreover, show that $[f * g](x) \to 0$ as $x \to \infty$.

Problem 3a. Note that
$$|[f*g](x) - [f*g](y)| \leq \int_{\mathbb{R}^3} |f(x-z) - f(y-z)||g(z)|dz$$
$$\leq ||f(x-y) - f(y-z)||_{L^p(\mathbb{R}^3, dz)}||g(z)||_{L^q}$$
and from the translation continuity of the integrals we deduce that this is a continuous map.

Problem 3b. Let $f_n, g_n \in C_c^\infty(\mathbb{R}^3)$ such that $||f_n - f||_{L^p} < \varepsilon$ and $||g_n - g||_{L^p} < \varepsilon$ where $\varepsilon > 0$ is arbitrary. Then
$$|[f*g](x) - [f_n * g_n](x)| \leq \int_{\mathbb{R}^3} |f(x-y)g(y) - f_n(x-y)g_n(y)|dy$$
$$\leq \int_{\mathbb{R}^3} |f(x-y)g(y) - g(y)f_n(x-y)| + |g(y)f_n(x-y) - f_n(x-y)g_n(y)|dy$$
$$\leq ||g||_{L^q}||f - f_n||_{L^p} + ||f_n||_{L^p}||g - g_n||_{L^q}$$
$$M_1\varepsilon + M_2\varepsilon$$
where we used $g \in L^q$ and f_n converges in L^p so its bounded in L^p. As this bound was independent of x we conclude that
$$f_n * g_n \to f * g \text{ uniformly}$$
but as f_n, g_n are compactly supported we have for $K_1 := \text{supp}(g_n)$ and $K_2 := \text{supp}(f_n)$ that
$$f_n * g_n = \int_K f_n(x-y)g_n(y)dy$$
which implies $\text{supp}(f_n * g_n) \subset K_1 + K_2 := \{k_1 + k_2 : k_1 \in K_1, k_2 \in K_2\}$ which implies $\lim_{x\to\infty} f_n * g_n = 0$ for any n so uniform convergence implies $\lim_{x\to\infty} f * g = 0$.

Problem 4. Let $f \in C^\infty([0,\infty) \times [0,1])$ such that
$$\int_0^\infty \int_0^1 |\partial_t f(x,t)|^2(1+t^2)dxdt < \infty$$
Prove there is a function g such that $f(t,\cdot)$ converges to $g(\cdot)$ in $L^2([0,1])$ as $t \to \infty$.

Proof. Notice by the fundamental theorem of Calculus that we have for $t_1 < t_2$ that
$$f(t_2, x) - f(t_1, x) = \int_{t_1}^{t_2} \frac{1+s^2}{1+s^2}\partial_t f(s,t)ds$$
so we have
$$|f(t_2,x) - f(t_1,x)| \leq \int_{t_1}^{t_2} |1+s^2|\frac{\partial_t f(s,t)}{|1+s^2|}ds \leq \left(\int_{t_1}^{t_2} |\partial_t f(s,x)|^2|1+s^2|^2 ds\right)^{1/2} \left(\int_{t_1}^{t_2} \frac{1}{|1+s^2|^2}ds\right)^{1/2}$$
so it follows that
$$\int_{x=0}^1 |f(t_2,x) - f(t_1,x)|^2 dx \leq \left(\int_{t_1}^{t_2} \frac{1}{|1+s^2|^2}ds\right)\left(\int_{t=0}^\infty \int_{x=0}^1 |\partial_t f(s,x)|^2|1+s^2|^2 ds\right)$$
$$\leq C\left(\int_{t_1}^{t_2} \frac{1}{|1+s^2|^2}ds\right)$$

and as $1/(1+x^2)^2 \in L^1([1,\infty))$ it follows that the integral term can be made arbitrarily small as $t_1, t_2 \to 0$. So we have $\{f(t,x)\}_{t\in[0,\infty)}$ is Cauchy as $t \to \infty$ so it converges to some function $g(x) \in L^2([0,1])$ since L^2 is complete. □

Problem 5. For a function $f \in L^1(\mathbb{R})$, we define

$$(Mf)(x) := \sup_{h>0} \frac{1}{2h} \int_{x-h}^{x+h} |f(y)| dy$$

Prove that there is the following property: There is a constant $A > 0$ such that for any $\lambda > 0$

$$m(\{x \in \mathbb{R} : Mf(x) > \lambda\}) \leq \frac{A}{\lambda} ||f||_{L^1}$$

If you use a covering lemma, you should prove it.

Proof. **Vitali Covering Lemma** Let $\{B_i\}_{i=1}^N$ be a finite collection of balls. Then there exists a subcollection of balls B_{i_j} that are disjoint such that

$$\bigcup_{i=1}^N B_i \subset \bigcup_{j=1}^M 3B_{i_j}$$

Indeed, let B_i be the ball with maximal radius of this finite collection. Then if $B_j \cap B_i \neq \emptyset$ we also remvoe B_j from this collection. Now we have a smaller subcollection and we repeat our algorithim of choosing the balls with maximum radius. It's clear from construction that the new balls $\{B_{i_j}\}_{j=1}^M$ are disjoint and as if $B_j \cap B_k \neq \emptyset$ with B_j being the circle with the biggest radius of the two balls then

$$B_k \subset 3B_j$$

so we have found such a subcollection.

Now fix λ and let m denote the Lebesgue measure then $F(\lambda) := \{x \in \mathbb{R} : (Mf) > \lambda\}$ now let $K \subset F(\lambda)$ be compact. Then we claim that $F(\lambda)$ is open; indeed, if $x \in F(\lambda)$ then there is an $r > 0$ such that

$$\lambda < \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| dy := A_r(x)$$

Notice that $A_r(x)$is continuous in x since

$$A_r(x) = \frac{1}{2r} \int_{\mathbb{R}} |f(y)| \chi_{[x-r,x+r]}(y) dy$$

so DCT implies continuity. Therefore, by continuity there exists a small ball around x such that for any y in this ball $A_r(y) > \lambda$. Therefore, for any $x \in K$ there is a ball $B_x \subset F(\lambda)$. Now compactness lets us find a subcover say $\{B_1, .., B_N\}$ and the covering lemma lets us find a subcollection of balls $\{\tilde{B}_1, .., \tilde{B}_M\}$ that are disjoint and $K \subset \bigcup_{j=1}^N B_j \subset \bigcup_{j=1}^M 3\tilde{B}_j$. So in particular,

$$m(K) \leq 3 \sum_{j=1}^M m(\tilde{B}_j) \leq \frac{3}{\lambda} \sum_{j=1}^M \int_{\tilde{B}_j} |f| \leq \frac{3}{\lambda} ||f||_{L^1(\mathbb{R})}$$

where the last inequality is due to the balls are disjoint. Now we use that the lebesgue measure is a radon measure so

$$m(F(\lambda)) = \sup_{K \subset F(\lambda)} m(K)$$

to get the desired result □.

□

Problem 6. Let (X, d) be a compact metric space. Let μ_n be a sequence of positive Borel Measures on X that weak* converge to a finite positive Borel measure μ, that is,

$$\int_X f d\mu_n \to \int_X f d\mu \text{ for all } f \in C(X)$$

Show that if K is compact then

$$\mu(K) \geq \limsup_{n\to\infty} \mu_n(K)$$

Proof. As K is compact, it is closed so χ_K is upper semi-continuous, so there is a sequence of functions $C(X) \ni f_n(x) \geq \chi_K$ with $f_n(x) \to \chi_K$ pointwise. Now we have

$$\mu_n(K) = \int_X \chi_K d\mu_n \leq \int_X f_m d\mu_n$$

so we have

$$\limsup_{n\to\infty} \mu_n(K) \leq \int_X f_m d\mu$$

where we used weak* convergence. Now as μ is finite and f_1 is bounded that $f_1 \in L^1(d\mu)$ so by the dominated convergence theorem we have

$$\lim_{m\to\infty} \int_X f_m d\mu = \int_X \chi_K d\mu$$

so

$$\limsup_{n\to\infty} \mu_n(K) \leq \int_X \chi_K d\mu = \mu(K)$$

as desired. □

Problem 7. Compute $\int_0^\infty \frac{\cos(x)}{(1+x^2)^2} dx$. Justify all steps!

Proof. Define $f(z) := \frac{e^{iz}}{(1+z^2)^2}$ and notice that it has a pole of order 2 at $z = i$ and $z = -i$. Then $f(z)$ is meromorphic with poles of order 2 at $z = i, -i$. Let $\gamma_R := \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$ and $\gamma := \{-R(1-t) + Rt : t \in [0, 1]\}$ then

$$\int_{\gamma_R} f(z)dz = \int_{\theta=0}^{\pi} \frac{e^{iRe^{i\theta}}}{(1 + R^2e^{2i\theta})^2} iRe^{i\theta} d\theta$$

so

$$|f(z)| \leq \frac{R}{(1-R^2)^2} \int_{\theta=0}^{\pi} e^{-R\sin(\theta)} d\theta \leq \frac{R}{(1-R^2)^2} \int_{\theta=0}^{\pi} e^{-CR\theta} d\theta$$

so the integral of $f(z)$ over γ_R converges to 0 as $R \to \infty$. Therefore, by the residue theorem and $\mathrm{Re(f(z))} = \frac{\cos(x)}{(1+x^2)^2}$ is symmetric that

$$\int_0^\infty \frac{\cos(x)}{(1+x^2)^2} = \pi i \mathrm{Res}(f, i) = \frac{\pi}{2e}$$

□

Problem 8. Determine the number of solutions of

$$z - 2 - e^{-z} = 0$$

with z in the right half plane $H = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$.

Proof. Observe that we have for any solutions of $z - 2 - e^{-z} = 0$ that

$$|z| \leq 2 + |e^{-z}| \leq 3$$

since z is in the upper half plane. Therefore, it suffices to consider the region $B_3(0) \cap \{x \geq 0\} := U$ to find our zeros. Notice on ∂U we have

$$|z - 2| > |e^{-z}|$$

since on the circle part we have

$$|z - 2| \geq |z| - 2 = 1$$

and on $\{x = 0\}$ we have $|z - 2| = |iy - 2| = |y| + 2 > 1$ so we have

$$|z - 2| > |e^{-z}| \text{ on } \partial U$$

so Rouche's theorem implies they have the same number of zeros inside U which is exactly 1. □

Problem 9. Let f be holomorphic on $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ such that $f \in L^2(\mathbb{D}, dxdy)$. Show that f has a holomorphic extension to $\mathbb{D}$.

Proof. Fix an $0 < \varepsilon \ll 1$ then on $A_{\varepsilon,1/2} := \{z : \varepsilon < |z| < 1/2\}$ then for any $z_0 \in A_{\varepsilon,1/2}$ there is a $\delta > 0$ such that $B_\delta(z) \subset \mathbb{D}^*$. So fix a $\rho < \delta$ and $z \in A_{\varepsilon,1/2}$ then we have from the Mean Value Theorem that

$$f(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z + \rho e^{i\theta}) d\theta$$

so

$$\int_{\rho=0}^{\delta} \rho f(z) d\rho = \int_{\rho=0}^{\delta} \rho \int_{\theta=0}^{2\pi} f(z + \rho e^{i\theta}) d\theta d\rho$$

which gives

$$f(z) = \frac{1}{\delta^2 \pi} \int_{B_\delta(z)} f(z) d\lambda(z)$$

so in particular,

$$|f(z)| \leq \frac{1}{\pi \delta^2}$$

and from Holder's we get

$$|f(z)| \leq ||f||_{L^2(B_\delta(z))}$$

so we have on $A_{\varepsilon,1}$ that $f(z)$ is bounded and this bound is uniform so we have $|f(z)| \leq C$ on $B_{1/2}(0)$. This allows us to use Riemann's Theorem on removable singularities to conclude $f(z)$ has a removable singularity at zero. □

Problem 10. Let $\Omega \subset \mathbb{C}$ be simply connected with $\Omega \neq \mathbb{C}$ and $f : \Omega \to \Omega$ is a holomorphic mapping. Suppose there exists $z_1 \neq z_2$ such that $f(z_i) = z_i$ for $i = 1, 2$. Show that $f(z) = z$ for all $z \in \Omega$.

Proof. By Riemann's Mapping Theorem there exists a conformal $\psi : \Omega \to \mathbb{C}$ and by composing with a Mobius Transformation we can assume $\psi(z_1) = 0$ so it follows that

$$g := \psi \circ f \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}$$

is conformal. Then we have $g(0) = \psi \circ f(z_1) = \psi(z_1) = 0$ and $g(\psi(z_2)) = \psi \circ f(z_2) = \psi(z_2)$. Therefore, by schwarz lemma as $g(0) = 0$ and there is a $p \neq 0$ such that $|g(p)| = |p|$ we have $g(z) = e^{i\theta} z$ for some θ and from equality we conclude $g(z) = z$. This implies

$$f(z) = z$$

as desired. □

Problem 11. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic with $f(z) \neq 0$ for all $z \in \mathbb{C}$. Define $U = \{z \in \mathbb{C} : |f(z)| < 1\}$. Show that all connected components of U are unbounded.

Proof. Assume that there existed a connected component of U denoted by Ω that is bounded. Then this implies Ω is compact, so in particular we have on $\partial\Omega$ that $|f(z)| = 1$ and since $f(z)$ is never zero, we know $1/f$ is holomorphic from which we deduce from the maximum modulus principle $|f(z)| = 1$ in Ω which contradicts the definition of U. □

Problem 12. A holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is of exponential type if there are c_1 and $c_2 > 0$ such that

$$|f(z)| \leq c_1 e^{c_2|z|} \text{ for all } z \in \mathbb{C}$$

Show that f is of exponential type iff f' is of exponential type.

Proof. For any $z \in \mathbb{C}$ we have $B_1(z) \in \mathbb{C}$ so by Cauchy's Estimate

$$|f'(z)| \leq \max_{w \in \overline{B}_1(z)} |f(w)| \leq \max_{\theta \in [0,2\pi]} c_1 \exp(|c_2 z + c_2 e^{i\theta}|)$$

$$\leq \exp(|c_2|) c_1 \exp|c_2 z|$$

so f' is of exponential type.

Now observe

$$f(z) = \int_{\gamma_{0\to z}} f'(z) + f(0)$$

where $\gamma_{0\to z} = \{tz : t \in [0,1]\}$

$$= \int_{t=0}^{1} z f'(tz) dt + f(0)$$

so

$$|f(z)| \leq |z||c_1 \exp(c_2|z|)) + |f(0)| \leq |c_1| \exp(c_3|z|) + |f(0)| \leq |c_4| \exp(c_3|z|)$$

so f is of exponential type □

5. Spring 2012

Problem 1. Some of the following statements for functions f_n in $L^3([0,1])$ are false. Indicate these and provide an appropriate counter example.

(1) If f_n converges a.e. to f then a subsequence of f_n converges to f in L^3.
(2) If f_n converges to f in L^3 then a subsequence converges almost everywhere.
(3) If f_n converges to f in measure then the sequence converges to f in L^3.
(4) If f_n converges to f in L^3 then the sequence converges to f in measure.

Proof. (1) is false since we can take $f_n := n^{1/3}\chi_{[0,1/n]}$ then $f_n \to 0$ everywhere except for $x = 0$. But

$$\int_0^1 f_n^3 dx = \int_0^{1/n} n dx = 1$$

so f_n does not converge along any subsequence to 0.

(2) is true. Indeed, as $f_n \to f$ in L^3 this means we can find a subsequence f_{n_k} such that we have

$$||f_{n_k} - f||_{L^3([0,1])} \le 2^{-k}$$

for any $k \in \mathbb{N}$. Now define

$$g_N(x) := \sum_{k=0}^{N} |f - f_{n_k}|^3$$

then notice that due to the monotone convergence theorem that we have

$$\lim_{N\to\infty} \int_0^1 |g_N(x)| dx = \int_0^1 \lim_{N\to\infty} |g_N(x)| = \int_0^1 \sum_{k=0}^{\infty} |f - f_{n_k}|^3$$

and for each N we have the uniform upper bound

$$\int_0^1 |g_N(x)| \le \int_0^1 \sum_{k=0}^{N} 2^{-k} \le 2$$

so it follows that

$$L^3([0,1]) \ni g(x) := \sum_{k=0}^{\infty} |f - f_{n_k}|^3$$

so it must be finite a.e., which implies the sum converges a.e., so $f_{n_k} - f \to 0$ as $k \to \infty$ a.e. along this subsequence.

(3) is false. Indeed, let m denote the Lebesgue Measure, then if $\varepsilon > 0$ we have for $f_n := n^{1/3}\chi_{[0,1/n]}$

$$m(\{x : |f_n(x)| > \varepsilon\}) \le m(\{x : |f_n(x)| > 0\}) \le 1/n$$

since these functions are supported on $[0, 1/n]$. So we have f_n converges to 0 in measure, but arguing as in (1) there is no subsequence of f_n that converges to 0 in L^3.

(4) is true. Notice if $\varepsilon > 0$ then

$$\int_0^1 |f_n(x) - f(x)| dx \ge \int_{\{x:|f_n(x)-f(x)|>\varepsilon\}} |f_n(x) - f(x)| dx \ge \varepsilon m(\{x : |f_n(x) - f(x)| > \varepsilon\})$$

so we have

$$\frac{1}{\varepsilon} \int_0^1 |f_n(x) - f(x)| \ge m(\{x : |f_n(x) - f(x)| > \varepsilon\})$$

But by Holder's Inequality we have

$$\int_0^1 |f_n(x) - f(x)| dx \le ||f_n(x) - f(x)||_{L^3([0,1])}$$

so by choosing n sufficiently large we have for $\delta > 0$ that $||f_n(x) - f(x)||_{L^3([0,1])} \leq \delta$ so we have

$$\frac{\delta}{\varepsilon} \geq m(\{x : |f_n(x) - f(x)| > \varepsilon\})$$

and as ε is fixed we can let $\delta \to 0$ to conclude $\lim_{n\to\infty} m(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0$. □

Problem 2. Let X and Y be topological spaces and let $X \times Y$ the Cartesian product endowed with the product topology. $\mathcal{B}(X)$ denotes the Borel Sets in X and similarly, $\mathcal{B}(Y)$ and $\mathcal{B}(X \times Y)$.

(1) Suppose $f : X \to Y$ is continuous. Prove that $E \in \mathcal{B}(Y)$ implies $f^{-1}(E) \in \mathcal{B}(X)$.
(2) Suppose $A \in \mathcal{B}(X)$ and $E \in \mathcal{B}(Y)$. Show that $A \times E \in \mathcal{B}(X \times Y)$.

Proof. Let $A := \{E \in \mathcal{B}(Y) : f^{-1}(E) \in \mathcal{B}(X)\}$. Then we claim that A is a σ-algebra that contains the open subsets of Y. Indeed, it is clear that $Y \in A$ since $f^{-1}(Y) = X$ and similarly for the empty set. Now if $\{X_i\}_{i=1}^{\infty} \in A$ then as $\mathcal{B}(Y)$ is a σ-algebra we have $\bigcup_{i=1}^{\infty} X_i \in \mathcal{B}(Y)$ with $f^{-1}(\bigcup_{i=1}^{\infty} X_i) = \bigcup_{i=1}^{\infty} f^{-1}(X_i) \in \mathcal{B}(X)$ since each $f^{-1}(X_i)$ is borel and borel sets are closed under countable unions. Finally if $E \in A$ then $E^c \in \mathcal{B}(Y)$ with

$$f^{-1}(E^c) = (f^{-1}(E))^c \in \mathcal{B}(X)$$

since borel sets are closed under complements. Therefore, A is a $\sigma-$algebra. And it contains the open sets since f is continuous so if $E \subset Y$ is open then $f^{-1}(E)$ is open i.e. $f^{-1}(E) \in \mathcal{B}(X)$. So if we denote the collection of open sets in Y as G then we have

$$G \subset A \Rightarrow \sigma(G) = \mathcal{B}(Y) \subset A$$

since the σ-algebra generated by the open sets is the borel sets and A is a σ-algebra. Therefore, we have proven (1).

For (2) we know the canonical projection map $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous since we are in the product topology. Then if $A \in \mathcal{B}(X)$ and $E \in \mathcal{B}(Y)$ then using part 1) gives

$$\mathcal{B}(X \times Y) \ni \pi_X^{-1}(A) = A \times Y$$

$$\mathcal{B}(X \times Y) \ni \pi_Y^{-1}(E) = X \times E$$

so we deduce that

$$\mathcal{B}(X \times Y) \ni \pi_Y^{-1}(E) \cap \pi_X^{-1}(A) = A \times E$$

as desired.

□

Problem 3. Given $f : [0,1] \to \mathbb{R}$ belonging to $L^1(dx)$ and $n \in \{1, 2, 3, ..., \}$ define

$$f_n(x) := n \int_{k/n}^{(k+1)/n} f(y)dy \quad \text{for } x \in [k/n, (k+1)/n) \text{ and } k = 0, .., n-1$$

Prove $f_n \to f$ in $L^1(dx)$

Proof. We first recall that compactly supported continuous functions are dense in $L^1([0,1], dx)$. In particular, if $\varepsilon > 0$ then there exists an $g \in C_c([0,1])$ such that $||f - g||_{L^1([0,1])} \leq \varepsilon$. We will first prove the theorem is true for this dense subclass then extend it to $f \in L^1(dx)$. Indeed, observe that

$$||g - g_n||_{L^1} = \int_0^1 |g(x) - g_n(x)|dx = \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} |g(x) - n \int_{k/n}^{(k+1)/n} g(y)dy|dx$$

$$= \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} n \left| \int_{k/n}^{(k+1)/n} g(x) - g(y)dy \right| dx \leq \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \int_{k/n}^{(k+1)/n} n \left| g(x) - g(y)dy \right| dx$$

So by uniform continuity if n is sufficiently large then we have $|g(x)-g(y)|<\varepsilon$ for $x,y\in[k/n,(k+1)/n]$ so if n is large enough

$$\leq n(1/n)(1/n)n\varepsilon=\varepsilon$$

so we have $g_n\to g$ in $L^1(dx)$.

Now we also have by the traingle inequality

$$||f-f_n||_{L^1(dx)}\leq ||f-g||_{L^1(dx)}+||g_n-f_n||_{L^1(dx)}+||g_n-g||_{L^1(dx)}$$

so if n is large

$$\leq 2\varepsilon+||g_n-f_n||_{L^1(dx)}$$

Now we compute

$$\int_0^1|g_n-f_n|dx=\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}|g_n(x)-f_n(x)|dx=\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}\int_{k/n}^{(k+1)/n}n|g(y)-f(y)|dydx$$

$$=\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n}|g(y)-f(y)|dy=\int_0^1|g(y)-f(y)|dy$$

so

$$\leq 3\varepsilon$$

so we are done $\square$.

$\square$

Problem 4. Let $S=\{f\in L^1(\mathbb{R}^3):\int fdx=0\}$
(1) Show that S is closed in the L^1 topology
(2) Show that $S\cap L^2(\mathbb{R}^3)$ is a dense subset of $L^2(\mathbb{R}^3)$

Proof. For (1) observe that if $f_n\in S$ such that $f_n\to f\in L^1(\mathbb{R}^3)$ where the convergence is in the L^1 sense, then

$$\left|\int_{\mathbb{R}^3}f\right|=\left|\int_{\mathbb{R}^3}f-f_n\right|\leq\int_{\mathbb{R}^3}|f-f_n|dx\to 0 \text{ as } n\to\infty$$

which gives us $f\in S$.

For (2) it suffices to show the problem for $f\in C_c(\mathbb{R}^3)$ since this is a dense subclass of $L^2(\mathbb{R}^3)$. Then say f is supported on $B_R(0)$ then define $I:=\int_{\mathbb{R}^3}f(x)dx$ then for $\varepsilon>0$ choose $M(\varepsilon)>R$ such that $m(B(0,M)\setminus B(0,R))=1/\varepsilon$ i.e. $4/3\pi(M(\varepsilon)^3-R^3)=1/\varepsilon$

$$f_\varepsilon(x):=\begin{cases}f(x) \text{ for } x\in B(0,R)\\ -\varepsilon I \text{ for } x\in B(0,M(\varepsilon))\setminus B(0,R)\end{cases}$$

then $\int f_\varepsilon(x)=0$ and

$$\int_{\mathbb{R}^d}|f(x)-f_\varepsilon(x)|^2dx=\varepsilon^2I^2\int_{B(0,M(\varepsilon))\setminus B(0,R)}1dx=\varepsilon I^2\to 0$$

as desired.

$\square$

Problem 5. State and prove the Risez Representation Theorem for linear functionals (on a separable) Hilbert Space.

Proof. Let H be our Hilbert space with inner product $(\cdot, \cdot)$ and $\Lambda \in H^*$ i.e. Λ is a continuous linear functional. Then there is a unique $y \in H$ such that

$$\Lambda(x) = (x, y)$$

Indeed, notice as Λ is continuous we must have $\text{Ker}(\Lambda)$ be a closed subset of H, so as it is a closed subspace we have the decomposition

$$H = \text{Ker}(\Lambda) \bigoplus (\text{Ker}(\Lambda))^{\perp}$$

So as long as Λ is not the trivial functional i.e. $\Lambda(x) = 0$ from which the theorem follows trivially with $y = 0$ there must exist an $z \in (\text{Ker}(\Lambda))^{\perp}$. Now notice that for any $x \in H$ we have

$$z\Lambda x - x\Lambda z \in \text{Ker}(\Lambda)$$

so we have

$$(z\Lambda x - x\Lambda z, z) = 0 \Rightarrow \Lambda x||z||^2 - (x\Lambda z, z) = 0 \Rightarrow \Lambda x = \frac{\Lambda z}{||z||^2}(x, z)$$

So by defining $y := \overline{\frac{\Lambda z}{||z||^2}} z$ then we have

$$\Lambda x = (x, y)$$

for any $y \in H$. Uniqueness follows from if

$$(x, y) = (x, z) \text{ for all } x \in H \Rightarrow (x, y - z) = 0 \text{ for all } x \in H$$

which implies $y = z$.

□

Problem 6. Suppose $f \in L^2(\mathbb{R})$ and that the Fourier transform obeys $\hat{f}(\xi) > 0$ for almost every ξ. Show that the set of finite linear combinations of translates of f is dense in the Hilbert Space $L^2(\mathbb{R})$.

Proof. Define S as the closure of the set of finite linear combinations of translates of f then we know that $L^2(\mathbb{R}) = S \bigoplus S^{\perp}$. So it suffices to show $S^{\perp} = \emptyset$. Indeed, observe that if $g \in S^{\perp}$ then by Plancherel, we have that

$$0 = (f(x - a), g) = (e^{-ita}\hat{f}(t), \hat{g}(t)) = \int_{\mathbb{R}} e^{-ita}\hat{f}(t)\overline{\hat{g}}(t)dt$$

and as $\hat{f}, \hat{g} \in L^2$ we know $\hat{f}\overline{\hat{g}} \in L^1$, so its Fourier Transform is well defined and we have

$$0 = \mathcal{F}(\hat{f}\overline{\hat{g}})(a)$$

So as $\mathcal{F}(\hat{f}\overline{\hat{g}}) = 0 \in L^1(\mathbb{R})$ and $\hat{f}\overline{\hat{g}} \in L^1$ we can apply the Fourier Inverse Formula to get

$$\hat{f}\overline{\hat{g}}(a) = 0$$

for a.e. a. This implies from $\hat{f} \geq 0$ that $\hat{g}(a) = 0$ a.e.

□

Problem 7. Let $\{u_n(z)\}$ be a sequence of real-valued harmonic functions on $\mathbb{D}$ that obey

$$u_1(z) \geq u_2(z) \geq u_3(z) \geq ... \geq 0 \quad \text{for all } z \in \mathbb{D}$$

Prove that $z \mapsto \inf_n u_n(z)$ is a harmonic function on $\mathbb{D}$.

Proof. Notice that if $n \geq m$ then we have $u_n - u_m \geq 0$ is a harmonic function and Harnack's inequality implies

$$0 \leq u_n(z) - u_m(z) \leq \frac{r + |z|}{r - |z|}(u_n(0) - u_m(0))$$

where $z \in D(r, 0) \subset D(1, 0)$ and we know that $\{u_n(0)\}$ is a cauchy sequence since it converges to $u(0) := \inf_n u_n(0)$. Therefore, $\{u_n\}$ converges locally uniformly (i.e. on every compact subset of $\mathbb{D}$). By the mean value property equivalence, we see that the limiting function is harmonic. And as the sequence is decreasing it must pointwise converge to $u(z) := \inf_n u_n(z)$, which implies $u(z)$ is harmonic on every compact subset, so it is harmonic on $\mathbb{D}$.

□

Problem 8. Let $\Omega := \{x + iy : x > 0, y > 0, xy < 1\}$. Give an example of an unbounded harmonic function on Ω that continuously extends to 0 on $\partial\Omega$.

Proof. Notice that by squaring Ω that the domain becomes $U := \{x + iy : x \in \mathbb{R}, 0 < y < 2\}$. Define $g(z) := \mathrm{Im}(e^{\pi z})$ then $g(\partial\Omega) = 0$ since the function becomes real valued. Also $g(x + i/2) = e^{\pi x}$ which is unbounded, so the function $\mathrm{Im}(e^{\pi z^2})$ works.

□

Problem 9. Prove Jordan's Lemma: If $f(z) : \mathbb{C} \to \mathbb{C}$ is meromorphic, $R > 0$, and $k > 0$, then

$$\left|\int_\Gamma f(z)e^{ikz}dz\right| \leq \frac{100}{k}\sup_{z\in\Gamma}|f(z)|$$

where Γ is the quarter-circle $z = Re^{i\theta}$ with $0 \leq \theta \leq \pi/2$.

Proof. Note that

$$\int_\Gamma f(z)e^{ikz}dz = \int_{\theta=0}^{\pi/2} f(Re^{i\theta})e^{ikRe^{i\theta}}iRe^{i\theta}$$

so we have from Holder

$$\left|\int_\Gamma f(z)e^{ikz}dz\right| \leq \sup_{z\in\Gamma}|f(z)|\int_{\theta=0}^{\pi/2} Re^{-kR\sin(\theta)}d\theta$$

Now using that $\sin(x) > x/2$ for $x \in [0, \pi/2]$ we have

$$\int_{\theta=0}^{\pi/2} Re^{-kR\sin(\theta)} \leq \int_{\theta=0}^{\pi/2} Re^{-kR\theta/2}d\theta = \frac{2}{k}\int_{\theta=0}^{kR\pi/4} e^{-\theta}d\theta \leq \frac{2}{k}\int_0^\infty e^{-k} = \frac{2}{k}$$

so we conclude that

$$\left|\int_\Gamma f(z)e^{ikz}dz\right| \leq \sup_{z\in\Gamma}|f(z)|\frac{2}{k} \leq \frac{100}{k}\sup_{z\in\Gamma}|f(z)|$$

□

Problem 10. Let us define the Γ function via

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$$

at least when the integral is absolutely converges. Show that this function extends to a meromorphic function in the whole complex plane. You cannot use any particular properties of the Γ function unless you derive it from this definition.

Proof. Let us show first that $\Gamma(z)$ is holomorphic in the region $U := \{z : \mathrm{Re}(z) > 0\}$. Indeed, observe first

$$|\Gamma(z)| \leq \int_0^\infty |t^{z-1}|e^{-t}dt \leq \int_0^\infty |t|^{\mathrm{Re}(z)-1}e^{-t}$$

so $\Gamma(z)$ is absolutely convergent in the region $\mathrm{Re}(z) > 0$ since the $|t|^{1-\varepsilon}$ for $\varepsilon > 0$ is integrable near the origin and e^{-t} gives enough decay factor at ∞. Also by the dominated convergence theorem, it follows that $\Gamma(z)$ is continuous in U.

In particular, continuity implies that if $R \subset U$ is a rectangle then $\Gamma \in L^1(R)$ so we have

$$\int_R \Gamma(z)dz = \int_R\int_0^\infty t^{z-1}e^{-t}dt = \int_0^\infty\int_R t^{z-1}e^{-t}dt = 0$$

where in the last step we used the integrand is holomorphic in z. Therefore, Morrera's theorem implies $\Gamma(z)$ is holomorphic on U.

Now we claim that $\Gamma(z)$ has a simple pole at $z = 0$. Indeed, observe by integration by parts that for z with $\text{Re}(z) \geq 1$ that we have from integration by parts

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt = \frac{1}{z}\int_0^\infty t^z e^{-t}dt$$

so by analytic continuation $\Gamma(z) = \frac{1}{z}\int_0^\infty t^z e^{-t}dt$ on U. Therefore, it follows that $\Gamma(z)$ extends to a meromorphic function on $\{z : \text{Re}(z) > -1\}$ with a simple pole at $z = 0$. Now we can also keep iterating this process infinitely many times to see $\Gamma(z)$ extends to a meromorphic function on $\mathbb{C}$ with simple poles at the negative integers, or we can use the identity

$$z\Gamma(z) = \Gamma(z+1)$$

to define $\Gamma(z) := \frac{1}{z}\Gamma(z+1)$ for $z \in \{z : \text{Re}(z) > -2\}$ then on $\{z : \text{Re}(z) > -3\}$ and inductively to define it on all of $\mathbb{C}$. So it suffices to justify this formula. Indeed, observe on U we have

$$z\Gamma(z) = z\int_0^\infty t^{z-1}e^{-t}dt = \int_0^\infty t^z e^{-t} = \Gamma(z+1)$$

by integration by parts; therefore, by analytic continuation this property holds over $\{z : \text{Re}(z) > 0\}$ then we can use this as the definition of $\Gamma(z)$ for $\{z : \text{Re}(z) \leq 0\}$

□

Problem 11. Let $P(z)$ be a polynomial. Show that there is an integer n and a second polynomial $Q(z)$ so that

$$P(z)Q(z) = z^n|P(z)|^2 \quad \text{whenever } |z| = 1$$

Proof. Say P has degree m then observe that when $P(z) \neq 0$

$$z^n|P(z)|/P(z) = z^n\overline{P(z)}$$

so when $|z| = 1$ and $P(z) \neq 0$ we get that this is equal to

$$e^{in\theta}\sum_{k=0}^m \overline{a_j}e^{-ik\theta} = \sum_{k=0}^m \overline{a_j}e^{i(n-k)\theta}$$

so taking $n = m$ we get

$$z^n\frac{|P(z)|}{P(z)} = \sum_{k=0}^m \overline{a_j}e^{i((m-k)\theta)} := Q(e^{i\theta})$$

Observe $z = \sum_{k=0}^m \overline{a_j}z^{m-k}$ is a polynomial and when $P(z) \neq 0$ we have

$$P(z)Q(z) = z^m|P(z)|^2$$

and when $P(z) = 0$ the equality is trivial. □

Problem 12. Show that the only entire function $f(z)$ obeying both

$$|f'(z)| \leq \exp(|z|) \text{ and } f(\frac{n}{\sqrt{1+|n|}}) = 0 \text{ for all } n \in \mathbb{Z}$$

is the zero function.

Proof. We first claim that f is an entire function of order 1. Indeed, by the fundamnetal theorem of calculus we have

$$f(w) = \int_\gamma f'(z)dz$$

where $\gamma = \{tw : t \in [0,1]\}$ so

$$f(w) = \int_{t=0}^{1} wf'(tw)dt \Rightarrow |f(w)| \leq \int_0^1 |w| \exp(|tw|)dw = \int_0^{|w|} \exp(t)dt = \exp(|w|) - 1 \leq \exp(|w|)$$

Therefore, we $f(z)$ is an entire function of order 1. Therefore, by Jensen's Formula unless $f \equiv 0$ then we must the number of zeros in a circle of radius R must be of order $C'R$. However, as $n/\sqrt{1+|n|} \sim \sqrt{n}$ this implies there should roughly N^2 zeros in a circle of radius $N \in \mathbb{N}$ for large N. Therefore, f is the zero function. $\square$

6. Fall 2012

Problem 1. Let $1 < p < \infty$ and let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\limsup ||f_n||_{L^p} < \infty$. Show that if f_n converges almost everywhere, then f_n converges weakly in L^p.

Proof. Note that as $1 < p < \infty$ we have that L^p is reflexive, so weak-* convergence is the same as weak convergence, so in particular as $\limsup ||f_n||_{L^p} < \infty$ this is a bounded sequence in L^p so by Banach-Alagou, there exists a sub-sequence f_{n_k} and a f such that $f_{n_k} \rightharpoonup f$ but as $f_{n_k} \to g$ a.e.

We then claim $f = g$. Indeed, fix any compact set $K \subset \mathbb{R}^3$ then by egorov for any $\varepsilon > 0$ there is a compact set $E \subset K$ such that $m(K \setminus E) < \varepsilon$ and $f_n \to g$ uniformly on E. Then for any $\psi \in L^q$ where q is the dual conjuagte of p we have

$$\int_K f_{n_k}\psi = \int_E f_{n_k}\psi + \int_{K\setminus E} f_{n_k}\psi$$

and observe that by uniform convergence we have for k sufficiently large we have

$$\int_E |g\psi - f_{n_k}\psi| \leq \varepsilon$$

and as $g\psi$ is in $L^1(K)$. We also observe

$$\int_{K\setminus E} |f_{n_k}\psi| \leq ||f_{n_k}||_{L^p}||\psi\chi_{K\setminus E}||_{L^q} = o(\varepsilon)$$

and

$$\int_{K\setminus E} |g\psi| = o(\varepsilon)$$

so it follows that

$$\lim_{k\to\infty}\int_K f_{n_k}\psi = \int_K f\psi$$

then this implies by DCT that

$$\lim_{k\to\infty}\int_{\mathbb{R}^3} f_{n_k}\psi = \int_K f\psi$$

which implies $f = g$ by uniqueness of weak limits. So this implies every sub-sequence has a further sub-sequence that converges to the same limit g; therefore, the whole sequence converges to g. □ □

Problem 2. Suppose $d\mu$ is a probability measure on the unit circle in the complex plane such that

$$\lim_{n\to\infty}\int_{S^1} z^n d\mu(z) = 0$$

For $f \in L^1(d\mu)$ show that

$$\lim_{n\to\infty}\int_{S^1} z^n f(z) d\mu(z) = 0$$

Proof. By Stone Weiestrass we know that trigonometirc polynomials i.e. $P(z) = \sum_{n=-N}^{M} a_n z^n$ are dense in S^1. So one has for any fixed trigonometric polynomial that

$$\int_{S^1} z^n P(z) d\mu(z) = \sum_{j=-N}^{M}\int_{S^1} a_j z^{j+n} d\mu(z) \to 0$$

Therefore, for $f \in L^1(d\mu)$ and $\varepsilon > 0$ we can find a trignometric polynomial such that $||P(z)-f(z)||_{L^\infty(S^1)} < \varepsilon$ then

$$\left|\int_{S^1} f(z)z^n\right| \leq \int_{S^1} |f(z) - P(z)| d\mu(z) + \left|\int_{S^1} P(z)z^n d\mu(z)\right| \leq \varepsilon + \left|\int_{S^1} P(z)z^n d\mu(z)\right|$$

so this goes to 0 as $n \to \infty$. ☐

Problem 3. Let H be a Hilbert Space and let E be a closed convex subset of H. Prove that there exists a unique element $x \in E$ such that

$$||x|| = \inf_{y \in E} |y||$$

Proof. Let $\alpha := \inf_{y \in E} ||y||$ and let $y_n \in E$ be a minimizing sequence i.e. $||y_n|| \to \alpha$. Notice by convexity that $\frac{y_n + y_m}{2} \in E$ so by the Parallelogram Law

$$||\frac{y_n + y_m}{2}||^2 + ||\frac{y_n - y_m}{2}||^2 = \frac{1}{2}||y_n||^2 + \frac{1}{2}||y_m||^2$$

and we have

$$\alpha \leq ||\frac{y_n + y_m}{2}||^2 \leq ||y_n||/2 + ||y_m||/2 \to \alpha$$

from which we deduce that $||y_n - y_m|| \to 0$ i.e. $\{y_n\}$ is a Cauchy sequence, so as we are on a Hilbert space and E is closed we know there exists a $y \in E$ such that $y_n \to y$. Now as the norm is continuous, we know that $||y|| = \alpha$ i.e. y is our minimizer.

Uniqueness arises since if $\alpha = ||y||, ||z||$ for $y, z \in E$ then we know

$$\alpha \leq ||\frac{y+z}{2}|| \leq ||y||/2 + ||z||/2 = \alpha$$

so again by the Parallelogram law we deduce that $||y - z|| = 0$ i.e. $y = z$.

☐

Problem 4. Fix $f \in C(\mathbb{T})$ where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let s_n denote the n-th partial sum of the Fourier Series of f. Prove that

$$\lim_{n \to \infty} \frac{||s_n||_{L^\infty(\mathbb{T})}}{\log(n)} = 0$$

Proof. Recall that we have

$$s_N = \sum_{n=-N}^{N} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \right) e^{inx} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^{N} e^{in(x-t)} dt$$
$$= (f * \mathcal{D}_N)(t)$$

where $\mathcal{D}_N(t) = \frac{1}{\pi} \sum_{n=-N}^{N} e^{int}$ and notice that its a geometric sum so we have

$$\pi \mathcal{D}_N(t) = \frac{e^{-iNt} - e^{i(N+1)t}}{1 - e^{it}} = \frac{e^{-it(N+1/2)} - e^{i(N+1/2)t}}{e^{-i/2t} - e^{i/2t}} = \frac{\sin((N+1/2)t)}{\sin(t/2)}$$

Therefore, we have the formula

$$s_N = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x-t) \frac{\sin((N+1/2)t)}{\sin(t/2)}$$

Therefore, we have from Holder's Inequality that

$$||s_N||_{L^\infty} \leq \frac{1}{\pi} ||f||_{L^\infty(\mathbb{T})} \left(\int_{-\pi}^{\pi} \left| \frac{\sin((N+1/2)t}{\sin(t/2)} \right| \right)$$

and the inner integral can be approximated by using concavity of log to deduce on $x \in [0, \pi]$ that

$$\sin(t/2 + 0/2) \geq t \sin(1/2) := \alpha t$$

and oddness of sin gives us the bound $|\sin(t/2)| \geq \alpha |t|$ for $t \in [-\pi, \pi]$, so we conclude

$$\int_{-\pi}^{0} \left| \frac{\sin((N+1/2)t}{\sin(t/2)} \right| \leq \int_{0}^{\pi} \frac{|\sin((N+1/2)t)|}{\alpha t} dt = \frac{1}{\alpha} \int_{0}^{(N+1/2)\pi} \frac{|\sin(t)|}{t}$$

$$\leq \frac{1}{\alpha}\sum_{k=0}^{N}\int_{k\pi}^{(k+1)\pi}\frac{|\sin(t)|}{t} \leq \frac{1}{\alpha}\int_0^{\pi}\frac{|\sin(t)|}{t} + \sum_{k=1}^{N}\frac{1}{\alpha}\frac{1}{k\pi}$$

so we have since $\sin(t)/t$ is continuous on $[0,\pi]$ that

$$\frac{||s_N||_{L^\infty}}{\log(N)} \leq ||f||_{L^\infty}\left(\frac{C}{\log(N)} + \frac{1}{\alpha}\sum_{k=1}^{N}\frac{1}{\log(N)k\pi}\right) \leq K||f||_{L^\infty}$$

since the Harmonic Series grows like $\log(N)$. Therefore, the family of linear operators

$$\Lambda_n(f) := \frac{s_n(f)}{\log(N)}$$

is uniformly bounded.

Now by Stone Weiestrass we can find a trigonoemetirc polynomial P such that if $\varepsilon > 0$ then $||f - P||_{L^\infty(T)} \leq \varepsilon$. Then notice that

$$\int_{-\pi}^{\pi} P(x)e^{inx} = 0$$

for all but finitely many n since $\{e^{inx}\}$ are orthogonal. Therefore,

$$|\Lambda_n(f)| \leq |\Lambda_n(f) - \Lambda_n(P)| + |\Lambda_n(P)| \leq K||f - P||_{L^\infty} + |\Lambda_n(P)| \leq \varepsilon + |\Lambda_n(P)|$$

and we know $|\Lambda_n(P)| \to 0$ since we have only a uniform amount of finitely many terms are non-zero and they are being scaled by $1/\log(N)$. Therefore,

$$\lim_{n\to\infty}|\Lambda_n(f)| = 0$$

□

Problem 5. Let $f_n : \mathbb{R}^3 \to \mathbb{R}$ be a sequence of functions such that $\sup_n ||f_n||_{L^2} < \infty$. Show that if f_n converges almost everywhere to a function $f : \mathbb{R}^3 \to \mathbb{R}$, then

$$\int_{\mathbb{R}^3} \left||f_n|^2 - |f_n - f|^2 - |f|^2\right| dx \to 0$$

Proof. Notice that by expanding we have

$$|f_n|^2 - |f_n - f|^2 - |f|^2 = f_n^2 - (f_n^2 - 2f_nf + f^2) - f^2 = 2f_nf - 2f^2 = 2f(f_n - f)$$

Therefore, we have

$$\int_{\mathbb{R}^3}\left||f_n|^2 - |f_n - f|^2 - |f|^2\right| dx = \int_{\mathbb{R}^3}|2f(f - f_n)|$$

And notice by Fatou's Lemma that

$$\int_{\mathbb{R}^3}|f|^2 \leq \liminf_{n\to\infty}\int_{\mathbb{R}^3}|f_n|^2dx \leq C$$

Therefore, $f \in L^2(\mathbb{R}^3)$, so there exists a compact set K such that on $\mathbb{R}^3 \setminus K$ we have $\int_{\mathbb{R}^3\setminus K}|f|^2 \leq \varepsilon$. By Egorov's Theoremm there exists a compact subset $K_1 \subset K$ with $m(K \setminus K_1) \leq \varepsilon$ such that $f_n \to f$ uniformly on K_1. In particular,

$$\int_{\mathbb{R}^3}|2f(f - f_n)| = \int_{K_1}|2f(f - f_n)| + \int_{K\setminus K_1}|2f(f - f_n)| + \int_{\mathbb{R}^3\setminus K}|2f(f - f_n)| := (I) + (II) + (III)$$

Notice that Cauchy-Schwarz gives

$$(I) \leq 2||f||_{L^2(\mathbb{R}^3)}||f - f_n||_{L^2(K_1)} \leq K\varepsilon$$

due to uniform convergence and our prior estimates. Also

$$(II) \leq 2||f||_{L^2(K\setminus K_1)}||f - f_n||_{L^2(\mathbb{R})} = o(\varepsilon)$$

since f is uniformly integrable as its in L^2 and the second term is bounded by a constant. Also

$$(III) \leq 2||f||_{L^2(\mathbb{R}^3 \setminus K)}||f - f_n||_{L^2(\mathbb{R})} \leq \tilde{K}\varepsilon$$

Therefore, we have

$$\int_{\mathbb{R}^3} |2f(f - f_n)| \to 0$$

as desired $\square$

Problem 6. Let $f \in L^1(\mathbb{R})$ and $\mathcal{M}f$ denote its maximal function, that is,

$$(\mathcal{M}f)(x) = \sup_{0<r<\infty} \frac{1}{2r} \int_{-r}^{r} |f(x-y)|dy$$

By the Hardy-Littlewood maximal function theorem,

$$|\{x \in \mathbb{R} : (\mathcal{M}f)(x) > \lambda\}| < 3\lambda^{-1}||f||_{L^1}$$

Using this show that

$$\limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy = 0 \text{ for a.e. } x \in \mathbb{R}$$

Proof. First we recall that $C_c(\mathbb{R}) \subset L^1(\mathbb{R})$ is a dense subclass and is $g \in C_c(\mathbb{R})$ then g is uniformly continuous. So if $\varepsilon > 0$ then there exists a $\delta > 0$ such that if $|x - y| < \delta$ implies $|g(x) - g(y)| \leq \varepsilon$. Take $r < \delta$ then

$$\frac{1}{2r} \int_{-r}^{r} |g(y) - g(x)|dy \leq \frac{1}{2r} \int_{-r}^{r} \varepsilon = \varepsilon$$

so we have

$$\limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |g(y) - g(x)|dy = 0 \quad \text{for all } g \in C_c(\mathbb{R})$$

Now we will use the maximal inequality to extend this to $f \in L^1(\mathbb{R})$. By density, there is a sequence $\{g_n\} \subset C_c(\mathbb{R})$ such that $||f - g||_{L^1(\mathbb{R})} < \frac{1}{n}$. Therefore, we have

$$\frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy \leq \frac{1}{2r} \int_{-r}^{r} |f(y) - g_n(y)|dy + \frac{1}{2r} \int_{-r}^{r} |g_n(y) - g_n(x)|dy + \frac{1}{2r} \int_{-r}^{r} |g_n(x) - f(x)|dy$$
$$\leq \frac{1}{2r} \int_{-r}^{r} |f(y) - g_n(y)|dy + \frac{1}{n} + |g_n(x) - f(x)|$$

Since $g_n \to f$ in $L^1(\mathbb{R})$ there exists a sub sequence such that $g_n \to f$ a.e., so by replacing an n in this subsequence we can assume $g_n(x) \to f(x)$ for a.e. x. Denote E as the set x such that along this subsequence we have $g_n(x) \to f(x)$. Then if $x \in E$ and $\varepsilon > 0$ arbitrary then we have by taking n sufficiently large

$$\frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy \leq \delta + \frac{1}{2r} \int_{-r}^{r} |f(y) - g_n(y)|dy$$

$$R_\delta := \{x \in E : \limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy > 2\delta\} \subset \{x \in E : \frac{1}{2r} \int_{-r}^{r} |f(y) - g_n(y)| > \delta\}$$
$$\subset \{x \in \mathbb{R} : (\mathcal{M}(f - g_n))(x) > \delta\}$$

so

$$|R_\delta| \leq 3\delta^{-1}||f - g_n||_{L^1} \leq 3\delta^{-1}n^{-1} \to 0 \text{ as } n \to \infty$$

Therefore, R_δ is a null set. But notice that

$$\{x \in \mathbb{R} : \limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy > 2\delta\} \subset R_\delta \cup E^c$$

and E^c is a null set, so it follows from

$$E := \{x \in \mathbb{R} : \limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy > 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : \limsup_{r \to 0} \frac{1}{2r} \int_{-r}^{r} |f(y) - f(x)|dy > n^{-1}\}$$

that E is a null set. So this implies the problem statement.

□

Problem 7. Let f be a function holomorphic in $\mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$ with $f(\mathbb{D}) \subset \mathbb{D}$. Show that

(1) $f'(1) \in \mathbb{R}$
(2) $f'(1) \geq 1$

Proof. Assume for the sake of contradiction that $f'(1) = a + ib$ where $b \neq 0$ and $a, b \in \mathbb{R}$. Then by the chain rule one has

$$f'(1)v = \lim_{t\to 0} \frac{f(1+tv) - f(1)}{t} = \lim_{t\to 0} \frac{f(1+tv) - 1}{t}$$

where we choose v such that $1 + tv \in \mathbb{D}$ for small enough t. So one has

$$\mathrm{Re}(f'(1)v) = \lim_{t\to 0} \frac{\mathrm{Re}(f(1+tv)) - 1}{t} \leq 0$$

where we are using $f(\mathbb{D}) \subset \mathbb{D}$. But as $f'(1)$ has an imaginary component this means we can rotate and find a v (thanks to rotating) such that $\mathrm{Re}(f'(1)v) > 0$. In particular, we can choose $v = -\varepsilon - i\delta b$ where $0 < \varepsilon \ll 1$ and $\delta > 0$ is a fixed constant chosen to ensure $1 + tv \in \mathbb{D}$ for small enough t. And we have $f'(1)v = -a\varepsilon + \delta^2 b^2$ so if ε is sufficiently small compared to δ we have arrived at a contradiction.

Notice that for $0 < t < 1$

$$f'(1) = \lim_{t\to 0} \frac{1 - f(1-t)}{t} = \lim_{t\to 0} \frac{1 - \mathrm{Re}(f(1-t))}{t}$$

and we see from Schwarz Lemma that

$$|f(1-t)| \leq 1 - t \Rightarrow -\mathrm{Re}(f(1-t)) \geq t - 1$$

so we deduce that

$$f'(1) \geq 1$$

as desired □.

□

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function such that every zero of f has even multiplicity. Show that f has a holomorphic square root.

Proof. As the zeros of $f(z)$ are isolated, we know there are only countably many of them. Enumerate them as $\{z_n\}_{n\in\mathbb{N}}$ where $z_n \neq z_m$ unless $n = m$ and with multiplicity $2m_n$ for $m_n \in \mathbb{N}$ then define

$$E_m(z) := \exp(z + z^2/2 + ... + z^m/m)$$

Then either $\{z_n\}$ is finite or $|z_n| \to \infty$, so by the Weiestrass Factorization Theorem we can find an entire function $g(z)$ with zeros only at z_n with m_n multiplicity. Then we have $f(z)/g^2(z)$ is an entire function with no zeros. Therefore, there is a entire function $h(z)$ such that

$$f(z)/g^2(z) = \exp(h(z)) \Rightarrow f(z) = g^2(z)\exp(h(z)) = (g(x)\exp(h(z)/2))^2$$

so f has a holomorphic square root $g(z)\exp(h(z)/2)$.

□

Problem 9. Suppose f is analytic in the unit disk $\mathbb{D}$ and $\{x_n\}$ is a sequence of real numbers satisfying $0 < x_{n+1} < x_n < 1$ for all n with $\lim_{n\to\infty} x_n = 0$. Show that if $f(x_{2n+1}) = f(x_{2n})$ for all $n \in \mathbb{N}$, then f is constant.

Proof. Assume by subtracting off by a constant that $f(0) = 0$. Now decompose $f(z) = u(z) + iv(z)$ where u and v are the real and imaginary parts of f. Now we restrict the domain of u and v to the real axis. Therefore, as $u(x_{2n+1}) = u(x_{2n})$ we can apply Rolle's theorem to conclude there is an $y_{2n}^{(1)}$ such that $x_{2n+1} \leq y_{2n}^{(1)} \leq x_{2n}$ for which $u'(x_{2n+1}) = 0$. This lets us find a sequence $\{y_{2n}^{(1)}\}$ that converges to 0 with $u'(y_{2n}^{(1)}) = 0$ (we use prime to denote x derivatives since we are viewing u as a function on the reals). So in particular continuity gives us $u'(0) = 0$ so using $f' = u_x + iv_x$ we get that $f'(0) = 0$ since we can do an identical argument on v. Now we can again repeating the above argument of using Rolle's we can find a decreasing sequence $y_{2n}^{(2)}$ for which $u''(y_{2n}^{(2)}) = 0$ to get $f''(0) = 0$. We can keep iterating this argument for all n to deduce that $f(0) = 0$ and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$, which means

$$A := \{z : f(z) = 0, f^{(n)}(z) = 0 \text{ for all } n \in \mathbb{N}\}$$

is a non-empty closed and open subset of a connected subset $\mathbb{D}$ so it is the entire space i.e. $f(z) \equiv 0$ in $\mathbb{D}$. □

Problem 10. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ satisfying $|f_n(z)| \leq 1$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Let $A \subset \mathbb{D}$ be the set of all $z \in \mathbb{D}$ for which $\lim_{n\to\infty} f_n(z)$ exists. Show that if A has an accumulation point in $\mathbb{D}$, then there exists a holomorphic function f on $\mathbb{D}$ such that $f_n \to f$ locally uniformly on $\mathbb{D}$ as $n \to \infty$.

Proof. Fix a subsequence $\{f_{n_k}\}$ and another subsequence $\{f_{m_k}\}$. By Montel's theorem both of these subsequence have a further subsequence that converges locally uniformly to some holomorphic functions f and g respectively. As A has an accumulation point we deduce $f = g$ on an accumulation point, so we must have $f \equiv g$ on $\mathbb{D}$. Therefore, every subsequence has a further subsequence that converges locally uniformly to f, which implies the entire sequence converges locally uniformly to f.

□

Problem 11. Find all holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ satisfying $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$.

Proof. Notice that $f(z) := \exp(-2\pi iz)$ satisfies $f(z+1) = f(z)$ and $f(z+i) = e^{2\pi} f(z)$. Let g be another entire function satisfying the periodicity conditions, then $h := g/f$ is an entire function since f never vanishes.And h satisfies $h(z+1) = h(z)$ and $h(z+i) = h(z)$. In particular, let $M := \max_{z\in[0,1]^2} |h(z)| < \infty$, then from the periodicity condition, we see that this bounds h everywhere. So in particular, $h(z) \equiv C$ for some constant C by Liouville, so $g = C \exp(-2\pi iz)$ and this classifies all such functions.

□

Problem 12. Let $M \in \mathbb{R}$ and $\Omega \subset \mathbb{C}$ be bounded open set, and $u : \Omega \to \mathbb{R}$ be a harmonic function.

(1) Show that if
$$\limsup_{z\to z_0} u(z) \leq M$$
for all $z \in \partial\Omega$, then $u(z) \leq M$ for all $z \in \Omega$

(2) Show that if u is bounded from above and there exists a finite set $F \subset \partial\Omega$ such that the inequality in (1) is satisfied for all $z_0 \in \partial\Omega \setminus F$ then the conclusion of (1) is still true.

Proof. If we fix an $\varepsilon > 0$ then due to the inequality, for any $z_0 \in \partial\Omega$ there exists a $r > 0$ such that on $\overline{B_r(z_0)} \cap \overline{\Omega}$ we have $u(z) \leq M + \varepsilon$. By compactness we can find a finite subcover of these balls say $\{B_{r_i}(z_i)\}_{i=1}^N$ then on $\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^N B_{r_i}(z_i)$ then we know that we have $u(z) \leq M + \varepsilon$ on $\partial\Omega_\varepsilon$, so the max principle implies $u(z) \leq M + \varepsilon$ on Ω_ε. But on the union of these balls we also have this inequality, so we deduce that $u(z) \leq M + \varepsilon$ on Ω and letting $\varepsilon \to 0$ concludes (1).

Part (ii) will follow from the standard $\varepsilon \log$ trick. Indeed, let $d := \mathrm{diam}(\Omega)$ and enumerate these finite points as $\{z_i\}_{i=1}^N$ and define $h(z) := -\log|\frac{z-z_0}{d}| - -\log|\frac{z-z_1}{d}| - ... - -\log|\frac{z-z_N}{d}|$ then we have $h(z)$ is harmonic on Ω since its locally the real part of a holomorphic function with $h(z_i) := \infty$. Then we consider

$$u(z) - \varepsilon h(z)$$

and thanks to the bounded above condition

$$\limsup_{z \to z_i} u(z) - \varepsilon h(z) = -\infty < M$$

and if $\delta > 0$ then for any $w \notin F$ we have

$$\limsup_{z \to w} u(z) - \varepsilon h(z) \leq M + \delta$$

since $h(z) \geq 0$. This lets us conclude with part (1) that

$$u(z) - \varepsilon h(z) \leq M + \delta \text{ on } \Omega$$

and letting $\varepsilon \to 0$ since $h(z)$ is finite on Ω we deduce that

$$u(z) \leq M + \delta$$

and finally letting $\delta \to 0$ gives the claim

□

7. Spring 2013

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is bounded, Lebesgue Measurable, and
$$\lim_{h\to 0}\int_0^1 \frac{|f(x+h)-f(x)|}{h}dx = 0$$
Show that f is a.e. constant on $[0,1]$.

Proof. Observe that for $F(x) := \int_0^x f(y)dy$ that we have
$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)dy \to f(x) \text{ a.e.}$$
thanks to Lebesgue's Differentiation Theorem since f bounded implies $f \in L_l^1oc(\mathbb{R})$. We also have for $x < y$
$$\begin{aligned}\frac{F(x+h)-F(y+h)+F(y)-F(x)}{h} &= \frac{1}{h}\int_{x+h}^{y+h} F(z)dz - \frac{1}{h}\int_x^y F(z)dz \\ &= \frac{1}{h}\int_x^y F(z+h)-F(z)dz\end{aligned}$$
so
$$\left|\frac{F(x+h)-F(y+h)+F(y)-F(x)}{h}\right| \le \frac{1}{h}\int_x^y |F(z+h)-F(z)|dz \le \frac{1}{h}\int_0^1 |F(z+h)-F(z)|dz$$
So we have
$$\lim_{h\to 0}\left|\frac{F(x+h)-F(y+h)+F(y)-F(x)}{h}\right| = 0$$
but by Lebesgue's Differentiation Theorem we know that for a.e. x and u
$$\lim_{h\to 0}\frac{F(x+h)-F(y+h)+F(y)-F(x)}{h} = f(x)-f(y)$$
so we have
$$f(x) = f(y) \text{ a.e.}$$
so f is constant $a.e.$ on $[0,1]$.

□

Problem 2. Consider the Hilbert Space $\ell^2(\mathbb{Z})$. Show that the Borel σ-algebra $\mathcal{N}$ on $\ell^2(\mathbb{Z})$ associated to the norm topology agrees with the Borel σ-algebra $\mathcal{W}$ on $\ell^2(\mathbb{Z})$ associated to the weak topology.

Proof. We first recall that the weak topology is the coarsest topology for which linear functionals are continuous so we have every open subset of the weak topology of $\ell^2(\mathbb{Z})$ is contained in the norm topology of $\ell^2(\mathbb{Z})$. This implies $\mathcal{W} \subset \mathcal{N}$, For the reverse direction recall that $\ell^2(\mathbb{Z})$ is separable for instance take finite rational linear combinations of $\{e_i\}$ where e_i is 0 everywhere except for a 1 on the ith coordinate so every open set is a countable union of balls. So it suffices to prove if $x \in \ell^2(\mathbb{Z})$ then $B_r(x) \in \mathcal{W}$. Observe that $y \in B_r(x) = \bigcup_{n\in\mathbb{N}}\{y : |y| \le r - 1/n\}$. And notice
$$||x-y||^2 \le r^2 \iff ||x||^2 - 2\text{Re}(x,y) + ||y||^2 \le r^2$$
and recall
$$||y||^2 = \sup_{||z||=1}(y,z)$$
so $y \in B_r(x)$ iff for all $z \in \ell^2(\mathbb{Z})$ with $||z|| = 1$ such that
$$||x||^2 - 2\text{Re}(x,y) + (y,z) \le r^2$$

As $\ell^2(\mathbb{Z})$ is separable, there is a countable dense subset which we label as v_n and we can assume $0 \neq v_n$ for any n. Then by continuity of the inner product we have for all $||z|| = 1$

$$||x||^2 - 2\text{Re}(x,y) + (y,z) \leq r^2 \iff ||x||^2 - 2\text{Re}(x,y) + (y, \frac{v_n}{||v_n||}) \leq r^2 \text{ for all } n$$

So for each $n \in \mathbb{N}$ define $\ell_n(y) := ||x||^2 - 2\text{Re}(x,y) + (y, \frac{v_n}{||v_n||})$. Then we have for any $n \in \mathbb{N}$

$$B_r(x) = \bigcup_{n\in\mathbb{N}} \{y : |y| \leq r - 1/n\} = \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \ell_m^{-1}([0, r-1/n]) \in \mathcal{W}$$

where we used $\ell_m^{-1}([0, r-1/n]) \in \mathcal{W}$ since linear functionals generate the topology. (Note implicitly the sum over $n \in \mathbb{N}$ is only taken for $r - 1/n \geq 0$). Therefore, $\mathcal{W} = \mathcal{N}$.

□

Problem 3. Given $f : \mathbb{R}^2 \to \mathbb{R}$, continuous, we define

$$[A_r f](x,y) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x + r\cos(\theta), y + r\sin(\theta))d\theta \quad \text{and} \quad [Mf](x,y) := \sup_{0<r<1} [A_r f](x,y)$$

By a theorem of Bourgain, there is an absolute constant C so that

$$||Mf||_{L^3(\mathbb{R}^2)} \leq C||f||_{L^3(\mathbb{R}^2)} \text{ for all } f \in C_c(\mathbb{R}^2)$$

Use this to show the following: If $K \subset \mathbb{R}^2$ is compact, then $[A_r\chi_K](x,y) \to 1$ as $r \to 0$ at almost every point $(x,y) \in K$ (w.r.t to the Lebesgue measure).

Proof. We first extend the inequality to $f = \chi_K - \psi_\varepsilon$ where ψ_ε where will be an approximation of χ_K. Indeed for $0 < \varepsilon \leq 1$ define

$$K_\varepsilon := \{x \in \mathbb{R}^2 : d(x,K) \leq \varepsilon\}$$

then K_ε is compact. Then by Uroshyn's lemma there is a continuous function ψ_ε such that $\psi_\varepsilon \equiv 1$ on $K_{\varepsilon/2}$ and $\psi_\varepsilon \equiv 0$ on K_ε^c. Then we have for any $0 < \varepsilon \leq 1$ that $\psi_{\varepsilon/2} \leq \psi_\varepsilon \in C_c(\mathbb{R}^2)$ so we have from construction that $\chi_K - \psi_\varepsilon \leq \psi_\varepsilon - \psi_{\varepsilon/2}$, $\chi_K \leq \psi_\varepsilon$, and $\psi_\delta \in C_c(\mathbb{R}^2)$ so we have

$$[A_r(\chi_K - \psi_\varepsilon)] \leq [A_r(\psi_\varepsilon - \psi_{\varepsilon/2})] \Rightarrow [M(\chi_K - \psi_\varepsilon)] \leq [M(\psi_\varepsilon - \psi_{\varepsilon/2})]$$

$$\Rightarrow ||[M(\chi_K - \psi_\varepsilon)]||_{L^3} \leq ||[M(\psi_\varepsilon - \psi_{\varepsilon/2})]||_{L^3} \leq C||\psi_\varepsilon - \psi_{\varepsilon/2}||_{L^3}$$

then using that $\psi_\varepsilon \leq \chi_{K_1} \in L^3(\mathbb{R}^2)$ since it is a compact set, so we have from the dominated convergence theorem that

$$||\psi_\varepsilon - \psi_{\varepsilon/2}||_{L^3} \to 0 \text{ as } \varepsilon \to 0$$

since by construction $\psi_\varepsilon \to \chi_K$ pointwise. Therefore, for any $\varepsilon > 0$ by Chebyshev inequality and our above inequality we have

$$m(\{x : |Mf(\chi_K - \psi_\varepsilon)| > \alpha\}) \leq \frac{C}{\alpha^3}||\psi_\varepsilon - \psi_{\varepsilon/2}||_{L^3}$$

Now one repeats the proof of Lebesgue Differentiation Theorem to conclude the problem using the approximation scheme $\psi_\varepsilon \in C_c(\mathbb{R}^2)$ since they converge in L^3 and L^1 to χ_K by DCT.

□

Problem 4. Let K be a non-empty compact subset of $\mathbb{R}^3$. For any Borel probability measure μ on K, define the Newtonian energy $I(u) \in (0, +\infty]$ by

$$I(\mu) := \int_K \int_K \frac{1}{|x-y|} d\mu(x)d\mu(y)$$

and let R_K be the infimum of $I(\mu)$ over all Borel probability measures μ on K. Show that there exists a Borel probability measure μ such that $I(\mu) = R_K$.

Proof. Take a minimizing sequence, that is a sequence $\{\mu_n\}$ of borel probability measures such that

$$I(\mu_n) \to R_K$$

which exists thanks to the definition of inf. So now as this is a sequence of Borel probability measures that there exists a subsequence and a borel measure μ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ i.e. for all $f \in C(K)$ we have

$$\int_K f d\mu_n \to \int_K f d\mu$$

note that as 1 is continuous we immediately obtain $\mu(K) = 1$ so it is a probability measure on K. Now we claim that $\mu_n \otimes \mu_n \overset{*}{\rightharpoonup} \mu \otimes \mu$. Indeed, by Stone Weiestrass we know that continuous functions of the form $f(x)g(y)$ are dense in $C(K \times K)$, from which we get the desired claim by a standard density argument. But notice that $1/|x-y|$ is not continuous, but it is lower semi-continuous, so we can approximate it from above by continuous functions to get that the functional is lower semi continuous with respect to weak convergence

$$I(\mu) = R_K$$

□

Problem 5. Define the Hilbert Space

$$H := \{u : \mathbb{D} \to \mathbb{R} : u \text{ is harmonic and } \int_{\mathbb{D}} |f(x,y)|^2 dxdy < \infty\}$$

with inner product $(f, g) := \int_{\mathbb{D}} fg dxdy$.

(1) Show that $f \mapsto \frac{\partial f}{\partial x}(0,0)$ is a bounded linear operator on $\mathbb{D}$.
(2) Compute the norm of this operator.

Proof. Note by the linearity of Δ that we have $\partial_x f$ is another harmonic function and by the mean value theorem we have that for $0 < r < 1$

$$\partial_x f(0,0) = \frac{1}{\pi r^2} \int_{B_r(0)} \partial_x f dS = \frac{1}{\pi r^2} \int_{\partial B_r(0)} f(x,y) \frac{x}{r} dS$$

where the last inequality is due to Green's theorem since the first component of the normal if $x/r = \cos(\theta)$. We compute to see

$$= \frac{1}{\pi r} \int_{\theta=0}^{2\pi} \cos(\theta) f(r\cos(\theta), r\sin(\theta)) d\theta$$

Therefore, we have from Cauchy-Schwarz that

$$|\partial_x f(0,0)|^2 \le \frac{1}{\pi r^2} \left(\int_{\theta=0}^{2\pi} |f(r\cos(\theta), r\sin(\theta))|^2 d\theta \right)$$

so multiplying both sides by πr^3 gives

$$\pi r^3 |\partial_x f(0,0)|^2 \le \int_{\theta=0}^{2\pi} r |f(r\cos(\theta), r\sin(\theta))|^2 d\theta$$

so integrating again in r from $r = 0$ to $\rho < 1$

$$|\partial_x f(0,0)|^2 \le \frac{4}{\pi \rho^4} \int_{B_\rho(0)} |f(x,y)| dxdy \le \frac{4}{\pi \rho^4} \int_{B_1(0)} |f(x,y)| dxdy$$

for all $0 < \rho < 1$ so we can take $\rho \to 1$ to yield

$$|\partial_x f(0,0)|^2 \le \frac{4}{\pi} \int_{B_1(0)} |f(x,y)| dxdy$$

i.e.

$$|\partial_x f(0,0)| \le \frac{2}{\sqrt{\pi}} ||f||_{L^2(B_1(0))}$$

so it is a continuous linear operator.

For part (2) taking $f(x,y) = x$ gives

$$\int_{B_1(0)} x^2 dxdy = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^3 \cos^2(\theta) dr d\theta = \frac{\pi}{4}$$

so

$$\frac{2}{\sqrt{\pi}} ||f||_{L^2(B_1(0))} = 1 = |\partial_x f(0,0)|$$

so the sharp constant is $2/\sqrt{\pi}$.

□

Problem 6. Let

$$X := \{\xi \mapsto \int_{\mathbb{R}} e^{ix\xi} f(x) dx | f(x) \in L^1(\mathbb{R})\}$$

(1) X is a subset of $C_0(\mathbb{R})$
(2) X is a dense subset of $C_0(\mathbb{R})$
(3) $X \neq C_0(\mathbb{R})$

Proof. For (1) observe that if $f(x) \in L^1(\mathbb{R})$ then

$$g(\xi) := \int_{\mathbb{R}} e^{ix\xi} f(x) dx = -\int_{\mathbb{R}} e^{i\xi(x+\frac{\pi}{\xi})} f(x) dx = -\int_{\mathbb{R}} e^{i\xi x} f(x - \pi/\xi)$$

Hence, one has

$$g(\xi) = \frac{1}{2} \int_{\mathbb{R}} e^{i\xi x} (f(x) - f(x - \pi/\xi))$$

so we have

$$|g(\xi)| \leq ||f(x) - f(x - \pi/\xi)||_{L^1(dx)}$$

which converges to 0 as $\xi \to \infty$ due to translation continuity of the Lebesgue integral. And g is continuous due to the translation continuity of the Lebesgue integral.

For (2) observe that if $f \in \mathcal{S}$ where $\mathcal{S}$ is in the Schwarz class then its fourier transform $\hat{f}$ is also in the Schwarz class. In particular, the fourier inverison formula holds for $f \in \mathcal{S}$. Then for any $f \in \mathcal{S}$ one has

$$f(\xi) = \int_{\mathbb{R}} e^{ix\xi} \hat{f}(x) dx$$

since $\hat{f} \in L^1$. Therefore, X contains the Schwarz class, which contains $C_c^\infty(\mathbb{R})$. And the $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$. So X is a dense subset of $C_0(\mathbb{R})$.

Note that if we define for $f \in L^1(\mathbb{R})$

$$Lf := \int_{\mathbb{R}} e^{ix\xi} f(x) dx$$

then this is operator is injective since due to Fourier Theory we have that

$$\mathcal{F}(Lf) = f$$

so if $Lf = 0$ then we deduce $f = 0$, which by linearity of the operator implies it is injective. So now if Lf was surjective to $C_0(\mathbb{R})$ then $L : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ is bijective so by the open mapping theorem we have that its inverse is continuous. Hence, we have constants $C_1, C_2 > 0$ such that

$$||Lf||_{L^\infty} \leq C_1 ||f||_{L^1} \text{ and } ||L^{-1}g||_{L^1} \leq C_2 ||g||_{L^\infty}$$

for any $f \in L^1(\mathbb{R})$ and $g \in C_0(\mathbb{R})$. Taking $g = Lf$ gives that

$$C_2 ||f||_{L^1} \leq ||Lf||_{L^\infty} \leq C_1 ||f||_{L^1}$$

Taking $f_n = (1/n)\chi_{[0,n]}(x) - (1/n)\chi_{[-n,0]}$ gives for $\xi \neq 0$

$$Lf_n(\xi) = (1/n) \int_0^n e^{ix\xi} dx - (1/n) \int_{-n}^{0} e^{ix\xi} dx$$

$$= \frac{1}{in\xi}(e^{in\xi} + e^{-in\xi} - 2)$$

which converges to 0 as $\xi \to 0$. Notice that

$$||\frac{e^{i\xi} + e^{-i\xi} - 2}{\xi}||_{L^\infty(\mathbb{R})} \leq C$$

since the expression is bounded near the origin since it converges to 0 and the expression decays to 0 as $\xi \to \infty$. Hence, we have

$$||Lf_n||_{L^\infty} \leq C/n \to 0 \text{ as } n \to \infty$$

but we have

$$||f_n||_{L^1} = 1 \text{ for all } n$$

which contradicts the contiuity of the inverse.

□

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $\log|f|$ is absolutely integrable with respect to the planar Lebesgue measure

Proof. As f is holomorphic we know $\log|f|$ is subharmonic i.e. we have the mean value inequality for any $z \in \mathbb{C}$

$$\log|f|(z) \leq \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \log|f|(z + re^{i\theta})d\theta$$

so multiplying by r and integrating in r from 0 to R gives

$$\log|f(z)| \leq \frac{1}{\pi R^2}\int_{B_R(z)} \log|f(z)|d\lambda(z) \leq \frac{1}{\pi R^2}||(\log|f|)||_{L^1(\mathbb{R})}$$

so we have for any z by taking $R \to \infty$ that

$$\log|f(z)| \leq 0 \Rightarrow |f(z)| \leq 1$$

so by Liouville's Theorem we have that f is constant.

□

Problem 8. Let A and B be real positive definite $n \times n$ symmetric matrix with the property

$$||BA^{-1}x|| \leq ||x|| \text{ for all } x \in \mathbb{R}^d$$

(1) Show that for each pair $x, y \in \mathbb{R}^n$

$$z \mapsto (y, B^z A^{-z} x)$$

admits an analytic continuation from $0 < z < 1$ to the entire complex plane.

(2) Show that

$$||B^\theta A^{-\theta} x|| \leq ||x||$$

for all $0 \leq \theta \leq 1$

Proof. As A and B are real positive definite $n \times n$ symmetric matrix, the Spectral Theorem tells us there are $\lambda_i > 0$ and $\sigma_i > 0$ with orthogonal matrix S, V such that

$$A = S^\top \text{diag}(\lambda_1, ..., \lambda_n)S \text{ and } B = V^\top \text{diag}(\sigma_1, ..., \sigma_n)V$$

Then using that A, B are symmetric, we have for any $x, y \in \mathbb{R}^n$ that

$$(y, B^z A^{-z} x) = (B^z y, A^{-z} x) = (S^\top \text{diag}(\lambda_1^z, ..., \lambda_n^z)Sy, V^\top \text{diag}(\sigma_1^{-z}, ..., \sigma_n^{-z})V)$$

where $a^z := \exp(z \log(a))$ where we use the standard complex log with a branch cut on the negative real axis. So in particular, $(y, B^z A^{-z} x)$ is some polynomial combination of $\{\lambda_i^z\}$ and $\{\sigma_i^z\}$, so $z \mapsto (y, B^z A^{-z} x)$ is an entire function for any $z \in \mathbb{C}$.

Now observe for any $0 \le \theta \le 1$ that

$$||B^\theta A^{-\theta}x|| = \sup_{||y||=1} (y, B^\theta A^{-\theta}x)$$

Define the entire function for $||y|| \le 1$

$$f_{x,y}(z) := (y, B^z A^{-z}x)$$

for $z \in \{x + iy : 0 \le x \le 1, y \in \mathbb{R}\}$ and define

$$M(s) := \sup_t |f_{x,y}(s+it)|$$

Now observe that

$$M(1) \le \sup_t ||B^{1+it}A^{-1-it}x|| = ||BA^{-1}x|| \le ||x||$$

Now observe

$$M(0) \le \sup_t ||B^{it}A^{-it}x|| \le (||B^{it}||)(||A^{-it}||)(||x||) = ||x||$$

since the eigenvalues have magnitude 1. Therefore, by Hadamard's three lines theorem we have for any $\theta \in [0,1]$

$$M(\theta) = M(\theta + (1-\theta)0) \le M(0)^\theta M(1)^{1-\theta} \le ||x||^\theta ||x||^{1-\theta} = ||x||$$

Hence, we have

$$||B^\theta A^{-\theta}x|| = \sup_{||y||\le 1} f_{x,y}(\theta) \le \sup_{||y||\le 1} M(\theta) \le ||x||$$

as desired. □

Problem 9. Let $P(z)$ be a non-constant complex polynomial, all of whose zeros lie in a half plane $\{z \in \mathbb{C} : \text{Re}(z) < \sigma\}$. Show that all the zeros of $P'(z)$ also lie in the half plane $\{z \in \mathbb{C} : \text{Re}(z) < \sigma\}$.

Proof. By the fundamental theorem of calculus we have that

$$P(z) = \alpha(z - z_1)(z - z_2)...(z - z_n)$$

where z_i are the zeros of P and $\alpha \in \mathbb{C} \setminus \{0\}$ and $\text{Re}(z_i) < \sigma$. Then it follows that the log derivative

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - z_i}$$

We already know that if w is a repeated root of P then $\text{Re}(w) < \sigma$ since $P(w) = 0$, so it suffices to assume that w is a root of P' but not a root of P. So this implies $P'(w) = 0 \iff P'(w)/P(w) = 0$ (since $P(w) \neq 0$). And we obtain

$$0 = \sum_{i=1}^n \frac{1}{w - z_i} = \sum_{i=1}^n \frac{\overline{w} - \overline{z_i}}{|w - z_i|^2} \Rightarrow 0 = \sum_{i=1}^n \frac{\text{Re}(w) - \text{Re}(z_i)}{|w - z_i|^2}$$

as $\text{Re}(z_i) < \sigma$ we obtain

$$0 = \sum_{i=1}^n \frac{\text{Re}(w) - \text{Re}(z_i)}{|w - z_i|^2} > \sum_{i=1}^n \frac{\text{Re}(w) - \sigma}{|w - z_i|^2}$$

from which it follows that $\text{Re}(w) < \sigma$. Therefore, all the roots of $P'(z)$ also lie in the half plane $\{z \in \mathbb{C} : \text{Re}(z) < \sigma\}$ □

Problem 10. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Without using either of the Picard theorems, show that there exists arbitrarily large complex numbers z for which $f(z)$ is positive real.

Proof. Let f be an entire function such that for arbitrarily large complex numbers $f(z)$ is not positive real. That is there exists a $M \in \mathbb{N}$ such that

$$f|_{\{z:|z|>M\}} : \{z : |z| > M\} \to \mathbb{C} \setminus \{x : x \geq 0\}$$

But by compactness anf continuity we know that there is a K such that $|\text{Re}(f)| \leq K$ on $\overline{B_M(0)}$. Therefore f is an entire function that maps the simply connected domain $\Omega := \mathbb{C} \setminus \{x : x > K + 1\}$. Ω is simply connected since it is star shaped, so by Riemann's theorem there is a conformal map $\varphi : \Omega \to \mathbb{D}$ such that

$$\varphi \circ f : \mathbb{C} \to \mathbb{D}$$

so it follows from Liouville's theorem that $\varphi \circ f$ is constant. So it follows from φ being bijective that f is constant. □

Problem 11. Let $f(z) := -\pi z \cot(\pi z)$ be a meromorphic function on $\mathbb{C}$.

(1) Locate all the poles of f and determine their residues.
(2) Show that for each $n \geq 1$ the coefficient of z^{2n} in the Taylor expansion of $f(z)$ about $z = 0$ coincides with

$$a_n := \sum_{k=1}^{\infty} \frac{2}{k^{2n}}$$

Proof. By using $\cot(\pi z) = \cos(\pi z)/\sin(\pi z)$ its clear that the poles of $f(z)$ are at $j \in \mathbb{Z} \setminus \{0\}$. Then for $j \in \mathbb{Z} \setminus \{0\}$ we have the pole is simple by taylor expansion of $\sin(\pi z)$ so

$$\text{Res}(f, j) = \lim_{z \to j} -\pi z \frac{(z-j)\cos(\pi z)}{\sin(\pi z)} = -j$$

where the last equality is due to L'hopitals rule (which extends to holomorphic functions thanks to the taylor series expansion). So $f(z)$ has a pole at each $j \in \mathbb{Z} \setminus \{0\}$ with residue $-j$.

Now we recall that $\pi \cot(\pi z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$ so it follows that

$$f(z) = -1 - \sum_{n=1}^{\infty} \frac{2z^2}{z^2 - n^2}$$

Now if we define

$$g(z) := \sum_{n=1}^{\infty} \frac{2z}{z - n^2}$$

then $-1 - g(z^2) = f(z)$ so the z^{2n} taylor coefficient of $f(z)$ is the nth taylor coefficient of $-g(z)$. Now we see

$$g'(z) = \sum_{n=1}^{\infty} -\frac{2n^2}{(z - n^2)^2}$$

and in general

$$g^{(n)}(z) = (-1)^n \sum_{n=1}^{\infty} \frac{2n^2 n!}{(z - n^2)^{n+1}} \Rightarrow \frac{g^{(n)}(0)}{n!} = -\sum_{n=1}^{\infty} \frac{2}{n^{2n}}$$

which gives the desired result.

□

8. Fall 2013

Problem 1. Let U and V be open and connected sets in $\mathbb{C}$ and $f : U \to \mathbb{C}$ be a holomorphic function with $f(U) \subset V$. Suppose that f is proper map from U into V i.e. $f^{-1}(K) \subset U$ is compact whenever $K \subset V$ is compact. Show then that f is surjective.

Proof. Let us show that $f(U)$ is open and closed. Indeed, if $z_0 \in f(U)$ then there exists a z such that $f(z) = z_0$ then as the proper condition implies that $f(z)$ is non-constant, so the open mapping theorem tells us that if $\varepsilon > 0$ is so small such that $B_\varepsilon(z) \subset U$ then $f(U)$ so the image is open.

Now let $\{y_n\}_{n\in\mathbb{N}} \in f(U)$ such that $y_n \to y \in V$ where $f(z_n) = y_n$ then $K := \{y_n\} \cup \{y\}$ is compact so

$$f^{-1}(K) \text{ is compact by properness}$$

so the sequence $\{z_n\} \subset f^{-1}(K)$, so it has a limit denoted z_0 along a subsequence z_{n_j}. Now we claim $f(z_0) = y$ indeed as $f(z_{n_j}) = y_{n_j} \to y$ it follows from continuity that $f(z_0) = y$. Therefore, $f(U)$ is a non-empty open and closed subset of a connected subset V so $f(U) = V$. ☐

Problem 2. Show that there is no function f that is holomorphic near $0 \in \mathbb{C}$ and satisfies

$$f(1/n^2) = \frac{n^2 - 1}{n^5}$$

Proof. Assume $f(z)$ is holomorphic then $f(z^2)$ is holomorphic. Notice that if we define $g(z) = z^{5/2}\left(\frac{1}{z} - 1\right) = z^{3/2} - z^{5/2}$ then $g(1/n^2) = f(1/n^2)$ and that $g(z^2) = z^3 - z^5$ is holomorphic such that $g(z^2) = f(z^2)$ on an accumulation point so $g(z^2) = f(z^2)$ for all z. But then observe we have the following contradiction for $z \neq 0$

$$f(z^2) = f((-z)^2) = -z^3 + z^5 = -f(z^2)$$

Therefore, $f(z)$ cannot be holomorphic.

☐

Problem 3. Does there exist a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ such that

$$\lim_{n\to\infty} |f(z_n)| = +\infty$$

for all sequence $\{z_n\} \subset \mathbb{D}$ with $\lim_{n\to\infty} |z_n| = 1$? Justify your answer.

Proof. No such functions exist. Indeed, as for any $z_0 \in \partial D$ we have $\limsup_{z\to z_0} |f(z)| = +\infty$, we can find a ball $B(z_0)$ such that on $\mathbb{D} \cap B(z_0)$ we have $|f(z)| > 1$. By compactness we can find a finite sub-cover denoted by $B_1, .., B_N$ that cover ∂D and on the compliment of these balls within $\mathbb{D}$ we know that f can only have finitely many zeros, but they have no zeros in these balls. This implies that f has only finitely many zeros on $\mathbb{D}$, say $\{z_i\}$ with multiplicities m_i then define

$$g(z) := f(z)/(z - z_i)^{m_i}$$

which is a new holomorphic function that is non-zero everywhere. And as these zeros are all δ for some $\delta > 0$ distance away from the boundary, we know that we still have

$$\limsup_{z\to z_0} |g(z)| = +\infty$$

Now fix an $M > 0$ large then for an $z_0 \in \partial D$ we can find a ball $B(z_0)$ with radius $\leq 1/M$ such that on $B(z_0)$ we have $|g(z)| \geq M$. By compactness we can find a finite collection of balls $B_1, .., B_N$ (with all of their radius $\leq 1/M$) that cover ∂D and on $\overline{B_i}$ we have $|g(z)| \geq M$. Define $\Omega_M := \mathbb{D} \setminus (\bigcup_{i=1}^N \overline{B_i})$ which is open, and on $\partial\Omega$ we have $|g(z)| \geq M$ and since g never vanishes we can apply the Minimum Modulus Principle (by looking at $1/g(z)$ which is holomorphic and applying the max-principle) to conclude that $|g(z)| \geq M$ on Ω_M. As $M \to \infty$ we know that $\Omega_M \to \mathbb{D}$ so we conclude that $|g(z)| = +\infty$ everywhere which is a contradiction, since $g(z)$ then cannot be continuous, but it was holomorphic. ☐

Problem 4. Let u be a non-negative continuous function on $\overline{\mathbb{D}} \setminus \{0\}$ that is subharmonic on $\mathbb{D} \setminus \{0\}$. Suppose that $u|\partial\mathbb{D} \equiv 0$ and

$$\lim_{r\to 0+} \frac{1}{r^2 \log(1/r)} \int_{\{z\in\mathbb{C}:0<|z|<r\}} u(z)d\lambda(z) = 0$$

where integration is with respect to Lebesgue measure λ on $\mathbb{C}$. Show that then $u \equiv 0$.

Proof. Observe that for $\varepsilon > 0$ that we have for sufficiently small $r > 0$ that

$$\frac{1}{\pi r^2} \int_{B_r(0)} u(z)d\lambda(z) \le \varepsilon \log(\frac{1}{|r|})$$

so we know that $z \in B_{r/2}(0)$ we have $B_{r/4}(z) \subset B_r(0) \setminus \{0\}$ so we have from the mean value inequality and non-negativity that

$$u(z) \le \frac{16}{\pi r^2} \int_{B_{r/4}(z)} u(w)d\lambda(w) \le \frac{16}{\pi r^2} \int_{B_r(0)} u(w)d\lambda(w) \le 16\varepsilon \log(\frac{1}{|r|}) \le 16\varepsilon \log\frac{1}{|z|}$$

so $u(z) = o(\log(\frac{1}{|z|})$.

Now fix any $\alpha > 0$ then we know $u(z) + \alpha\log(\frac{1}{|z|}) \to -\infty$ as $z \to 0$ thanks to $u(z) = o(\log(1/|z|))$. So there exists an $R_\alpha = R > 0$ such that on $B_{R_\alpha}(0)$ we have $v(z) = u(z) + \alpha\log(1/|z|) \le 0$. Therefore, on the annulus $A_{1,R} = \{|z| : R < |z| < 1\}$ we have that $v|_{\partial A_{1,R}} \le 0$ but as $v(z)$ is sub-harmonic it follows that the max is obtained on the boundary so we have $v(z) \le 0$ on $A_{1,R}$ which by letting $R \to 0$ implies $v(z) \le 0$ on $\mathbb{D}\setminus\{0\}$. Therefore, letting $\alpha \to 0$ gives $u(z) \le 0$ on $\mathbb{D}\setminus\{0\}$, which implies from non-negativity that $u(z) \equiv 0$ □ □

Problem 5. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ and suppose that

$$\int_{\mathbb{D}} |f_n(z)|d\lambda(z) \le 1$$

for all $n \in \mathbb{N}$ where $d\lambda$ denotes integration with respect to Lebesgue measure λ on $\mathbb{C}$. Show that then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on all compact subsets of $\mathbb{D}$.

Proof. Fix a compact subset $K \subset \mathbb{D}$ and let $\text{dist}(K, \partial D) := 2\delta$ then for any $z \in K$ we have $B_\delta(z) \subset \mathbb{D}$. So by the Mean Value Property we have for any $z \in K$ and $0 < r < \delta$

$$f_n(z) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z + re^{i\theta})d\theta$$

so we have

$$\int_{r=0}^{\delta} rf_n(z)dr = \int_{r=0}^{\delta} r\frac{1}{2\pi} \int_{\theta=0}^{2\pi} f(z + re^{i\theta})drd\theta = \frac{1}{2\pi} \int_{B_\delta(z)} f_n(z)d\lambda(z)$$

so we have

$$f_n(z) = \frac{1}{\pi\delta^2} \int_{B_\delta(z)} f_n(z)d\lambda(z)$$

so in particular, on K we have the uniform bound

$$\sup_{z\in K} |f_n(z)| \le \frac{1}{\pi\delta^2}$$

This implies on any compact subset the family $\{f_n\}$ is uniformly bounded. Therefore, for any $n \in \mathbb{N}$ with $\Omega_n := B_{1-\frac{1}{n}}(0)$ we can by Montel's theorem find a uniformly convergent subsequence on Ω_n. We will now use a diagonal argument: On Ω_1 we can find a subsequence $\{f_{n_k^{(1)}}\}$ such that it convegres uniformly on Ω_1 to a limiting function which we denote by f. Then on Ω_2 we can a subsequence $n_k^{(2)} \subset n_k^{(1)}$ for which $f_{n_k^{(2)}}$ converges uniformly on Ω_2 to f. We repeat this for all n. Define the index $n_k := n_k^{(k)}$ i.e.

the diagonal subsequence then we have $f_{n_k} \to f$ uniformly on Ω_j for any j. Indeed, if $\varepsilon > 0$ then there exists an $K \in n$ such that if $k \neq N$ then

$$||f_{n_k^{(j)}} - f||_{L^\infty(\Omega_j)} \leq \varepsilon$$

and since n_k is a subsequence of $n_k^{(j)}$ for $k \geq j$ we can take $K_1 := \max\{K, n_k\}$ and obtain

$$||f_{n_k} - f||_{L^\infty(\Omega_j)} \leq \varepsilon$$

Therefore, $f_{n_k} \to f$ uniformly on Ω_n for all n. Now if K is compact it must live in an Ω_n for some n which implies f_{n_k} converges uniformly to f on K . □

Problem 6. Let $U \subset \mathbb{C}$ be a bounded open set with $0 \in U$ and $f : U \to \mathbb{C}$ be a holomorphic map with $f(U) \subset U$ and $f(0) = 0$. Show that $|f'(0)| \leq 1$

Proof. We first observe by the chain rule that if $f^n := f \circ f \circ ... \circ f$ where we do n compositions, then we have

$$\frac{d}{dz}(f^n(z))(0) = (f'(0))^n$$

And we have that for all n that $f^n(U) \subset U$ and $f^n(0) = 0$. Since U is bounded there exists an $R > 0$ such that $U \subset B_R(0)$. As U is open there exists an $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U$, so by Cauchy's theorem we have for $g_n := f^n$

$$|g_n'(0)| \leq \frac{1}{\varepsilon}||g_n||_{L^\infty}^n \leq \frac{R}{\varepsilon} \Rightarrow |f'(0)| \leq \left(\frac{R}{\varepsilon}\right)^{1/n}$$

where $U \subset B_R(0)$. Letting $n \to \infty$ gives

$$|f'(0)| \leq 1$$

as desired. □

Problem 7. Show that there is a dense set of functions $f \in L^2([0,1])$ such that $x \mapsto x^{-1/2}f(x) \in L^1([0,1])$ and $\int_0^1 x^{-1/2}f(x)dx = 0$

Proof. As $C_c([0,1])$ is a dense subclass of $L^2([0,1])$ it will suffice to show the claim for $f \in C_c([0,1])$. So fix $f \in C_c([0,1])$ this implies there exists an $\delta = \delta(f) > 0$ such $f = 0$ on $[0,\delta]$. Define $I := \int_0^1 x^{-1/2}f(x)dx$ which is finite since $x^{-1/2} \in L^1([0,1])$ and f is bounded and for $0 < \varepsilon < \delta$ define

$$g_\varepsilon := -\frac{I}{\delta(\varepsilon)}x^{-1/2+\varepsilon}$$

where

$$\delta(\varepsilon) = \int_0^\varepsilon x^{-1+\varepsilon}$$

and we write

$$f^\varepsilon(x) := \begin{cases} f(x) \text{ for } x \in [\varepsilon, 1] \\ g_\varepsilon(x) \text{ for } x \in [0, \varepsilon] \end{cases}$$

so observe

$$\int_0^1 x^{-1/2}g_\varepsilon dx = \int_0^\varepsilon -I/\delta(\varepsilon)x^{-1+\varepsilon}dx + \int_\varepsilon^1 x^{-1/2}f(x)$$
$$= -I + \int_\varepsilon^1 x^{-1/2}f(x) = 0$$

And it is clear that $x^{-1/2}f^\varepsilon \in L^1([0,1])$ and its clear $f^\varepsilon \in L^2([0,1])$. Then observe that

$$\int_0^1 |f^\varepsilon - f|^2 = \int_0^\varepsilon \frac{I^2}{\delta^2(\varepsilon)}x^{-1+2\varepsilon} = \frac{I^2}{\delta^2}\frac{\varepsilon^{2\varepsilon}}{2\varepsilon}$$

and we observe

$$\delta = \frac{\varepsilon^\varepsilon}{\varepsilon} \Rightarrow \delta^2 = \frac{(\varepsilon^\varepsilon)^2}{\varepsilon^2}$$

so

$$\int_0^1 |f^\varepsilon - f|^2 = O(\varepsilon)$$

so $f^\varepsilon \to f$ and f so the desired class is dense.

□

Problem 8. Compute the following limits and justify your calculations!

(1) $\lim_{k\to\infty} \int_0^k x^n(1-\frac{x}{k})^k$
(2) $\lim_{k\to\infty} \int_0^\infty (1+\frac{x}{k})^{-k}\cos(x/k)$

Proof. We first compute

$$\int_0^\infty \lim_{k\to\infty} x^n(1-\frac{x}{k})^k dx = \int_0^\infty x^n \exp(-x) = n!$$

So we claim that

$$\lim_{k\to\infty} \int_0^k x^n(1-\frac{x}{k})^k = n!$$

which we will justify by swapping the limits with the integral. Indeed, observe

$$\int_0^k x^n(1-\frac{x}{k})^k = \int_0^\infty x^n(1-\frac{x}{k})^k \chi_{[0,k]}$$

and pointwise we have

$$x^n(1-\frac{x}{k})^k \chi_{[0,k]} \to x^n \exp(-x)$$

Notice by the AMGM inequality we have that

$$\Big(1\cdot(1-\frac{x}{k})^k\Big)^{1/(k+1)} \le \frac{1+k(1-\frac{x}{k})}{k+1} = \frac{1+k-x}{k+1} = 1-\frac{x}{k+1}$$

i.e.

$$(1-\frac{x}{k})^k \le (1-\frac{x}{k+1})^{k+1}$$

so the family is increasing and we can apply the Monotoe Convergence Theorem to interswap limits with the derivative to get the integral is $n!$.

Notice pointwise we have

$$\lim_{k\to\infty}(1+\frac{x}{k})^{-k}\cos(x/k) \to e^{-x}$$

so we should have

$$\lim_{k\to\infty}\int_0^\infty (1+\frac{x}{k})^{-k}\cos(x/k)dx = \int_0^\infty e^{-x} = 1$$

Notice that for $k \ge 2$

$$(1+\frac{x}{k}) \ge (1+\frac{x}{2}) \Rightarrow (1+\frac{x}{k})^{-k} \le (1+\frac{x}{2})^{-k} \le (1+\frac{x}{2})^{-2} \in L^1([0,\infty])$$

so by the DCT we can swap limits since

$$(1+\frac{x}{k})^{-k}|\cos(x/k)| \le (1+\frac{x}{2})^{-2} \in L^1([0,\infty])$$

so

$$\lim_{k\to\infty}\int_0^\infty (1+\frac{x}{k})^{-k}\cos(x/k)dx = \int_0^\infty e^{-x} = 1$$

□

Problem 9. Let X be a Banach Space, Y a normed linear space, and $B : X \times Y \to \mathbb{R}$ be a bilinear function. Suppose that for each $x \in X$ there exists a constant $C_x \geq 0$ such that $|B(x,y)| \leq C_x||y||$ for all $y \in Y$, and for each $y \in Y$ there exists a constant $C_y \geq 0$ such that $|B(x,y)| \leq C_y||x||$ for all $x \in X$.

Show that then there exists a constant $C \geq 0$ such that $|B(x,y)| \leq C||x|| \cdot ||y||$ for all $x \in X$ and $y \in Y$

Proof. Fix a $y \in Y$ and define the linear operator $B_y : X \to \mathbb{R}$ and with $B_y(x) := B(x,y)$ then by assumption we have

$$|B_y(x)| \leq C_y||x||$$

so $B_y \in X^*$. So we consider the family $F := \{B_y\}_{y \in Y}$ with $||y|| = 1$ of continuous linear operators. Fixing an $x \in X$ we see that

$$|B_y(x)| \leq C_x||y|| = C_x \text{ since } ||y|| = 1$$

Therefore,

$$\sup_{y \in Y, ||y||=1} |B_y(x)| < \infty$$

so by the uniform boundness principle since X is a banach space, the family is uniformly bounded i.e. there exists a $C > 0$ such that for any x

$$|B_y(x)| \leq C||x|| \text{ for all } ||y|| = 1$$

Now for $y \neq 0$ we have that

$$B(x,y) = B(x, ||y||\frac{y}{||y||}) = ||y||B(x, \frac{y}{||y||}) \leq C||y|| \cdot ||x||$$

as desired $\quad \square$. $\square$

Problem 10. Let $f \in L^2(\mathbb{R})$ and define $h(x) := \int_{\mathbb{R}} f(x-y)f(y)dy$ for $x \in \mathbb{R}$. Show then that there exists a function $g \in L^1(\mathbb{R})$ such that

$$h(\xi) = \int_{\mathbb{R}} e^{-i\xi x} g(x)dx$$

for $\xi \in \mathbb{R}$ i.e. h is the fourier transform of some function in $L^1(\mathbb{R})$.

Conversely, show that if $g \in L^1(\mathbb{R})$, then there is a function $f \in L^2(\mathbb{R})$ such that the fourier transform of g is given by $x \mapsto h(x) := \int_{\mathbb{R}} f(x-y)f(y)dy$

Proof. Let $f_n \subset L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ be such that $f_n \to f$ in $L^2(\mathbb{R})$. Then define

$$h_n(x) := \int_{\mathbb{R}} f_n(x-y)f_n(y)dy$$

we then claim $h_n(x) \to h(x)$ uniformly. Note that $h(x)$ is continuous since $h(x) = \int_{\mathbb{R}} f(x-y)f(y)dy = \int_{\mathbb{R}} f(x+y)f(-y) = (\tau_{-x}f(y), f(-y))$ where $\tau_x f(y) := f(y-x)$. This implies continuity since

$$|h(x) - h(z)| = |(\tau_{-x}f(y) - \tau_{-z}f(y), f(-y))| \leq ||f(y+x) - f(y+x)||_{L^2}||f||_{L^2}$$

Now observe that

$$|h(x) - h_n(x)| \leq \int_{\mathbb{R}} |f_n(x-y)f_n(y) - f_n(x-y)f(y)| + |f_n(x-y)f(y) - f(x-y)f(y)|dy$$

$$\leq ||f_n||_{L^2}||f_n - f||_{L^2} + ||f||_{L^2}||f_n - f||_{L^2} \to 0$$

so we have uniform convergence.

Now for each f_n we know that $\mathcal{F}^{-1}(h_n) = \int_{\mathbb{R}} (\hat{h_n})e^{itx}dx$ by computation since $f_n \in L^1 \cap L^2$. But again since $f_n \in L^1$ we have

$$\hat{h_n} = (\hat{f_n})^2$$

which implies $\mathcal{F}^{-1}(h_n) = \int_{\mathbb{R}} (\hat{f_n})^2 e^{itx}d$. So it follows that

$$h_n(t) = \mathcal{F}(\mathcal{F}^{-1}(h_n))(t) = \int_{\mathbb{R}} g_n(x)e^{-itx}dx \text{ where } g_n(x) := \int_{\mathbb{R}} (\hat{f_n}(t))^2 e^{itx}dt$$

and notice that by Plancherel Theorem that $\hat{f_n} \in L^2(\mathbb{R})$ so $g_n(x) \in L^1(\mathbb{R})$. Also notice that for $g(x) := \int_{\mathbb{R}} (\hat{f})^2 e^{itx}dx$

$$|g_n(x) - g(x)| = \left| \int_{\mathbb{R}} \left[(\hat{f_n}(t) + \hat{f}(t))(\hat{f_n}(t) - \hat{f_n}(t)) \right] e^{-itx} \right| \leq ||f_n + f||_{L^2} ||f_n - f||_{L^2} \to 0$$

so g_n uniformly converges to g. So it follows that from taking $n \to \infty$ on

$$h_n(t) = \int_{\mathbb{R}} g_n(t)e^{-itx}$$

that

$$h(t) = \int_{\mathbb{R}} g(t)e^{-itx}dx$$

and its clear by Cauchy-Schwarz that $g \in L^1(\mathbb{R})$ as desired.

Let us formally derive what f should be first. Indeed, observe if we have such an f then

$$\int_{\mathbb{R}} g(x)e^{-i\xi x} = \int_{\mathbb{R}} f(\xi - y)f(y)dy = \int_{\mathbb{R}} F(x)e^{-i\xi x}dx$$

where

$$F(x) := \int_{\mathbb{R}} (\hat{f}(t))^2 e^{itx}$$

so we expect

$$g(x) = \int_{\mathbb{R}} (\hat{f}(t))^2 e^{itx}dt = \mathcal{F}^{-1}(\hat{f}^2) \Rightarrow \mathcal{F}(g(x)) = \hat{f}^2 \Rightarrow f = \mathcal{F}^{-1}(\sqrt{\mathcal{F}(g(x))})$$

where

$$\sqrt{\mathcal{F}(g(x))} := \sqrt{|\mathcal{F}(g(x))} \frac{\mathcal{F}(g(x))|}{|\mathcal{F}(g(x))|}$$

and 0 when $\mathcal{F}(g(x)) = 0$. □

Problem 11. Consider the space $C([0,1])$ of real-valued continuous functions on the interval $[0,1]$. We denote $||f||_\infty := \sup_{x\in[0,1]} |f(x)|$ the supremium norm and by $||f||_2 := \left(\int_0^1 |f(x)|^2\right)^{1/2}$ the L^2 norm of the function $f \in C([0,1])$.

Let S be a subspace of $C([0,1])$. Show that if there existed a constant $K > 0$ such that $||f||_\infty \leq K||f||_2$ for all $f \in S$, then S is finite dimensional.

Proof. Notice that if we endow S with $||\cdot||_2$ then this is an equivalent norm, that the evaluation linear functional (for $x \in [0,1]$)

$$L_x(f) := f(x)$$

is a continuous linear functional since

$$|L_x(f)| \leq ||f(x)||_{L^\infty} \leq K||f||_{L^2}$$

so it extends to a continuous linear operator on $\overline{S}$. So by Risez Representation Theorem (the equivalent norm implies $\overline{S}$ is a Hilbert Space), we can find a $g_x \in \overline{S}$ such that

$$L_x(f) = (f, g_x) = \int_0^1 f g_x dy$$

Fix N orthonormal vectors $\{e_i(x)\}_{i=1}^N$, then we compute from Bessel's Inequality that

$$\sum_{i=1}^N |e_i(x)|^2 = \sum_{i=1}^N |(e_i, g_x)|^2 \leq ||g_x||_2$$

and we have

$$||g_x||_2^2 = (g_x, g_x) = g_x(x) \leq ||g_x||_\infty \leq K||g_x||_2 \Rightarrow ||g_x||_2 \leq K$$

so we have

$$\sum_{i=1}^N |e_i(x)|^2 \leq K^2 \Rightarrow \sum_{i=1}^N \int_0^1 |e_i(x)|^2 dx \leq K^2$$

i.e.

$$N \leq K^2$$

so we can have at most K orthnormal vectors i.e. at most K linearly independent vectors since we can preform Grahm-Schmit, so the space is fintie dimensional.

□

Problem 12. Let $f : [0,1] \to \mathbb{R}$ be a continuous function that is absolutely continuous on each interval $[\varepsilon, 1]$ for $0 < \varepsilon \leq 1$.

(1) Show that f is not necessarily absolutely continuous on $[0,1]$.

(2) Show that if f is of bounded variations on $[0,1]$, then f is absolutely continuous on $[0,1]$.

Proof. For (1) we take $f(x) := x\sin(1/x)$ with $f(0) := 0$ then $f \in C([0,1])$ by the squeeze theorem and $f(x) \in C^1((0,1))$ since

$$f'(x) = \sin(1/x) - \frac{\cos(1/x)}{x}$$

which is continuous on the open interval $(0,1)$. So in particular, if $1 \geq \varepsilon > 0$ then on $[\varepsilon, 1]$ we have $||f'||_{L^\infty} \leq C(\varepsilon)$ so $f(x)$ is lipschitz on $[\varepsilon, 1]$ so it is absolutely continuous on $[\varepsilon, 1]$. But f is not absolutely continuous on $[0,1]$ since it is not of bounded variation. Indeed, notice that at $x_n := \frac{2}{\pi(2n+1)}$ and $x_n \to 0$ as $n\infty$ where $n \geq 1$ and $\sin(1/x_n) = (-1)^n$ so we have

$$\sum_{n=1}^N |f(x_n) - f(x_{n+1})| = \sum_{n=1}^N |(-1)^n x_n - (-1)^{n+1} x_{n+1}| = \sum_{n=1}^N |x_n + x_{n+1}| \geq \frac{2}{\pi} \sum_{n=1}^N \frac{1}{2n+1} \to \infty$$

Therefore, the total variation is unbounded over $[0,1]$ is unbounded since

$$T_f([0,1]) = \sup\{\sum_{n=1}^N |f(x_n) - f(x_{n+1})| : 0 = x_1 < ... < x_{N+1} = 1\}$$

and we can always adjoin to the above sum $x_{N+2} = 0$ which only increases the sum size. So in particular, f is not of bounded variation so it cannot be absolutely continuous.

For (2) we claim that the total variation is continuous i.e.

$$T_f([x,y]) := \sup\{\sum_{n=1}^N |f(x_n) - f(x_{n+1})| : x = x_1 < ... < x_{N+1} = y\}$$

for any f with the above conditions. Indeed, by uniform continuity if $\varepsilon > 0$ then there exists a $\delta > 0$ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. So notice we can find a partiton $0 = x_1 < ... < x_{N+1} = 1$ such that

$$T_f([0,1]) \leq \varepsilon + \sum_{i=1}^N |f(x_{i+1}) - f(x_i)|$$

Now fixing $x \in (0,1]$ we can assume that $x = x_M$ and $|x_M - x_{M+1}| < \delta$ for some M since adding a point to our partiton only increases the sum. So in particular,

$$T_f([0,1]) \leq \varepsilon + |f(x) - f(x_{M+1})| + \sum_{i=1}^{M-1} |f(x_{i+1}) - f(x_i)| + \sum_{M+1}^{N} |f(x_{i+1}) - f(x_i)|$$
$$\leq 2\varepsilon + T_f([0,x]) + T_f([x_{M+1},1])$$

Therefore, as

$$T_f([0,1]) = T_f([0,x]) + T_f([x, x_{M+1}] + T_f([x_{M+1},1]$$

we conclude that

$$T_f([x, x_{M+1}] \leq 2\varepsilon$$

so in particular T_f is right continuous. But an identical argument using the if $|x - x_{M-1}| < \delta$ shows that T_f is left continuous, so its continuous.

Now as $T_f([0,x])$ is uniformly continuous since $[0,1]$ is compact we can find a $\delta > 0$ such that if $\varepsilon > 0$ is given then

$$T_f([0,x]) \leq \varepsilon \text{ when } |x| < \delta$$

Then on $[\delta/2, 1]$ we know that f is absolutely continuous so we can find a $\eta > 0$ such that if $\sum_{i=1}^N |x_i - y_i| < \eta$ then $\sum_{i=1}^N |f(x_i) - f(y_i)| < \varepsilon$ where $\{x_i\}_{i=1}^N$ and $\{y_i\}_{i=1}^N$ are in the interval $[\delta, 1]$. Let $\hat{\delta} := \min\{\delta, \eta\}/4$ then if we are given

$$\sum_{i=1}^{M} |x_i - y_i| < \hat{\delta}$$

where the $x_i, y_i \in [0,1]$ then we know that if $x_i \in [0, \delta/2]$ then as

$$|x_i - y_i| < \delta/4 \Rightarrow y_i \in [0, \delta/2]$$

So we relabel our sequence to $\{x_i\}_{i=1}^M, \{y_i\}_{i=1}^M$ and $\{x_i\}_{i=M+1}^N, \{y_i\}_{i=M+1}^N$ where $x_i, y_i \in [0, \delta/2]$ for $1 \leq i \leq M$ and $x_i, y_i \in [\delta/2, 1]$ for $M+1 \leq i \leq N$. Then we have

$$\sum_{i=1}^{M} |f(x_i) - f(y_i)| + \sum_{i=M+1}^{N} |f(x_i) - f(y_i)| \leq \sum_{i=1}^{M} |f(x_i) - f(y_i)| + \varepsilon$$

where the second inequality is due to absolute continuity of $[\delta/2, 1]$ and observe

$$\leq T_f([0, \delta/2]) + \varepsilon \leq 2\varepsilon$$

by uniform continuity of the total variation. So f is uniformly continuous.

□

9. Spring 2014

Problem 1. Consider a measure space $(X, \mathcal{X})$ with a σ-finite measure μ and, for each $t \in \mathbb{R}$, let e_t denote the characteristics function of the interval (t, ∞). Prove that if $f, g : X \to \mathbb{R}$ are $\mathcal{X}$-measurable, then $||f - g||_{L^1(X)} = \int_{\mathbb{R}} ||e_t \circ f - e_t \circ g||_{L^1(X)} dt$

Proof. Notice that

$$\int_{\mathbb{R}} ||e_t \circ f - e_t \circ g||_{L^1(X)} = \int_{\mathbb{R}} \int_X |e_t \circ f - e_t \circ g| d\mu dt$$

so by Tonelli's since the integrand is non-negative and μ is σ-finite we have

$$= \int_X \int_{\mathbb{R}} |e_t \circ f - e_t \circ g| dt d\mu = \int_X \int_{\mathbb{R}} |\chi_{f(x) \geq t} - \chi_{g(x) \geq t}| dt d\mu$$

Now we compute

$$\int_{\mathbb{R}} |\chi_{f(x) \geq t} - \chi_{g(x) \geq t}| dt = \int_{\min\{f(x), g(x)\}}^{\max\{f(x), g(x)\}} 1 dt = |f(x) - g(x)|$$

so this gives us

$$\int_{\mathbb{R}} ||e_t \circ f - e_t \circ g||_{L^1(X)} = ||f - g||_{L^1(X)}$$

□

Problem 2. Let $f \in L^1(\mathbb{R}, dx)$ and $\beta \in (0, 1)$. Prove that

$$\int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^\beta} dx < \infty$$

for (Lebesgue) a.e. $a \in \mathbb{R}$.

Proof. Fix $n \in \mathbb{N}$ then by Tonelli since $|f(x)|/|x - a|^\beta \geq 0$ we can justify the following computation

$$\int_{a=-n}^{n} \int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^\beta} dx da = \int_{\mathbb{R}} \int_{a=-n}^{n} \frac{|f(x)|}{|x - a|^\beta} da dx = \int_{\mathbb{R}} |f(x)| \int_{a=-n}^{n} \frac{1}{|x - a|^\beta} da dx$$

and observe that

$$\int_{a=-n}^{n} \frac{1}{|x - a|^\beta} da = \int_{x-n}^{x+n} \frac{1}{|a|^\beta} da \leq \int_{-n}^{n} \frac{1}{|a|^\beta} da \leq C(\beta, n)$$

since $1/|x|^\beta \in L^1_{loc}(\mathbb{R})$ so we have

$$\int_{a=-n}^{n} \int_{\mathbb{R}} \frac{|f(x)|}{|x - a|^\beta} dx da \leq C(\beta, n) \int_{\mathbb{R}} |f(x)| dx < \infty$$

so we conclude that $\int_{\mathbb{R}} \frac{|f(x)|}{|x-a|^\beta} dx < \infty$ a.e. on $[-n, n]$, which by letting $n \to \infty$ implies this is true for a.e. $a \in \mathbb{R}$. □

Problem 3. Let $[a, b] \subset \mathbb{R}$ be a finite interval and $f : [a, b] \to \mathbb{R}$ be a bounded Borel Measurable function.

(1) Prove that $\{x \in [a, b] : f(x)$ is continuous at $x\}$ is Borel Measurable.
(2) Prove that f is Riemann Integrable if and only if its continuous almost everywhere.

Proof. Define $A := \{x \in [a, b] : f(x)$ is continuous at $x\}$ then we claim that A is a G_δ set. Indeed, define the oscillation

$$\omega(f, A) := \sup_{x, y \in A} |f(x) - f(y)|$$

then notice that f is continuous at x iff

$$\lim_{r\to 0+}\omega(f,B_r(x)) := \omega(f,x) = 0$$

where the limit exists since its an inf. Now observe

$$A = \{x : \omega(f,x) = 0\} = \bigcap_{n=1}^{\infty}\{x : \omega(f,x) < \frac{1}{n}\} := \bigcap_{n=1}^{\infty} A_n$$

Now we claim that each A_n is an open set. Indeed, if $\omega(f,x) < 1/n$ then there is a ball $B_r(x)$ such that

$$\omega(f,B_r(x)) < 1/n$$

Then for any $y \in B_{r/2}(x)$ we can find a $\delta_y > 0$ such that $B_{\delta_y}(y) \subset B_r(x)$ so it follows that

$$\omega(f,B_{\delta_y}(y)) \le \omega(f,B_r(x)) < 1/n$$

so $B_{r/2}(x) \subset A_n$ so A_n is open and the set of continuity is a G_δ, so it is Borel Measurable.

Now we prove f is Riemann Integrable iff it is continuous a.e. Notice if B is the set where $f(x)$ is discontinuous on $[a,b]$ then

$$B = \bigcup_{n=1}^{\infty}\{x \in [a,b] : \omega(f,x) > \frac{1}{n}\} := \bigcup_{n=1}^{\infty} B_n$$

so it suffices to show $m(B_n) = 0$ for all n.

For the second condition fix a partition P $a = x_0 < ... < x_n = b$. Then define the lower and upper Riemann Sums

$$U(f,P) := \sum_{i=1}^{n}\left(\sup_{x\in[x_{i-1},x_i]} f(x)\right)\Delta x_i \text{ and } L(f,P) := \sum_{i=1}^{n}\left(\inf_{x\in[x_{i-1},x_i]} f(x)\right)\Delta x_i$$

where $\Delta x_j = x_j - x_{j-1}$. Observe

$$U(f,P) = \sum_{i=1}^{n}\int_a^b (\sup_{x\in I_i} f(x))\chi_{I_i}(y)dy \text{ and } L(f,P) = \sum_{i=1}^{n}\int_a^b (\inf_{x\in I_i} f(x))\chi_{I_i}(y)dy$$

where $I_i = [x_{i-1},x_i]$. Then

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}\int_a^b \omega(f,I_i)\chi_{I_i}(y)dy$$

So if f is continuous a.e., then $m(A) = m(\{x : \omega(f,x) = 0\}) = b - a$. So in particular, $\omega(f,I_i) \to 0$ as $\Delta x \to 0$ where $\Delta x := \max_{1\le j\le n}\Delta x_j$, so the dominated convergence theorem implies since the oscillation is bounded since f is bounded that

$$U(f,P) - L(f,P) \to 0 \text{ as } \Delta x \to 0$$

i.e. f is Riemann Integrable.

Now for the reverse direction fix $\varepsilon > 0$, since f is Riemann Integrable there exists an $\delta > 0$ such that if $\Delta x < \delta$ then

$$0 \le U(f,P) - L(f,P) = \sum_{i=1}^{n}\omega(f,I_i)m(I_i) \le \varepsilon$$

Write $F_n := \{x : \omega(f,x) > 1/n\}$ Now write

$$I := \{k \in 0,1..,n : F_n \cap I_k \neq \emptyset\}$$

this implies

$$F_n \subset \bigcup_{i\in I} I_i \cup \{x_0,..,x_n\}$$

so it suffices to show $\sum_{i\in I} m(I_i) = O(\varepsilon)$. Notice as $I_i \cap F_n \neq \emptyset$ that $\omega(f, I_i) > 1/n$ so we have

$$\frac{1}{n}\sum_{i\in I} m(I_i) \leq \sum_{i\in I} \omega(f, I_i)m(I_i) \leq \sum_{i=1}^{n} \omega(f, I_i)m(I_i) \leq \varepsilon$$

so it follows that

$$\sum_{i\in I} m(I_i) \leq \varepsilon n$$

this implies $m(F_n) = 0$ so f is contiuous a.e. □

Problem 4. Consider a sequence $\{a_n : n \geq 1\} \subset [0, 1]$. For $f \in C([0, 1])$, define

$$\varphi(f) := \sum_{n=1}^{\infty} 2^{-n} f(a_n)$$

Orive that there is no $g \in L^1$ such that $\varphi(f) = \int f(x)g(x)dx$ is true for all $f \in C([0, 1])$.

(2) Each $g \in L^1$ defines a continuous functional $T_g \in L^\infty$ via

$$T_g(f) = \int f(x)g(x)dx$$

Show there are continuous functionals on $L^\infty([0, 1])$ that are not of this form.

Proof. Define $B([0, 1])$ to be the space of bounded functions on $[0, 1]$ endowed with the sup norm, now we extend φ to a map on this space via

$$\varphi(f) := \sum_{n=1}^{\infty} 2^{-n} f(a_n)$$

So now we can define a measure

$$\mu(E) := \varphi(E)$$

and it is easy to see μ is a measure and it is not absolutely continuous with respect to the Lebesgue measure since $\mu(\{a_n\}) = 2^{-n}$ and $\{a_n\}$ is a null set for the Lebesgue measure. Therefore, by Radon Nikodyn there exists a $m_f \ll m$ and $\lambda \perp m$ with $\lambda \neq 0$ such that

$$\mu = m_f + \lambda$$

And notice that

$$\varphi(f) = \int f d\mu$$

due to linearity and equality holding for simple functions. So it follows that $\varphi \ll \mu$ so φ is not absolutely continuous with the Lebesgue measure. □

For the second part, by Hanh-Banach as φ is a continuous linear functional on the subspace $C([0, 1])$ we can extend it to a continuous linear functional on $L^\infty([0, 1])$. And the desired result holds since if φ is a functional of that form it implies its restriction is as well, which we proved in a) was not.

Problem 5. Recall that a metric space is separable if it contains a countable dense subset.

(1) Prove that $\ell^1(\mathbb{N})$ and $\ell^2(\mathbb{N})$ are separable Banach Spaces, but ℓ^∞ is not.
(2) Prove there exists no linear bounded surjective map $T : \ell^2(\mathbb{N}) \to \ell^1(\mathbb{N})$

Proof. Denote e_i to be the vector that is 0 everywhere except for a 1 at the ith position. We claim that finite rational combinations of e_i are dense in ℓ^p for any $1 \leq p < \infty$. This immediately implies (1). Indeed, if $x = (x_1, x_2, ...) \in \ell^p$ and $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$\sum_{n=N}^{\infty} |x_n|^p \leq \varepsilon$$

Then for each $i = 1, .., N$ we can find a rational q_i such that $|x_i - q_i|^p \leq \varepsilon 2^{-i}$. This implies for $y := \sum_{i=1}^N q_i e_i$ with $y = (y_1, y_2, ...)$

$$\sum_{n=1}^\infty |x_n - y_n|^p = \sum_{n=1}^N |x_n - q_n|^p + \sum_{n=N+1}^\infty |x_n|^p \leq \sum_{n=1}^\infty \varepsilon 2^{-n} + \varepsilon = 2\varepsilon$$

Therefore, we have shown finite rational combinations of e_i are dense in ℓ^p for $1 \leq p < \infty$ as desired.

To show ℓ^∞ is not separable we will show there exists an uncountable sequence $\{x_\alpha\}_{\alpha \in A}$ such that $||x_\alpha||_{\ell^\infty} = 1$ and $||x_\alpha - x_\beta||_{\ell^\infty} \geq 1/2$ whenever $\alpha \neq \beta$. This implies ℓ^∞ is not separable since if $\{v_i\}$ is a countable sub-sequence of ℓ^∞ then if

$$||x_\alpha - v_i||_\infty < 1/4$$

then

$$||x_\beta - v_i||_\infty \geq ||x_\alpha - x_\beta|| - ||x_\beta - v_i|| \geq 1/2 - 1/4 = 1/4$$

so the v_i cannot be dense since there are uncountably many x_β. Now we construct such x_α. Notice that there are uncountably many binary strings i.e. sequences where every entry is 0 or 1. Observe that if x_α and x_β are different binary strings then

$$||x_\alpha - x_\beta||_{\ell^\infty} = 1$$

and $||x_\alpha||_{\ell^\infty} = 1$, so we are done.

For (2), if T is surjective then its adjoint $T^* : \ell^\infty(\mathbb{N}) \to \ell^2(\mathbb{N})$ is injective. So T^* is a linear isomorphism from ℓ^∞ to a subset of ℓ^2. This implies that $T^*(\ell^\infty(\mathbb{N}))$ is separable since it is a subset of ℓ^2. But then since $(T^*)^{-1} : T^*(\ell^\infty(\mathbb{N})) \to \ell^\infty$ is a homeomomorphism we see it preserves separability, which means ℓ^∞ is separable, which contradicts the first part.

□

Problem 6. Given a Hilbert Space $\mathcal{H}$, let $\{a_n\}_{n\geq 1} \subset \mathcal{H}$ be a sequence with $||a_n|| = 1$ for all $n \geq 1$. Recall that the convex hull of $\{a_n\}_{ng1}$ is the closure of the set of all convex combinations in $\{a_n\}_n$

(1) Show that if $\{a_n\}$ spans $\mathcal{H}$ linearly (i.e., any $x \in \mathcal{H}$ is of the form $\sum_{k=1}^m c_k a_{n_k}$, for some m and $c_k \in C$), then $\mathcal{H}$ is finite dimensional.
(2) Show that if $(a_n, \xi) \to 0$ for all $\xi \in \mathcal{H}$, then 0 is in the closed convex hull of $\{a_n\}_n$.

Proof. We argue by contradiction and assume that there are infinitely many a_n that are linearly independent. Denote the largest subset of $\{a_n\}$ such that every term is linearly independent as $\{b_n\}$. This automatically implies $\text{span}\{b_n\} = \text{span}\{a_n\}$ which exists thanks to Zorn's Lemma. Now we go Gram-Schmit on b_n to obtain a new sequence $\{\alpha_n\}$ that are orthonormal and $\text{span}\{\alpha_n\} = \text{span}\{a_n\}$. Note there are infinitely many α_n. So in particular define

$$y := \sum_{n=1}^\infty \frac{1}{n^2}\alpha_n \in \mathcal{H}$$

since it is the limits of $y_n := \sum_{k=1}^n (1/n^2)\alpha_n$ (and $\mathcal{H}$ is complete). But as α_n is a linear combination of a_n it must still linearly span $\mathcal{H}$, but our y is not any finite linear combination of α_n. Thus we have arrived at a contradiction. This implies $\mathcal{H}$ is finite dimensional.

For (2) notice that if $v_1, .., v_N \in \mathcal{H}$ with $||v_i|| = 1$ then

$$||\frac{1}{N}\sum v_i|| = \frac{1}{N} + \text{ sum of N inner products}$$

taking v_i to be some a_{n_k} we see that we can make their convex combination norm get arbitrarily close to 0 for a correct subsequence thanks to $(a_n, \xi) \to 0$.

□

Problem 7. Characterize all entire functions with $|f(z)| > 0$ for $|z|$ large and
$$\limsup_{z\to\infty} \frac{|\log|f(z)||}{|z|} < \infty$$

Proof. Notice that as zeros are isolated and $f(z)$ is non-zero for large z we know that f has only finitely many roots. So there is a polynomial $P(z)$ with the exact same zeros as f. It follows that $g(z) := f(z)/P(z)$ is an entire function with no zeros. Therefore, there is a $h(z)$ entire such that
$$f(z)/P(z) = \exp(h(z))$$
Then notice
$$\limsup_{z\to\infty} \frac{|\log|g(z)||}{|z|} = \limsup_{z\to\infty} \frac{|\log|f(z)| - \log|P(z)||}{|z|} \leq \limsup_{z\to\infty} \frac{|\log|f(z)|| + |\log|P(z)||}{|z|} < \infty$$
so it follows that from $\log|g(z)|$ being finite on every compact set since it is a continuous function that there is a $C > 0$ such that
$$|\log|g(z)|| \leq C|z|$$
This implies by taking exponentials that
$$|g(z)| = |\exp(h(z))| \leq \exp(C|z|)$$
i.e. g is an entire function of order 1 with no zeros, so by Hadamard's Factorization Theorem we know there is a linear polynomial $Az + B$ such that
$$g(z) = \exp(Az + B) \Rightarrow p(z) = P(z)\exp(Az + B)$$
so these functions are polynomials multiplied by the exponential of a linear function.

□

Problem 8. Construct a non-constant entire function $f(z)$ such that the zeros of f are simple and coincide with the set of all (positive) natural numbers.

Proof. We mimic the proof of Weiestrass Factorization Theorem. Define the canonical factors
$$E_n := (1 - z)\exp(z + z^2/2 + .. + z^n/n)$$
and define
$$g(z) := \prod_{j=1}^{\infty} E_j(\frac{z}{n})$$
then this is an entire function with simple zeros with zeros at the positive natural numbers.

□

Problem 9. Prove Hurwitz's Theorem. Let $\Omega \subset \mathbb{C}$ be connected open set and $f_n, f : \Omega \to \mathbb{C}$ be holomorphic functions such that $f_n(z)$ converges uniformly on compact sets to $f(z)$. Prove that if $f_n(z) \neq 0$ for all n then either f is identically zero or $f(z) \neq 0$ for any $z \in \Omega$.

Proof. Define $A := \{z \in \Omega : f(z) = 0\}$. Continuity implies A is closed, so we will show that A is open which will imply either $f \equiv 0$ or f is never zero due to connectedness.

Indeed, fix $z_n \in A$. Assume for the sake of contradiction that f is not identically 0 in a neighborhood of z_n. Then as all the zeros are isolated, this implies there exists an $\varepsilon > 0$ such that $\overline{B_\varepsilon(z_n)} \subset \Omega$ such that on $\partial B_\varepsilon(z_n)$ we have $f \neq 0$. Now by the argument principle we have that if we define $\gamma := \varepsilon e^{i\theta} + z_n$ for $\theta \in [0, 2\pi)$ then
$$\frac{1}{2\pi i}\int_\gamma \frac{f'}{f} dz \geq 1$$

since we have at least one zero in our domain. But as $f_n \to f$ uniformly on $\overline{B_\varepsilon(z_n)}$ this implies by Cauchy's Integral Formula that $f_n' \to f'$ uniformly on $\overline{B_\varepsilon(z_n)}$. Indeed, recall

$$f_m'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f_m(w)}{(w-z)^2}dw = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_m(z_n + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}$$

So we do have $f_n' \to f'$ uniformly. Therefore, we have

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{f_m'}{f_m}dz \to \frac{1}{2\pi i}\int_\gamma \frac{f'}{f}dz \geq 1$$

which is a contradiction. This implies f is identically zero in a neighborhood of 0. So A is open and closed, which implies by connectedness either $f \equiv 0$ or f is never zero. □

Problem 10. Let $\alpha \in [0,1] \setminus \mathbb{Q}$ and let $\{a_n\} \in \ell^1(\mathbb{N})$ with $a_n \neq 0$ for all $n \geq 1$. Show that

$$f(z) := \sum_{n \geq 1} \frac{a_n}{z - e^{i\alpha n}}, z \in \mathbb{D}$$

converges and defines a function that is analytic in $\mathbb{D}$ which does not admit an analytic continuation to any domain larger than $\mathbb{D}$

Proof. We will show $f_n := \sum_{j=1}^n a_n/(z - e^{i\alpha n})$ converges locally uniformly to f. Indeed, fix a compact set $K \subset \mathbb{D}$ then there is an $0 < r < 1$ such that $K \subset B_r(0)$. Then we have

$$\left|\frac{a_n}{z - e^{i\alpha n}}\right| \leq \frac{|a_n|}{1-r}$$

this implies since $a_n \in \ell^1(\mathbb{N})$ that $f(z)$ is absolutely convergent on K. This implies $f_n \to f$ uniformly on K and as f_n is a finite sum of holomorphic functions on $\mathbb{D}$ we see that f_n is holomorphic, so since we have uniform convergence this implies by Morrera's theorem that f is holomorphic on K. Then this implies by taking a compact exahusation of $\mathbb{D}$ that f is holomorphic on $\mathbb{D}$.

Now to see why there is no analytic continuation onto a larger domain that contains $\mathbb{D}$. If such an extension existed then it must contain a circle arc of $\partial\mathbb{D}$. So by density of $e^{i\alpha n}$ on the Torus since α is irrational we have that there is an $e^{i\alpha m}$ in this arc. But then $\lim_{r \to 1-} |f(re^{i\alpha m})| = +\infty$, so this function cannot be continuous on ∂D. Therefore, no such analytic continuation exists.

□

Problem 11. For each $p \in (-1,1)$ compute the improper Riemann integral

$$\int_0^\infty \frac{x^p}{x^2+1}dx$$

Proof. Fix $\varepsilon > 0$ and $R > 0$ and define $\gamma_1 := \{Re^{i\theta} : \theta \in [0,\pi]\}$, $\gamma_2 := \{-R(1-t) + \varepsilon t : t \in [0,1]\}$, $\gamma_3 := \{\varepsilon e^{i\theta} : \theta \in [0,\pi]\}$ (with γ_3 having clock wise orientation) and $\gamma_4 := \{\varepsilon(1-t) + Rt : t \in [0,1]\}$ where all these curves except γ_3 have counter clock wise orientation. Now define $\gamma := \sum_{i=1}^4 \gamma_i$ and now we compute

$$\int_\gamma \frac{z^p}{z^2+1}dz$$

where $z^p := \exp(p\log(z))$ where $\log(z) := \log(|z|) + i\arg(z)$ with $\arg(z) \in [-\pi/2, 3\pi/2]$ i.e. the log with a branch cut on the negative imaginary axis. On the big circular arc γ_1 we have

$$\int_{\gamma_1} \frac{z^p}{z^2+1}dz = \int_{\theta=0}^{\pi} \frac{R^p e^{pi\theta}}{R^2e^{2i\theta}+1}Rie^{i\theta}$$

so

$$\left|\int_{\gamma_1} \frac{z^p}{1+z^2} dz\right| \leq \int_0^\pi \frac{R^{1+p}}{1+R^2} = \pi \frac{R^{1+p}}{1+R^2}$$

since $p \in (-1,1)$ this goes to 0 as $R \to \infty$.

Now let us see what happens on the small semi-circle of radius ε. Observe that on this circle circle that $|z| = \varepsilon$ so we have

$$\int_{\gamma_3} \frac{z^p}{z^2+1} dz = \int_0^\pi \frac{\varepsilon^p e^{ip\theta}}{1+\varepsilon^2 e^{i\theta}} i\varepsilon^{i\theta}$$

so

$$\left|\int_{\gamma_3} \frac{z^p}{z^2+1} dz\right| \leq \pi \frac{\varepsilon^{1+p}}{1+\varepsilon^2}$$

which vanishes as $\varepsilon \to 0$ since $p \in [-1,1]$ so the numerator is a positive power of ε. So now we compute the residuals at $z = i$. to find the integral evaluation. Indeed,

$$\text{Res}(f,i) = \lim_{z \to i} \frac{z^p}{z+i} = \frac{i^p}{2i} = \frac{\exp(p(\frac{\pi}{2}i))}{2i}$$

and using if x is real then $(-x)^p = \exp(p \log|x|)\exp(pi\pi) = |x|^p \exp(ip\pi)$ to conclude

$$\int_0^\infty \frac{x^p}{1+x^2} dx = \frac{1}{1+\exp(ip\pi)} \lim_{\varepsilon \to 0, R \to \infty} \int_\gamma \frac{z^p}{1+z^2} dz = \frac{1}{1+\exp(ip\pi)} \pi \left(\exp(p(\pi/2i))\right)$$

□

Problem 12. Compute the number of zeros, including multiplicity, of $f(z) := z^6 + iz^4 + 1$ in the upper half plane in $\mathbb{C}$.

Proof. By Rouche's Theorem we know that if we can show $|iz^4| < |z^6+1|$ on the real axis, then z^6+1+iz^4 has the same number of zeros on the upper half plane as z^6+1 which has 3 namely 3 roots of unity. Indeed, it suffices to show for real x that

$$f(x) := x^6 + 1 - x^4 > 0$$

Observe as $x \to \pm\infty$ that $f(x) \to +\infty$ so if f was non-positive somewhere, its minimum exist and is in a compact set in $\mathbb{R}$ has to be non-positive. So differentiating, we have

$$f'(x) = 5x^4 - 4x^3 = x^3(5x-4)$$

so the potential zeros are $x = 0$ or $x = 4/5$. And it is clear $f(0), f(4/5) > 0$ so $f(x) > 0$. So there are exactly 3 roots in the upper half plane by Rouche's Theorem □

10. Fall 2014

Problem 1. Let
$$A := \{f \in L^3(\mathbb{R}) : \int_{\mathbb{R}} |f|^2 < \infty\}$$
Show that A is a Borel subset of $L^3(\mathbb{R})$.

Proof. Note that $f \in A$ if and only if there exists an $M > 0$ such that for all N we have $\int_{-N}^{N} |f|^2 \leq M$ i.e.
$$A = \bigcup_{m=0}^{\infty} \bigcap_{n=0}^{\infty} \{f \in L^3(\mathbb{R}) : \int_{-n}^{n} |f|^2 \leq m\}$$
so it suffices to show the linear functional
$$\Lambda(f) := \int_{-n}^{n} |f|^2$$
is Borel measurable since $\{f \in L^3(\mathbb{R}) : \int_{-n}^{n} |f|^2 \leq m\} = \Lambda^{-1}([0, m))$. In particular, it suffices to show it is continuous. Indeed, fix $f, g \in L^3(\mathbb{R})$ then
$$|\Lambda(f-g)| = \int_{-n}^{n} |f-g|^2 dx \leq \left(\int_{-n}^{n} |f-g|^3\right)^{\frac{2}{3}} \left(\int_{-n}^{n} 1^3\right)^{1/3} = 2^{1/3} n^{1/3} ||f-g||_{L^3(\mathbb{R})}^2$$
so in particular, Λ is continuous, so $\{f \in L^3(\mathbb{R}) : \int_{-n}^{n} |f|^2 \leq m\}$ is Borel, hence A is Borel.

Alternative one can use Fatou's Lemma to show that $\{f \in L^3(\mathbb{R}) : \int_{-n}^{n} |f|^2 \leq m\}$ is closed. $\square$

Problem 2. Construct an $f \in L^1(\mathbb{R})$ so that $f(x+y)$ does not converge a.e. to $f(x)$ as $y \to 0$. Prove that your f has this property.

Proof. Let C be a fat cantor set, then $m(C) > 0$ and it has no open intervals. Define $f(x) := \chi_C$ then this is not equal to 0 a.e. since $m(C) > 0$ and since C is measurable so is χ_C and χ_C is supported on a set of finite measure so it is in $L^1(\mathbb{R})$. Then notice for any $x \in C$ as C has no open intervals around x, we conclude there is a sequence $y_n \in K^c$ such that $y_n \to x$ to get
$$f(x + y_n) = 0 \text{ for all } n$$
but $f(x) = 1$ and this is true for all $x \in K$ so we do not have a.e. convergence of the translates pointwise. $\square$

Problem 3. Let (f_n) be a bounded sequence in $L^2(\mathbb{R})$ and suppose that $f_n \to 0$ Lebesgue a.e. Show that $f_n \rightharpoonup 0$ in the weak topology of $L^2(\mathbb{R})$.

Proof. Same argument as Fall 2012 Number 1. $\square$

Problem 4. Given $f \in L^2([0, \pi])$ we say $f \in \mathcal{G}$ if f admits a representation of the form
$$f(x) := \sum_{n=0}^{\infty} c_n \cos(nx) \quad \text{and} \quad \sum_{n=0}^{\infty} (1+n^2)|c_n|^2 < \infty$$
Show that if $f, g \in \mathcal{G}$ then $fg \in \mathcal{G}$.

Proof. Note that due to the summation condition we have that

$$\sum_{n=0}^{\infty}|c_n| = \sum_{n=0}^{\infty}\frac{\sqrt{1+n^2}}{\sqrt{1+n^2}}|c_n| \leq ||(1+n^2)^{-1}||_{\ell^2}||\sqrt{1+n^2}|c_n||_{\ell^2} < \infty$$

where we used Cauchy-Schwarz. Therefore, the sum that defines $f(x)$ converges uniformly. So in particular, there exists a representative of $f \in L^2([0,\pi])$ that is continuous and we choose to look at this representative. From now on we regard $\mathcal{G}$ as a subset of $L^2([0,\pi]) \cap C([0,\pi])$. We also may by taking an even extension i.e.

$$f(-x) := f(x) \text{ for } x \in [0,\pi]$$

may regard $\mathcal{G}$ as a subset of continuous even functions on $[-\pi,\pi]$. And we know that the basis of even functions on $C([-\pi,\pi])$ in the L^2 norm is $\{\cos(nx)\}$. So now fix $f, g \in \mathcal{G}$ then fg is continuous and even. Therefore, we have that the nth Fourier Coefficient in the L^2 sense is

$$\pi a_n := \int_{-\pi}^{\pi} f(x)g(x)\cos(nx) = \sum_{j,m=0}^{\infty}\int_{-\pi}^{\pi} b_j\cos(jx)c_m\cos(mx)\cos(nx)$$

$$= \pi\sum_{j=0}^{\infty} b_j c_{n-j}$$

where $f = \sum b_j\cos(jx)$ and $g = \sum c_j\cos(jx)$ and we swapped sum and integrals thanks to uniform convergence of each sum. So in particular, we have

$$a_n = \sum b_j c_{n-j}$$

Therefore, we have

$$\sum_n(1+n^2)|a_n|^2 \leq \sum_n\left(\sum_j\sqrt{1+n^2}|b_j||c_{n-j}|\right)^2$$

Now we use that

$$\sqrt{1+n^2} \lesssim \sqrt{1+(n-j)^2} + \sqrt{1+j^2}$$

to get

$$\sum_j\sqrt{1+n^2}|b_j||c_{n-j}| \lesssim \sum_j\sqrt{1+(n-j)^2}|c_{n-j}||b_j| + \sqrt{1+j^2}|b_j||c_{n-j}|$$

So it follows that $(x+y)^2 \leq 2x^2+2y^2$ with taking x, y as the above sums that

$$\left(\sum_j\sqrt{1+n^2}|b_j||c_{n-j}|\right)^2 \lesssim \left(\sum_j\sqrt{1+(n-j)}|c_{n-j}||b_j|\right)^2 + \left(\sqrt{1+j^2}|b_j||c_{n-j}|\right)^2$$

so by Young's Convolution Inequality we have that

$$\sum_n(1+n^2)|a_n|^2 \lesssim ||b_n||_{\ell^1}||\sqrt{1+n}|c_n|||_{\ell^2} + ||\sqrt{1+n^2}|b_n|||_{\ell^2}||c_n||_{\ell^1} < \infty$$

so $fg \in \mathcal{G}$ $\square$.

$\square$

Problem 5. Let $\phi : [0,1] \to [0,1]$ be continuous and $d\mu$ be a Borel Probability measure on $[0,1]$. Suppose $\mu(\phi^{-1}(E)) = 0$ for every Borel Set $E \subset [0,1]$ with $\mu(E) = 0$. Show that there is a Borel measurable function $w : [0,1] \to [0,\infty)$ so that

$$\int f\circ\phi(x)d\mu(x) = \int f(y)w(y)d\mu(y) \quad \text{for all continuous functions } f : [0,1] \to \mathbb{R}$$

Proof. Let E be Borel measurable subset of $[0,1]$ then define $f = \chi_E(x)$ then we have $f \circ \phi(x) = \chi_{\phi^{-1}(E)}$

$$\int_0^1 f \circ \phi(x) d\mu(x) = \int_0^1 \chi_{\phi^{-1}(E)}(x) d\mu = \mu(\phi^{-1}(E))$$

This implies for any continuous function $f : [0,1] \to \mathbb{R}$ that

$$\int_0^1 f \circ \phi(x) d\mu(x) = \int_0^1 f d\phi_*$$

where we define the push forward measure $\phi_*(E) := \mu(\phi^{-1}(E))$ by the definition of an integral. Now we see from the condition of $\phi_*(E) = 0$ whenever $\mu(E) = 0$ that there exists an $w \in L^1(d\mu)$ such that

$$d\phi_* = w d\mu$$

i.e. for any continuous function f we have

$$\int_0^1 f \circ \phi(x) d\mu(x) = \int_0^1 f(y) w(y) d\mu(y)$$

and since ϕ_* is a positive measure, we know $w(y) \geq 0$ and its set where its $+\infty$ is a μ null set, so we can redefine it on this null set to make $w(y) : [0,1] \to [0,\infty)$.

□

Problem 6. Let X be a Banach Space and let X^* its dual space. Suppose X^* is separable, show that X is separable. (You may assume the axiom of choice).

Proof. Let $\{f_n\}$ be a countable dense subset of X^*. For each $n \in \mathbb{N}$ we can find an $x_n \in X$ such that

$$f_n(x_n) \geq \frac{1}{2}||f_n||$$

So define D to be the set of all finite linear combinations of x_n and E to be the set of all finite rational linear combinations of D. It suffices to show E is dense since D is a countable dense subset of E. Indeed, assume for the sake of contradiction that $\overline{D} \neq X$, so there is an $x \in X \setminus \overline{D}$, so by Hanh Banach there is a linear functional $f \in X^*$ such that

$$f(x) \neq 0 \text{ and } f|_{\overline{D}} = 0$$

Now observe that

$$||f|| \leq ||f - f_n|| + ||f_n|| \leq ||f - f_n|| + 2f_n(x_n)$$

and as $x_n \in \overline{D}$ we see that

$$= ||f - f_n|| + 2f_n(x_n) - 2f(x_n) \leq 3||f - f_n||$$

Therefore, as f_n is dense we conclude that

$$||f|| = 0$$

i.e. f is the zero function, but this contradicts the fact that $f(x) \neq 0$. Therefore, we must have $\overline{D} = X$, so it follows that E is a countable dense subset of X.

□

Problem 7. Find an explicit conformal map from the Upper Half Plane-Slit along the vertical segment

$$\{z \in \mathbb{C} : \text{Im}(z) > 0\} \setminus (0, 0 + ih]$$

for $h > 0$ to the unit disk $\mathbb{D}$.

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Show that

$$|f(z)| \leq Ce^{a|z|}, \quad z \in \mathbb{C}$$

for some constants C, a if and only if we have

$$|f^{(n)}(0)| \leq M^{n+1}, \quad n = 0, 1, ..$$

for some constant M.

Proof. Note that as f is entire we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Now for the $\Leftarrow$ direction observe that $|f^{(n)}(0)| \leq M^{n+1}$ implies

$$|f(z)| \leq \sum_{n=0}^{\infty} \frac{M^{n+1}}{n!} |z|^n = M \sum_{n=0}^{\infty} \frac{|Mz|^n}{n!} = M \exp(|M||z|)$$

so this direction has been shown with $C = a = |M|$.

For the reverse direction, note that Cauchy's Estimate implies for any $R > 0$

$$\frac{|f^{(n)}(0)|}{n!} \leq \max_{z \in \partial B_R(0)} |f(z)|/R^n \leq C \frac{\exp(aR)}{R^n}$$

By taking derivatives on the final expression, we see that it is minimized when $R = n/a$ i.e.

$$a_n := |f^{(n)}(0)|/n! \leq C \frac{a^n}{n^n} \exp(n)$$

now stirling's approximation gives

$$a_n \lesssim \frac{\sqrt{n} a^n}{n!} \lesssim \frac{a^n}{(n-1)!} \text{ for } n \geq 1$$

Therefore, we have

$$|f(z) - f(0)| \leq \sum_{n=1}^{\infty} a_n |z|^n \lesssim \sum_{n=1}^{\infty} \frac{a^n}{(n-1)!} |z|^n$$

In particular,

$$|f^{(n)}(0)| \leq n a^n \lesssim (2a)^{n+1}$$

as desired. ☐

Problem 9. Let $\Omega \subset \mathbb{C}$ be open and connected. Let (f_n) be a sequence of injective holomorphic functions defined on Ω and suppose $f_n \to f$ locally uniformly in Ω. Show that if f is not constant, then f is injective in Ω.

Proof. Let $w \in \Omega$ and define $A := \{z \in \Omega \setminus \{w\} : f(z) = f(w)\}$. Notice that A is closed since $A = f^{-1}(f(w))$ and f is continuous. So it suffices to show A is open in $\Omega \setminus \{w\}$ to conclude the problem since $\Omega \setminus \{w\}$ is still connected (since Ω is open). Indeed, let $z_0 \in A$ then we have for $0 < \varepsilon \ll 1$ that $B_\varepsilon(z_0) \subset \Omega$ and $w \notin B_\varepsilon(z_0)$ and the argument principle tells us

$$\int_{|z-z_0|=\varepsilon} \frac{f(z) - f(w)}{f'(z)} dz \geq 1$$

where we are assuming for the sake of contradiction that $f(z)$ is not identically $f(w)$ in a small ball around z, so as zeros of holomorphic functions are isolated we can find a small enough $\varepsilon > 0$ such that $f(z) - f(w) \neq 0$ on $|z - z_0| = \varepsilon$ which allows us to apply the Argument Principle.

But notice that $f_n(z)-f_n(w) \to f(z)-f(w)$ (which by Cauchy's Integral Formula implies $f_n'(z) \to f'(z)$ uniformly on $\overline{B_\varepsilon(0)}$) uniformly on $\overline{B_\varepsilon(z_0)}$ and as f_n is injective the argument principle tells us that

$$\int_{|z-z_0|=\varepsilon} \frac{f_n(z)-f_n(w)}{f_n'(z)} dz = 0$$

so uniform convergence tells us

$$0 = \lim_{n\to\infty} \int_{|z-z_0|=\varepsilon} \frac{f_n(z)-f_n(w)}{f_n'(z)} dz = \int_{|z-z_0|=\varepsilon} \frac{f(z)-f(w)}{f'(z)} dz \geq 1$$

which is our contradiction. Therefore, it follows that A is open and closed. Therefore, as $\Omega \setminus \{w\}$ is connected we know that either $f(z) \equiv f(w)$ on $\Omega \setminus \{w\}$ or $f(z) \neq f(w)$ for any $z \in \Omega \setminus \{w\}$. Therefore, it follows that if f is not injective i.e. $f(z_0) = f(w)$ for some $z_0 \neq w$ in Ω then f is identically $f(w)$ in Ω. ☐

Problem 10. Let $\mathcal{B}$ be the vecor space defined as follows

$$\mathcal{B} := \{u : \mathbb{C} \to \mathbb{C} \quad u \text{ holomorphic and } \int\int_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)} dxdy < \infty\}$$

Show that $\mathcal{B}$ becomes complete when we introduce the norm

$$||u||^2 := \int\int_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)} dxdy$$

Proof. Let $0 < r < R < \infty$ then we claim that there is a $C = C(r,R)$ such that for all entire functions f that

$$||f||_{L^\infty(B_r(0))} \leq C||f||$$

Indeed, notice that if $\delta := \frac{R+r}{2}$ then if $z \in B_r(0)$ then $B_\delta(z) \subset B_R(0)$ and so we have by the mean value property that

$$u(z) = \frac{1}{\pi\delta^2} \int\int_{B_\delta(z)} u(x+iy) dxdy$$

so we have

$$|u(z)| \leq \frac{1}{\pi\delta^2} \int\int_{B_\delta(z)} |u(x+iy)| dxdy \leq \frac{1}{\pi\delta^2} \int\int_{B_R(0)} |u(x+iy)| dxdy$$

so Holder's inequality gives

$$|u(z)|^2 \leq \frac{R^2}{\pi\delta^4} \int\int_{B_R(0)} |u(x+iy)|^2 dxdy$$

which implies since the right hand side is independent of $z \in B_r(0)$ that

$$\sup_{z\in B_r(0)} |u(z)|^2 \leq \frac{R^2}{\pi\delta^4} \int\int_{B_R(0)} |u(x+iy)|^2 dxdy = \frac{R^2}{\pi\delta^4 e^{-R^2}} \int\int_{B_R(0)} |u(x+iy)|^2 e^{-R^2} dxdy$$

$$\leq \frac{R^2}{\pi\delta^4 e^{-R^2}} \int\int_{B_R(0)} |u(x+iy)|^2 e^{-(x^2+y^2)} dxdy \leq C(r,R)||u||^2$$

where $C(r,R) = \frac{R^2}{\pi\delta^4 e^{-R^2}}$ Now take $R = 2r$. This implies if $\{u_n\}$ is Cauchy in $\mathcal{B}$ that we have local uniform convergence since

$$\sup_{z\in B_r(0)} |u_n(z) - u_m(z)| \lesssim_r ||u_n - u_m|| \to 0 \text{ as } n,m \to \infty$$

this implies there is an entire function u such that $u_n \to u$ locally uniformly. Then as $\{u_n\}$ is cauchy, it is a bounded sequence so Fatou's lemma gives

$$\int\int_{\mathbb{C}} |u(x+iy)|^2 e^{-(x^2+y^2)} dxdy \leq \liminf_{n\to\infty} \int\int_{\mathbb{C}} |u_n(x+iy)|^2 e^{-(x^2+y^2)} dxdy \leq M < \infty$$

where $M := \sup_n ||u_n||$. That is $u \in \mathcal{B}$. Now by local uniform convergence we know that along any compact subset $K \subset \mathbb{C}$ that

$$\int\int_K |u(x+iy) - u_n(x+iy)|^2 e^{-(x^2+y^2)} dxdy \to 0$$

And then observe that from the triangle inequality that

$$\left(\int\int_{\mathbb{C}\setminus K} |u(x+iy) - u_n(x+iy)|^2 e^{-(x^2+y^2)} dxdy\right)^{1/2} \leq$$

$$\left(\int\int_{\mathbb{C}\setminus K} |u_m(x+iy) - u_n(x+iy)|^2 e^{-(x^2+y^2)} dxdy\right)^{1/2} + \left(\int\int_{\mathbb{C}\setminus K} |u(x+iy))|^2 e^{-(x^2+y^2)} dxdy\right)^{1/2}$$

The first term is small due to the sequence being Cauchy and the last term is small when the compact set is big since $ue^{-\sqrt{x^2+y^2}} \in L^2(\mathbb{C}, dxdy)$. Therefore, $||u - u_n|| \to 0$ and $\mathcal{B}$ is complete.

Alternatively It is also easy to see if we define $d\mu := e^{-(x^2+y^2)} dxdy$ to see that $\mathcal{B}$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$ due to the L^2 to interior L^∞ estimate. And since $\{u_n\}$ is cauchy in $L^2(\mathbb{C}, d\mu)$ it converges to some limit in L^2, which combined with pointwise convergence along a subsequence implies u is the limit in $L^2(d\mu)$, which allows us to skip our estimates above to show $||u - u_n|| \to 0$.

□

Problem 11. Let $\Omega \subset \mathbb{C}$ be open, bounded, and simply connected and u a harmonic function on Ω such that $u \geq 0$. Show the following: for each compact set $K \subset \Omega$ there is a constant $C_K > 0$ such that

$$\sup_{z\in K} u(z) \leq C_k \inf_{z\in K} u(z)$$

Proof. As Ω is simply connected open subset of $\mathbb{C}$ that is not all of $\mathbb{C}$ since Ω is bounded, we know by the Riemann's mapping theorem there is a conformal map $\varphi : \mathbb{D} \to \Omega$. Therefore, $v(z) := u \circ \varphi$ is a harmonic function on the disk. Hence, on any $0 < r < 1$ we know by the Poisson Kernal Formula that for $z \in B_r(0)$

$$v(z) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} v(re^{i\theta}) dz = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{r^2 - |z|^2}{(re^{i\theta} - z)(re^{-i\theta} - \overline{z})} v(re^{i\theta}) dz$$

so the triangle inequality implies

$$v(z) = |v(z)| \leq \frac{1}{2\pi}\int_{\theta=0}^{2\pi} v(re^{i\theta}) \frac{(r - |z|)(r + |z|)}{(r - |z|^2)} d\theta = \frac{r + |z|}{r - |z|} v(0)$$

where we used the mean value property. And similarily, we also have

$$v(z) \geq \frac{1}{2\pi}\int_0^{2\pi} \frac{(r + |z|)(r - |z|)}{(r + |z|)^2} v(re^{i\theta}) d\theta = \frac{r - |z|}{r + |z|} v(0)$$

i.e. for any $z \in B_r(0)$ we have

$$\frac{r - |z|}{r + |z|} v(0) \leq v(z) \leq \frac{r + |z|}{r - |z|} v(0)$$

so if $|z| < r/2$ this implies there are positive constants $C_1 = C_1(r)$ and $C_2 = C_2(r)$ such that

$$C_1 v(0) \leq v(z) \leq C_2 v(0)$$

Hence,

$$C_1 v(0) \leq \inf_{z\in B_{r/2}(0)} v(z)$$

so it follows that

$$\sup_{z\in B_{r/2}(0)} v(z) \leq \frac{C_2}{C_1} \inf_{z\in B_{r/2}(0)} v(z)$$

this immediately implies for any compact set $K \subset \mathbb{D}$ that

$$\sup_{z \in K} v(z) \leq C_k \inf_{z \in K} v(z)$$

Now as conformal maps map boundary to boundary imples if $F \subset \Omega$ is a compact set then there is an $1 > R > 0$ such that $F \subset \varphi(B_R(0))$ and taking $K = \overline{B_R(0)}$ above gives

$$\sup_{z \in \overline{B_R(0)}} u \circ \varphi(z) \leq C_R \inf_{z \in \overline{B_R(0)}} u \circ \varphi(z)$$

which implies since $F \subset \varphi(B_R(0))$ that

$$\sup_{z \in F} u \leq C_F \inf_{z \in F} u$$

as desired.

□

Problem 12. Let $\Omega := \{z \in \mathbb{C} : |z| > 1\}$. Suppose $u : \Omega \to \mathbb{C}$ is bounded and continuous on $\overline{U}$ and is subharmonic on Ω. Prove the following: if $u(z) \leq 0$ on $|z| = 1$ then $u(z) \leq 0$ on Ω.

Proof. This is the standard ϵ-log trick. Indeed, fix $\varepsilon > 0$ and notice $\log |z|$ is Harmonic on Ω then define

$$u_\varepsilon(z) := u(z) - \varepsilon \log |z|$$

and observe that as u is bounded that

$$\lim_{|z| \to \infty} u_\varepsilon(z) = -\infty$$

so we for R sufficiently large we have that the subharmonic function $u_\varepsilon(z) \leq 0$ on $\partial\{1 \leq |z| \leq R\}$ so $u_\varepsilon(z) \leq 0$ on $\{1 \leq |z| \leq R\}$ by the maximum principle. So letting $R \to \infty$ shows $u_\varepsilon(z) \leq 0$ on Ω so letting $\varepsilon \to 0$ lets us conclude that $u(z) \leq 0$ on Ω.

□

Problem 1. Let $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| = \int_{\mathbb{R}} |f(x)|dx$$

Proof. We will show this problem first for the dense subclass $C_c^\infty(\mathbb{R})$. Indeed, if $f \in C_c^\infty(\mathbb{R})$ let K be a compact set such that $\overline{\text{supp}(f)} \subset K$. Let $Z := \{x \in K : f(x) = 0\}$ and notice that since f is continuous and K is compact that Z is compact. So there exists $z_1, .., z_N \in Z$ such that $Z \subset \bigcup_{j=1}^N B(1, z_j)$.

Now notice that if

$$\left| \int_{k/n}^{(k+1)/n} f(x)dx \right| \neq \int_{k/n}^{(k+1)/n} |f(x)|dx$$

then by continuity that f must change signs on $[k/n, (k+1)/n]$, so $Z \cap [k/n, (k+1)/n] \neq \emptyset$. Fix an $n \in \mathbb{N}$ and let I be the index in $-n^2 \le k \le n^2$ where $Z \cap [k/n, (k+1)/n] \neq \emptyset$ and defining $I_k := [k/n, (k+1)/n]$. Notice that this implies

$$\bigcup_{k \in I} I_k \subset \bigcup_{j=1}^N B(2, z_j)$$

so this means $|I| \le 4Nn$ since each $B(2, z_j)$ covers at most $4n$ intervals of I_k since I_k has length $1/n$ and there's N of these balls. In particular, now we see

$$\left| \sum_{k=n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| - \int_{-n}^{n+1/n} |f(x)| \right| = \left| \sum_{k \in I} \left| \int_{I_k} f(x)dx \right| - \int_{I_k} |f(x)|dx \right|$$

so if $\varepsilon > 0$ is arbitrary, we can from uniform continuity find an N so large such that if $n \ge N$ then

$$|x - y| \le 1/n \Rightarrow |f(x) - f(y)| \le \varepsilon$$

Taking n to be sufficiently large, we see that for $x \in I_k$ that $|f(x)| \le \varepsilon$. Therefore, we obtain since each interval I_k is of length $1/n$ and height at most ε

$$\le \sum_{k \in I} 2\frac{\varepsilon}{n} = 2\frac{\varepsilon}{n}|I| \le 8\varepsilon N \to 0$$

since ε is independent of N. So we have

$$\lim_{n\to\infty} \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| = \lim_{n\to\infty} \int_{-n}^{n+1/n} |f(x)|dx = \int_{\mathbb{R}} |f(x)|dx$$

Therefore, the problem is true for the dense subclass $C_c^\infty(\mathbb{R})$. Now by density we can find $C_c^\infty(\mathbb{R}) \ni f_n \to f$ in $L^1(\mathbb{R})$. Then we have

$$\left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| - \int_{\mathbb{R}} |f(x)|dx \right|$$

$$\le \left| \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f_m(x)dx \right| - \sum_{k=-n^2}^{n^2} \left| \int_{k/n}^{(k+1)/n} f(x)dx \right| \right| + \left| \left| \sum_{k=-n^2}^{n^2} \int_{k/n}^{(k+1)/n} f_m(x)dx \right| - \int_{\mathbb{R}} |f_m(x)| \right| + ...$$

$$+ \left| \int_{\mathbb{R}} |f(x)| - |f_m(x)| \right|$$

notice that the second term can be made small for large n and the third time is small for large m, so it suffices to make the first term small. Indeed, observe the first term by the reverse triangle inequality

$$\le \int_{-n}^{n+1/n} |f(x) - f_m(x)|dx \le \int_{\mathbb{R}} |f(x) - f_m(x)|$$

which can be made small for large m, which finishes the proof.

Remark: It is probably significantly easier to prove this statement first for the dense subclass of step functions.

□

Problem 2. Let $f \in L^2_{loc}(\mathbb{R}^n)$ and $g \in L^3_{loc}(\mathbb{R}^n)$. Assume that for all real $r \geq 1$, we have

$$\int_{r \leq |x| \leq 2r} |f(x)|^2 \leq r^a \quad \int_{r \leq |x| \leq 2r} |g(x)|^3 dx \leq r^b$$

Here $a, b \in \mathbb{R}$ are such that $3a + 2b + n < 0$. Show that $fg \in L^1(\mathbb{R}^n)$

Proof. Notice that

$$\int_{\mathbb{R}^n} |fg| dx = \int_{|x| \leq 1} |fg| dx + \sum_{m=0}^{\infty} \int_{2^m \leq |x| \leq 2^{m+1}} |fg| dx = (I) + (II)$$

and Holder's gives

$$(II) \leq \sum_{m=0}^{\infty} ||f||_{L^2(2^m \leq |x| \leq 2^{m+1})} ||g||_{L^3(2^m \leq |x| \leq 2^{m+1})} ||1||_{L^6(2^m \leq |x| \leq 2^{m+1})}$$

$$\lesssim \sum_{m=0}^{\infty} 2^{m(a/2 + b/3 + n/6)} < \infty$$

where we used $||1||_{L^6(2^m \leq |x| \leq 2^{m+1})} \lesssim 2^{mn/6}$ since the volume of a sphere in $\mathbb{R}^n$ grows like r^n and that $(3a + 2b + n)/6 < 0$ to get the sum converges. And notice

$$\int_{|x| \leq 1} |fg| dx \leq ||f||_{L^2(|x| \leq 1)} ||g||_{L^3(|x| \leq 1)} ||1||_{L^6(|x| \leq 1)} < \infty$$

since $f \in L^2_{loc}(\mathbb{R}^n)$ and $g \in L^3_{loc}(\mathbb{R}^n)$, so it follows that

$$\int_{\mathbb{R}^n} |fg| dx < \infty$$

□

Problem 3. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and let

$$Mf(x) := \sup_{r > 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)| dy$$

be the Hardy-Littlewood maximal function.

(1) Show that

$$m(\{x : Mf(x) > s\}) \leq \frac{C_n}{s} \int_{|f| > s/2} |f(x)| dx, \quad s > 0$$

where the constant C_n depends on n only. The Hardy-Littlewood maximal theorem may be used.

(2) Prove that if $\varphi \in C^1(\mathbb{R})$, $\varphi(0) = 0$, and $\varphi' > 0$ then

$$\int \varphi(Mf(x)) dx \leq C_n \int |f(x)| \left(\int_{0 < t < 2|f(x)|} \frac{\varphi'(t)}{t} \right)$$

Proof. Let us fix $s > 0$ and decompose

$$f(x) = f(x)\chi_{|f(x)| \leq s/2} + f(x)\chi_{|f(x)| > s/2} := g(x) + h(x)$$

then notice that for any $r > 0$ we have

$$\frac{1}{m(B(r,x)}\int_{B(r,x)}|f(y)|dy \leq \frac{1}{m(B(r,x)}\int_{B(r,x)}|g(y)|dy + \frac{1}{m(B(r,x)}\int_{B(r,x)}|h(y)|dy$$
$$\leq \frac{s}{2} + \frac{1}{m(B(r,x)}\int_{B(r,x)}|h(y)|dy$$

since $|g| > s/2$. So if

$$\frac{1}{m(B(r,x)}\int_{B(r,x)}|f(y)|dy > s$$

this implies that $Mh(x) > s/2$, so in particular $\{x : Mf(x) > s\} \subset \{x : Mh(x) > s/2\}$. And by the Hardy-Little wood theorem we have there is a K_n a constant that depends only on n such that

$$m(\{x : Mh(x) > s/2\}) \leq \frac{2K_n}{s}\int_{\mathbb{R}}|h(x)|dx = \frac{C_n}{s}\int_{|f|>s/2}|f(x)|dx$$

by the definition of h where $C_n = 2K_n$. Thus we have

$$m(\{x : Mf(x) > s\}) \leq \frac{C_n}{s}\int_{|f|>s/2}|f(x)|dx$$

as desired.

For the second part, we have by the Fundamental Theorem of Calculus that and $\varphi(0) = 0$

$$\varphi(Mf(x)) = \int_0^{Mf(x)}\varphi'(t)dt$$

so

$$\int\varphi(Mf(x))dx \leq \int\int\varphi'(t)\chi_{[0,Mf(x)]}(t)dtdx = \int\varphi'(t)\int\chi_{[0,Mf(x)]}dxdt \text{ By Tonelli}$$

where Tonelli is justified by $\varphi' \geq 0$.

$$= \int_0^\infty\varphi'(t)m(\{x : Mf(x) > t\}) \leq C_n\int_0^\infty\frac{\varphi'(t)}{t}\int_{|f|\geq t/2}|f(x)|dxdt$$

$$= C_n\int_0^\infty\frac{\varphi'(t)}{t}\int|f(x)|\chi_{|f|>t/2}dxdt = C_n\int|f(x)|\left(\int_0^\infty\frac{\varphi'(t)}{t}\chi_{|f|>t/2}dt\right)dx = C_n\int|f(x)|\left(\int_{0<t<2|f(x)|}\frac{\varphi'(t)}{t}dt\right)dx$$

as desired.

□

Problem 4. Let $f \in L^1_{loc}(\mathbb{R})$ be 2π-periodic. Show that linear combinations of the translates $f(x-a)$, $a \in \mathbb{R}$ are dense in $L^1((0, 2\pi))$ iff each Fourier coefficient of f is $\neq 0$.

Proof. Denote $\hat{f}(n)$ as the nth Fourier coefficient of f.

$\Rightarrow$ Assume that $\{f(x-a)\}_{a\in\mathbb{R}}$ is dense in L^1. Assume for the sake of contradiction that $\hat{f}(n) = 0$. For any $\{a_i\}_{i=1}^N$ and $\{c_i\}_{i=1}^N$ let $g(x) := \sum_{i=1}^N c_i f(x-a_i)$. Then observe by Parsavel's identity that we have $\int_0^{2\pi}u(x)\overline{v(x)}dx = \sum_{n\in\mathbb{N}}\hat{u}(n)\overline{\hat{v}(n)}$ so

$$\int_0^{2\pi}g(x)e^{-inx}dx = \hat{g}(n) = 0$$

But this implies since functions of the form $g(x)$ are dense in L^1 that $||e^{inx}||_{L^2([0,2\pi])} = 0$ but its not 0, which is our contradiction.

$\Leftarrow$ Let $S := \text{span}\{f(x-a) : a \in \mathbb{R}\}$ then if $\overline{S} \neq L^1((0,2\pi))$ we have by Hanh Banach the existence of $\ell \in (L^1((0,2\pi))^* \cong L^\infty((0,2\pi))$ such that $\ell|_S = 0$ but ℓ is not the zero function. So in particular, we have by Risez Representation Theorem a $g \in L^\infty((0,2\pi))$ such that for any $u(x) \in L^1((0,2\pi))$

$$\ell(u) = \int_0^{2\pi} u(x)g(x)dx$$

Then we have

$$(f * g)(a) = \ell(f(x-a)) = \int_0^{2\pi} f(x-a)g(x)dx = 0$$

So we have that by taking the Fourier Series of $(f * g)$ and using uniqueness of the Fourier Series gives

$$\hat{f}(n)\overline{\hat{g}(n)} = 0$$

and as $\hat{f}(n) \neq 0$ for any n we deduce that $\hat{g}(n) = 0$ so $g \equiv 0$ i.e. ℓ is the zero function, which is a contradiction.

□

Problem 5. Let $u \in L^2(\mathbb{R})$ and let us set

$$U(x,\xi) := \int e^{-(x+i\xi-y)^2/2}u(y)dy, \quad x,\xi \in \mathbb{R}$$

Show that $U(x,\xi)$ is well defined on $\mathbb{R}^2$ and that there is a $C > 0$ such that for all $u \in L^2(\mathbb{R})$, we have

$$\int\int |U(x,\xi)|^2 e^{-\xi^2} dx d\xi = C \int |u(y)|^2 dy$$

Proof. Notice that by Cauchy-Schwarz that

$$|U(x,\xi)| \leq ||u||_{L^2(\mathbb{R})} \left(\int_{\mathbb{R}} |e^{-(x+i\xi-y)^2/2}|^2 dy \right)^{1/2} < \infty$$

since the Gaussian is in L^2, so U is well defined. Now we will use Plancheral's Theorem to prove the second statement. Observe that

$$U(x,\xi) = e^{-x^2/2 - ix\xi + \xi^2/2} \int e^{-y^2/2+yx}u(y)e^{iy\xi}dy$$

Define the $F_x(y) := e^{-y^2/2+yx}u(y)$ then

$$U(x,\xi) = e^{-x^2/2 - ix\xi + \xi^2/2}\hat{F}_x(-\xi)$$

so we have that

$$\int\int |U(x,\xi)|^2 e^{-\xi^2} dx d\xi = \int\int e^{-x^2}|\hat{F}_x(-\xi)|^2 d\xi dx$$

where we swapped integrals due to Tonelli since every term is non-negative. So Plancheral's Theorem gives

$$= \int\int e^{-x^2}F_x(y)^2 dy dx = \int\int e^{-x^2}e^{-y^2+2yx}|u(y)|^2 dy dx$$
$$= \int |u(y)|^2 \int e^{-(x-y)^2} dx dy$$
$$= C\int |u(y)|^2 dy$$

since $\int e^{-(x-y)^2}dx = \int e^{-(y)^2}dy$ and $C := \int e^{-y^2}dy$ as desired.

□

Problem 6. When B_1 and B_2 are Banach spaces, we say that a linear operator $T : B_1 \to B_2$ is compact if for any bounded sequence $(x_n) \in B_1$, the sequence (Tx_n) has a convergent subseqeuence. Show that if T is compact then ImT has a dense countable subset.

Proof. First note that T is continuous. Indeed, if it was not then there is a sequence $\{x_n\}$ such that $||x_n|| = 1$ and

$$||T(x_n)|| \geq n$$

but then $\{T(x_n)\}$ cannot converge along any sub-sequence, which is a contradiction to compactness. So T is continuous.

Now we claim if D_r is a ball of radius $r > 0$ then $T(D_r)$ is pre-compact. Indeed, given any $y_n \in T(D_r)$ we can find $x_n \in D_r$ such that $||x_n|| < r$ and $T(x_n) = y_n$. By compactness, we can find a subsequence x_{n_k} such that $T(x_{n_k})$ converges to some limit y. So in particular, $y \in \overline{T(D_r)}$, so $T(D_r)$ is pre-compact, so $\overline{T(D_r)}$ is totally bounded, which implies $T(D_r)$ is totally bounded, which implies its separable. So we have

$$T(B_1) = \bigcup_{n\in\mathbb{N}} T(D_n)$$

and each $T(D_n)$ is separable, so the entire space is separable.

□

Problem 7. Let $\mathbb{C}^+ := \{z \in \mathbb{C} : \text{Im}(z) > 0\}$. Suppose $f_n : \mathbb{D} \to \mathbb{C}^+$ is a sequence of holomorphic functions and $f_n(0) \to 0$ as $n \to \infty$. Show that $f_n(z) \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Proof. Write $f_n = u_n + iv_n$ where u_n and v_n are the real and imaginary parts of f_n respectively. Note that $v_n \geq 0$ since f_n maps to $\mathbb{C}^+$. Therefore, by Harnack's Inequality we have for any compact set the existence of a constant $C = C(K)$ that depends on the compact set K only such that

$$\sup_{z\in K} v_n(z) \leq C(K) \inf_{z\in K} v_n(z) \leq C(K)v_n(0) \to 0$$

so it follows that as $n \to \infty$, we have that $v_n(z) \to 0$ uniformly. Therefore, by Cauchy's Estimate it follows that $\nabla v_n(z) \to 0$ uniformly on K, so by the Cauchy-Riemann equations we get that $\nabla u_n(z) \to 0$ uniformly on K. In particular, the Fundamental Theorem of Calculus then implies $u_n \to 0$ uniformly on K since $\lim_{n\to\infty} u_n(0) = 0$, so we have that $f_n \to 0$ uniformly on any compact subset as desired.

□

Problem 8. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic and suppose

$$\sup_{x\in\mathbb{R}}\{|f(x)|^2 + |f(ix)|^2\} < \infty \quad \text{and} \quad |f(z)| \leq e^{|z|} \text{ for all } z \in \mathbb{C}$$

Deduce that $f(z)$ is a constant.

Proof. This will follow from the Phragmen-Lindeolf method. We will prove on each of the four half planes. Indeed, define $R_1 = \{z \in \mathbb{C} : \text{Re}(z) > 0, \text{Im}(z) > 0\}$. Then notice that for $\varepsilon > 0$ if we define $g_\varepsilon(z) := f(z)(\exp(-\varepsilon(e^{i\phi}z)^{3/2}))$ where $z^{3/2}$ is the branch with the negative real axis removed. Then $(e^{i\phi}z)^{3/2}) = \exp(3/2(\log|z| + i\text{Arg}(z + \phi))) = |z|^{3/2}\exp(3/2i\text{Arg}(z + \phi))$. So we have

$$|(\exp(-\varepsilon(e^{i\phi}z)^{3/2}))| = \exp(-\varepsilon|z|^{3/2}\cos(3/2\text{Arg}(z + \phi)))$$

Note that since $z \in R_1$ we have

$$0 < \text{Arg}(z) < \pi/2 \Rightarrow 3/2\phi < 3/2\text{Arg}(z + \phi) < 3\pi/4 + 3\phi/2$$

As we want $0 < \cos(3/2\text{Arg}(z + \phi))$ we see we need

$$3/2\phi > -\pi/2 \text{ and } 3\pi/4 + 3\phi/2 < \pi/2$$

For instance take $\phi = -\pi/4$ gives the desired bound, so we have a $\delta > 0$ such that $\delta < \cos(3/2\text{Arg}(z - \pi/4))$. This implies

$$|g_\varepsilon(z)| \leq |f(z)|\exp(-\varepsilon|z|^{3/2}\delta)$$

and we have that g_ε is bounded on ∂R_1 and as $|f(z)| \leq e^{|z|}$ it follows that $|g_\varepsilon(z)| \to 0$ as $|z| \to \infty$. So for $r > 0$ large enough we have that $|g_\varepsilon(z)| \leq M := \sup_{x\in\mathbb{R}}\{|f(x)|^2 + |f(ix)|^2\} < \infty$ on $\partial B_r(0) \cap R_1 \cup \partial R_1$, so it follows from the maximum principle that on $B_r(0) \cap R_1$ that $|g_\varepsilon(z)| \le M$ and letting $R \to \infty$ lets us conclude

$$|g_\varepsilon(z)| \le M \text{ on } R_1$$

and letting $\varepsilon \to 0$ using that $g_\varepsilon(z) \to f(z)$ lets us conclude that $|f(z)| \le M$ on R_1. Repeating this argument on the other 3 half planes lets us conclude that f is a bounded entire function, so it is constant. □

Problem 9.

Problem 10. Determine

$$\int_{-\infty}^{\infty} \frac{dy}{(1+y^2)(1+[x-y]^2)} dy$$

for all $x \in \mathbb{R}$. Justify all manipulations.

Proof. Note that this function has poles at $z = \pm i$ and $z = x \pm i$. And that

$$\text{Res}(f, i) = \lim_{z \to i} \frac{1}{(z+i)(1+[x-z]^2)} = \frac{1}{2i(1+[x-i]^2)}$$

$$\text{Res}(f, x+i) = \lim_{z \to x+i} \frac{1}{(z+i)(z-i)(z-(x-i))} = \frac{1}{(x+2i)(x)(2i)}$$

Define $\gamma_R := Re^{i\theta}$ for $\theta \in [0, \pi]$ and $\gamma_{-R\to R} := -R(1-t) + tR$ for $t \in [0,1]$ and $\gamma(R) := \gamma_R + \gamma_{-R \to R}$. Then

$$\int_{\gamma_R} \frac{dz}{(1+z^2)(1+[x-z]^2)} = \int_0^\pi \frac{Re^{i\theta}}{(1+R^2e^{2i\theta})(1+[x-Re^{i\theta}]^2)} d\theta =: I$$

and

$$|I| \le \int_0^\pi \frac{R}{(R^2-1)(|1-|x-Re^{i\theta}|^2)|}$$

and notice that as $R \to \infty$ that

$$\frac{1}{|1-|x-Re^{i\theta}|^2|} \to 0$$

since the denominator approahces ∞ as $R \to \infty$ so we can find a $C > 0$ thanks to continuity to get

$$|I| \le 2\pi C \frac{R}{R^2-1} \to 0$$

Therefore,

$$\lim_{R\to\infty} \int_{\gamma(R)} \frac{dz}{(1+z^2)(1+[x-z]^2)} = \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)(1+[x-y]^2)}$$

and by the residue theorem we have that

$$\int_{\gamma(R)} \frac{dz}{(1+z^2)(1+[x-z]^2)} dz = \pi \left(\frac{1}{(1+[x-i]^2)} + \frac{1}{(x+2i)(x)} \right) = \pi\left(\frac{1}{x^2-2ix} + \frac{1}{x^2+2ix}\right) = \frac{2\pi}{x^2+4}$$

□

Problem 11. Let $\Omega := \mathbb{D} \setminus \{0\}$. Prove that for every bounded harmonic function $u : \Omega \to \mathbb{R}$ there is a harmonic function $v : \Omega \to \mathbb{R}$ obeying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Proof. We will show that u is the real part of a holomorphic function. Define $g(z) := \partial_x u - i\partial_y u$ then g is holomorphic on Ω since it is real differentiable in the real sense and we have that it satisfies the Cauchy-Riemann equations. Indeed, we have

$$\frac{\partial g}{\partial x} = \partial_{xx}^2 u - i\partial_{yx}^2 u \text{ and } \frac{1}{i}\frac{\partial g}{\partial y} = -i\partial_{yx}^2 u - \partial_{yy}^2 u$$

and as u is harmonic we deduce that $\partial_x g = 1/i\partial_y g$ so g is holomorphic on Ω. Now let $c := \text{Res}(g, 0)$ which exists since g is holomorphic on the annulus $\{0 < |z| < 1\}$. Then we have from the Residue Theorem that for any closed curve $\gamma \subset \mathbb{D}$ that

$$\int_\gamma g(z) - c\frac{1}{z}dz = 0$$

so it follows that $h(z) := g(z) - c\frac{1}{z}$ has a holomorphic primitive f on $\mathbb{D}$. In particular, it follows that $h(z)$ is holomorphic on $\mathbb{D}$ since $h = f'(z)$. And notice that this implies

$$\text{Re}(f(z)) = u - c\log|z|$$

so $u - c\log|z|$ is harmonic on $\mathbb{D}$. So from the maximum modulus principle it follows that $|u - c\log|z|| \leq C$ where C is an upper bound for u since $\log|1| = 0$ and this is a bound on $\partial B_1(0)$. This implies $c = 0$ since if $c \neq 0$ we have

$$\lim_{z\to 0} |u - c\log|z|| = \infty$$

since u is bounded. Therefore, u is the real part of the holomorphic function $f(z)$, so it follows that it has a harmonic conjugate, namely $\text{Im}(f(z))$.

□

Problem 12. Find all entire functions $f : \mathbb{C} \to \mathbb{C}$ that obey

$$f'(z)^2 + f(z)^2 = 1$$

Prove that your list is exhaustive.

Proof. Note that this implies

$$(f' + if)(f' - if) = 1$$

so $f' + if$ and $f' - if$ omit the value 0, so there exists an entire function $h(z)$ and $g(z)$ such that $f' + if = \exp(h(z))$ and $f' - if = \exp(g(z))$. From

$$(f' + if)(f' - if) = 1 \Rightarrow g(z) = -h(z)$$

So we have that $f' + if = \exp(h(z))$ and $f' - if = \exp(-h(z))$. Therefore, we obtain that

$$f(z) = \frac{\exp(h(z)) - \exp(-h(z))}{2i} = \sin(h(z)/i)$$

plugging this into the ODE gives for $w := h(z)/i$ that

$$-\cos^2(w)(h'(z))^2 + \sin^2(w) = 1 \Rightarrow -\cos^2(w)(h'(z))^2 = \cos^2(w) \Rightarrow (h'(z))^2 = -1$$

so we have

$$h'(z) = \pm i \Rightarrow h(z) = \pm iz + c$$

where $c \in \mathbb{C}$, so we have that

$$f(z) = \sin(z + c) \text{ or } \sin(-z + c)$$

for any constant $c \in \mathbb{C}$. □

12. Fall 2015

Problem 1. Let g_n be a sequence of measurable functions on $\mathbb{R}^d$ such that $|g_n(x)| \leq 1$ for all x and assume that $g_n \to 0$ a.e. Let $f \in L^1(\mathbb{R}^d)$. Show that the sequence

$$f * g_n(x) := \int_{\mathbb{R}^d} f(x-y)g_n(y)dy \to 0$$

uniformly on every compact subset of $\mathbb{R}^d$.

Proof. We will show that when $R > 0$ is large enough that $f * g_n$ uniformly converges to 0 on $\overline{B}_R(0)$ which will imply the claim. As $f \in L^1(\mathbb{R}^d)$ we see that it is uniformly integrable. That is if $\varepsilon > 0$ then there is a $\delta > 0$ such that if $m(E) < \delta$ then

$$\int_E |f(x)|dx \leq \varepsilon$$

and by the translation invariance of the Lebesgue measure this implies

$$\int_E |f(x-y)|dx \leq \varepsilon$$

for all y. Now by Egorov's theorm we can find a $K \subset \overline{B}_R(0)$ with $m(\overline{B}_R(0) \setminus K) < \delta$ and $g_n \to 0$ uniformly on K. Then observe that if $x \in \overline{B}_R(0)$ that we have

$$|f * g_n|(x) \leq \int_K |f(x-y)g_n(y)|dy + \int_{\overline{B}_R(0)\setminus K} |f(x-y)|dy + \int_{\mathbb{R}^d \setminus \overline{B}_R(0)} |f(x-y)|dy$$

As $g_n \to 0$ uniformly on K we can find an N such that if $n \geq N$ then $||g_n||_{L^\infty(K)} \leq \varepsilon$ which implies

$$\leq \varepsilon ||f||_{L^1(\mathbb{R}^d)} + \varepsilon + \int_{\mathbb{R}^d \setminus \overline{B}_R(0)} |f(x-y)|dy$$

and we have $\int_{\mathbb{R}^d \setminus \overline{B}_R(0)} |f(x-y)|dy = \int_{\{|x-y| \geq R\}} |f(y)|dy$ then if $x \in \overline{B_r(0)}$ for $0 < r < R$ we have for $y \in \{|x-y| \geq R\}$

$$|y| \geq |x-y| - |x| \geq R - r = R - r$$

so we have $\{|x-y| \geq R\} \subset \{|y| \geq R - r\}$. Therefore,

$$\int_{\{|x-y| \geq R\}} |f(y)|dy \leq \int_{|y| \geq R-r} |f(y)|dy$$

So as $f \in L^1$ we can find an $\overline{R} > 0$ such that if $R > \overline{R}$ then

$$\int_{|x| \geq R} |f(x)|dx \leq \varepsilon$$

choosing such a large R implies for all $x \in \overline{B}_r(0)$ we have

$$|f * g_n(x)| \leq \varepsilon ||f||_{L^1} + \varepsilon + \varepsilon = C\varepsilon$$

which implies uniform convergence on every compact subset.

□

Problem 2. Let $f \in L^p(\mathbb{R})$, $1 < p < \infty$, and let $a \in \mathbb{R}$ be such that $a > 1 - 1/p$. Show that the series

$$\sum_{n=1}^{\infty} \int_n^{n+n^{-a}} |f(x+y)|dy$$

converges for a.e. $x \in \mathbb{R}$

Proof 1

Proof. Notice that by Tonelli and Holder's Inequality for sums that

$$\sum_{n=1}^{\infty}\int_{n}^{n+n^{-a}}|f(x+y)|dy=\sum_{n=1}^{\infty}n^{-a}\int_{0}^{1}|f(x+(n^{-a}z+n))|dz\leq\left(\sum_{n=1}^{\infty}n^{-aq}\right)^{1/q}\left(\sum_{n=1}^{\infty}\int_{0}^{1}|f(x+n^{-a}z+n|^{p}dz\right)^{1/p}$$

where $1/p+1/q=1$. Notice as $a>1-1/p$ this implies $-aq<-1$ i.e. the sum to the left converges so

$$=C\left(\sum_{n=1}^{\infty}\int_{0}^{1}|f(x+n^{-a}z+n)|^{p}dz\right)^{1/p}:=g(x)$$

So now it suffices to show that if $j\in\mathbb{Z}$ that

$$\int_{j}^{j+1}|g(x)|^{p}dx<\infty$$

Indeed, observe by Tonelli as all the integrand are positive that

$$\int_{j}^{j+1}|g(x)|^{p}dx=K\int_{j}^{j+1}\sum_{n=1}^{\infty}\int_{0}^{1}|f(x+n^{-a}z+n)|^{p}dzdx=K\sum_{n=1}^{\infty}\int_{0}^{1}\int_{j}^{j+1}|f(x+n^{-a}z+n)|^{p}dxdz$$

$$=K\sum_{n=1}^{\infty}\int_{0}^{1}\int_{j+n+n^{-a}z}^{j+1+n+n^{-a}z}|f(w)|^{p}dwdz\leq K\int_{0}^{1}\sum_{n=1}^{\infty}\int_{j+n+n^{-a}z}^{j+1+n+n^{-a}z}|f(w)|^{p}dwdz$$

Then notice that the intervals $[j+n+n^{-a}z,j+1+n+n^{-a}z]$ and $[j+n+1+(1+n)^{-a}z,j+2+n+(1+n)^{-a}z]$ are disjoint, so we have

$$\leq K\int_{0}^{1}\int_{\mathbb{R}}|f(w)|^{p}=K||f||_{L^{p}}^{p}<\infty$$

so the sum is finite. □

Proof 2

Proof. Fix $k\in\mathbb{N}$ then notice that it suffices to show that

$$\int_{k}^{k+1}\sum_{n=1}^{\infty}\int_{n}^{n+n^{-a}}|f(x+y)|dy<\infty$$

to get the desired claim. So Tonelli applies since all the theorems are non-negative to get

$$\int_{k}^{k+1}\sum_{n=1}^{\infty}\int_{n}^{n+n^{-a}}|f(x+y)|dydx=\int_{k}^{k+1}\sum_{n=1}^{\infty}\int_{\mathbb{R}}|f(x+y)|\chi_{[n,n+n^{-a}]}(y)dydx$$

$$=\int_{k}^{k+1}\sum_{n=1}^{\infty}\int_{\mathbb{R}}|f(z)|\chi_{[n,n+n^{-a}]}(z-x)dzdx=\int_{k}^{k+1}\sum_{n=1}^{\infty}\int_{\mathbb{R}}|f(z)|\chi_{[n+x,n+x+n^{-a}]}(z)dzdx$$

$$=\int_{\mathbb{R}}|f(z)|\int_{k}^{k+1}\sum_{n=1}^{\infty}\chi_{[n+x,n+x+n^{-a}]}(z)dxdz$$

Now we claim that we have the bound

$$\int_{k}^{k+1}\sum_{n=1}^{\infty}\chi_{[n+x,n+x+n^{-a}]}(z)dx\lesssim\min(|z-k|^{-a},1)$$

Indeed, observe that

$$\int_{k}^{k+1}\sum_{n=1}^{\infty}\chi_{[n+x,n+x+n^{-a}]}(z)dx=\int_{k}^{k+1}\sum_{n=1}^{\infty}\chi_{[n+x,n+x+n^{-a}]}(z)dx=\sum_{n=1}^{\infty}\int_{k}^{k+1}\chi_{[z-n-n^{-a},z-n]}(x)dx$$

Now we note that the integral is zero when $k>z-n$ or $k+1<z-n-n^{-a}$. So the region where the intgeral is non-zero is contained in $k+n\leq z$ and $(n+n^{-a})\geq z-(k+1)$. Notice that $n+n^{-a}\geq n$ so the region where the integral is non-zero is contained in

$$n\in[z-(k+1),z-k)]$$

so we have that

$$\sum_{n=1}^{\infty}\int_k^{k+1}\chi_{[z-n-n^{-a},z-n]}(x)dx \leq \sum_{z-(k+1)}^{z-k}\min\{|n|^{-a},1\} \leq \min\{|z-k|^{-a},1\}$$

From which it follows that by Holder

$$\int_{\mathbb{R}}|f(z)|\int_k^{k+1}\sum_{n=1}^{\infty}\chi_{[n+x,n+x+n^{-a}]}(x)dxdz \leq ||f||_{L^p}||\min\{|z-k|^{-a},1\}||_{L^q}$$

and $aq > (1-1/p)q = (1/q)q = 1$ where q is the Holder conjugate of p, so it follows that the right hand side is finite.

□

Problem 3. Let $f \in L^1_{loc}(\mathbb{R}^d)$ be such that for some $0 < p < 1$, we have

$$\left|\int f(x)g(x)dx\right| \leq ||g||_{L^p}$$

for all $g \in C_0(\mathbb{R}^d)$. Show that $f = 0$ a.e.

Proof. The key is to notice that if a set E had measure $m(E) = \delta < 1$ then $m(E)^{1/p} < m(E)$ so we want to first plug in g to be the characteristic of a nice set say a cube and keep cutting the cube up into smaller pieces which strengthens the bound to show $f = 0$ a.e.

Now let R be a rectangle unioned with its interior. So as R is closed, we know that χ_R is upper semi-continuous, so it can be approximated from above by continuous functions. But as R is compact we can make these approximations $g_n \in C_0(\mathbb{R}^d)$ and we can assume $g_n \geq 0$ with $\chi_R(x) = \inf_{n\geq 1} g_n(x)$. Therefore, by the monotone convergence theorem since $f_1 \in L^1(\mathbb{R}^d)$ we have

$$\lim_{n\to\infty}\left|\int f(x)g_n(x)dx\right| = \left|\int f(x)\chi_R dx\right|$$

and for each n we have again by the monotone convergence theorem that

$$\left|\int f(x)g_n(x)dx\right| \leq ||g_n||_p \to ||\chi_R||_p$$

so for any rectangle R

$$\left|\int_R f(x)dx\right| \leq m(R)^{1/p}$$

Now we decompose R into smaller rectangles. Indeed, fix an $N \in \mathbb{N}$ and cut R into equal 2^N pieces with each sub rectangle labeled R_i for $1 \leq i \leq 2^N$. Then we have

$$\left|\int_R f(x)dx\right| = \left|\sum_{i=1}^{2^N}\int_{R_i} f(x)dx\right| \leq \sum_{i=1}^{2^N}\left|\int_{R_i} f(x)dx\right| \leq \sum_{i=1}^{2^N} m(R_i)^{1/p} = 2^N((m(R)/2^N)^{1/p}) \to 0$$

as $N \to \infty$ since $1/p > 1$. Therefore, f integrates to zero on every rectangle, which implies since every open subset of $\mathbb{R}^d$ is a countable union of rectangles that f integrates to zero on every open set. So in particular, $f \equiv 0$ a.e. □

Alternative Proof

Proof. As $f \in L^1_{loc}(\mathbb{R}^d)$ we know that the lebesgue points of f are a set of full measure. Fix $y \in E$ then observe that if $g(x) = \frac{1}{m(B(r,y))}\chi_{B(r,y)}(x)$ then there is a sequence of functions $g_n \in C_0(\mathbb{R}^d)$ such that $g_n \leq g_{n+1} \leq g$ and $g_n \to g$ pointwise (since g is lower semi continuous). Then

$$\left|\int_{\mathbb{R}^d} f(x)g_n(x)dx\right| \leq \left(\int_{\mathbb{R}^d}|g_n(x)|^p dx\right)^{1/p}$$

so the monotone convergence theorem gives

$$\lim_{n\to\infty}\left|\int_{\mathbb{R}^d} f(x)g_n(x)dx\right| \le \left(\int_{\mathbb{R}^d}|g(x)|^p\right)^{1/p} = (m(B(r,y))^{1/p-1}$$

And note that $|f(x)g_n(x)| \le |f(x)g(x)| \in L^1(\mathbb{R}^d)$ so DCT gives us

$$\left|\int_{\mathbb{R}^d} f(x)g(x)dx\right| = \frac{1}{m(B(r,y))}\left|\int_{B(r,y)} f(x)dx\right| \le (m(B(r,y))^{1/p-1}$$

Note that as $0 < p < 1$ that $1/p - 1 > 0$ so we obtain

$$\lim_{r\to 0}\frac{1}{m(B(r,y))}\left|\int_{B(r,y)} f(x)dx\right| = 0$$

but as $y \in E$ we obtain $|f(y)| = 0$, so $f = 0$ a.e.

□

Problem 4. Let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space and assume that (e_n) is an orthonormal system in $\mathcal{H}$. Let (f_n) be another orthnormal system that is complete, i.e. the closure of the span of (f_n) is all of $\mathcal{H}$.

(1) Show that if $\sum_{n=1}^{\infty}||f_n - e_n||^2 < 1$ then the orthonormal system (e_n) is also complete.
(2) Assume that we only have $\sum_{n=1}^{\infty}||f_n - e_n||^2 < \infty$. Prove that it is still true that (e_n) is complete.

Proof. Let $E := \overline{\text{span}(e_1, e_2, ...)}$ then it suffices to show $E^\perp = \{0\}$ so fix any $x \in E$. Then as (f_n) is a complete orthonormal system we know that

$$x = \sum_{n=1}^{\infty}(x, f_n)f_n$$

so motivated by this, we define $y := \sum_{n=1}^{\infty}(x, f_n)e_n$. Now we know that $x \perp y$ so

$$||x - y||^2 = ||x||^2 + ||y||^2$$

but we also have

$$||x - y|| \le \sum_{n=1}^{\infty}||(x, f_n)(f_n - e_n)|| \le \left(\sum_{n=1}^{\infty}|(x, f_n)|^2\right)^{1/2}\left(\sum_{n=1}^{\infty}||f_n - e_n||^2\right)^{1/2}$$

so in particular, we have

$$||x - y||^2 \le \left(\sum_{n=1}^{\infty}|(x, f_n)|^2\right) = ||x||^2$$

where we used Cauchy-Schwarz and that (f_n) is a complete orthonormal system. In particular, this implies $||y||^2 = 0$, so by Bessels' Inequality we conclude $|(x, f_n)| = 0$ for all n, which means $x = 0$ i.e. $E^\perp = \{0\}$.

For (2) we define $E_N := \overline{\text{span}\{e_N, e_{N+1}, ...,\}}$ and $F_N := \overline{\text{span}\{f_N, f_{N+1}, ...,\}}$. We know that we have

$$\mathcal{H} = E_N \oplus E_N^\perp$$

We also know that $\{e_1, ..., e_N\} \subset E_N^\perp$ so it suffices to show there there is some N such that $\dim(E_N^\perp) \le N$ to conclude that $\{e_i\}$ are a complete orthonormal system. First we show that $E_N^\perp$ is finite dimensional when N is large enough.

Indeed, for any closed subspace $V \subset \mathcal{H}$ define π_V to be the orthogonal projection operator onto V. Then we have

$$||\pi_{E_N}(x) - \pi_{F_N}(x)|| = ||\sum_{n=N}^{\infty}(x, e_n)e_n - (x, f_n)f_n||$$

$$= ||\sum_{n=N}^{\infty}(x,f_n)e_n - (x,f_n)f_n + (x,e_n-f_n)e_n|| \leq \sum_{n=N}^{\infty}|(x,f_n)|(||e_n-f_n||) + ||\sum_{n=N}^{\infty}(x,e_n-f_n)f_n||$$
$$\leq \sum_{n=N}^{\infty}|(x,f_n)|(||e_n-f_n||) + (\sum_{n=N}^{\infty}||(x,e_n-f_n)||^2)^{1/2}$$
$$\leq \left(\sum_{n=N}^{\infty}|(x,f_n)|^2\right)^{1/2}\left(\sum_{n=N}^{\infty}||e_n-f_n||^2\right)^{1/2} + ||x||(\sum_{n=N}^{\infty}||e_n-f_n||^2)^{1/2} \to 0$$

so we see that the operator norm of $\pi_{E_N} - \pi_{F_N}$ converges to 0 as $N \to \infty$. Note that in the above inequalities we used Cauchy Schwarz and Pythagerous Theorem for sums.

Note that we also have
$$\mathcal{H} = F_N \oplus F_N^{\perp} \text{ and } \mathcal{H} = E_N \oplus E_N^{\perp}$$
where $\dim(E_N^{\perp}) = N$ since $\{f_n\}$ is complete and as $\pi_{F_N^{\perp}} = \text{id} - \pi_{F_N}$ and $\pi_{E_N} + \pi_{E_N^{\perp}} = \text{id}$ which implies we can choose an N so large such that
$$||\pi_{F_N^{\perp}} - \pi_{E_N^{\perp}}||_{op} = ||\pi_{F_N} - \pi_{E_N}||_{op} < 1/2$$
Now we claim this implies $\dim(F_N^{\perp}) \leq \dim(E_N^{\perp}) = N$ which lets us conclude. Indeed, observe that for any $N+1$ vectors $\{x_i\}_{i=1}^{N+1}$ in $F_N^{\perp}$ there is some $\alpha_1,..,\alpha_{N+1}$ such that $\pi_{E_N^{\perp}}(\sum_{i=1}^{N+1}\alpha_i x_i) = 0$ since $E_N^{\perp}$ has dimension N. But then
$$\frac{1}{2}||\sum_{i=1}^{N+1}\alpha_i x_i|| \geq ||\pi_{F_N}^{\perp}(\sum_{i=1}^{N+1}\alpha_i x_i) - \pi_{E_N}^{\perp}(\sum_{i=1}^{N+1}\alpha_i x_i)|| = ||\sum_{i=1}^{N+1}\alpha_i x_i||$$
i.e. $\sum_{i=1}^{N+1}\alpha_i x_i = 0$ so $F_N^{\perp}$ has dimension at most N linearly independent vectors which lets us conclude the problem since then $\{f_1,...,f_N\}$ is a basis for $F_N^{\perp}$ and the closure of $\{f_{N+1},...\}$ is a basis of $\overline{F_N}$. Hence, the closure of $\{f_1,f_2,...\}$ is a basis of $\mathcal{H}$ so it is also complete. □

Problem 5. Show that the Holder continuous functions form a set of first category (a meager set) in $C([0,1])$.

Proof. Use C^{α} to denote the space of α Holder continuous functions. Then if $\beta < \alpha$ and $f \in C^{\alpha}$ we have
$$\frac{|f(x)-f(y)|}{|x-y|^{\alpha}} = \frac{|f(x)-f(y)|}{|x-y|^{\beta+(\alpha-\beta)}} \leq C \Rightarrow \frac{|f(x)-f(y)|}{|x-y|^{\beta}} \leq C|x-y|^{\alpha-\beta} < K$$
since $\alpha - \beta > 0$ and $x, y \in [0,1]$, so we have $f \in C^{\beta}$. Therefore, the space of Holder continuous functions can be written as
$$\bigcup_{n=1}^{\infty} C^{1/n}([0,1])$$
Then notice that
$$C^{1/n}([0,1]) = \bigcup_{M=0}^{\infty}\{f : ||f||_{C^{1/n}} \leq M\} := \bigcup_{M=0}^{\infty} E_M^n$$
where $||f||_{C^{\alpha}} := ||f||_{L^{\infty}} + \sup_{x\neq y}\frac{|f(x)-f(y)|}{|x-y|^{\alpha}}$. And then we claim each E_M^n is closed. Indeed,
$$||f||_{L^{\infty}} + \frac{|f(x)-f(y)|}{|x-y|^{\alpha}} \leq ||f_n - f||_{L^{\infty}} + ||f_n||_{L^{\infty}} + \frac{|f(x)-f_n(x)|}{|x-y|^{\alpha}} + \frac{|f_n(x)-f_n(y)|}{|x-y|^{\alpha}} + \frac{|f(y)-f_n(y)|}{|x-y|^{\alpha}}$$
which can be made arbitrarily smaller than M thanks to uniform convergence. So E_M^n is closed. Therefore, if $f \in E_n^M$ we claim that $f + \varepsilon|x|^{1/(2n)} \notin E_n^M$. Indeed, observe at $x = 0$ that for any $x > 0$
$$\left|\frac{f(x)-f(0)+\varepsilon|x|^{1/(2n)}}{|x|^{1/n}}\right| \to \infty \text{ as } x \to 0$$
since $|f(x)-f(0)|/|x|^{1/n}$ is bounded and $\varepsilon|x|^{-1/n} \to \infty$. Therefore, E_n^M has empty interior, so the space of holder continuous functions is meager. □

Problem 6. Let $u \in L^2(\mathbb{R}^d)$ and let us say $u \in H^{1/2}(\mathbb{R}^d)$ if

$$(1+|\xi|^{1/2})\hat{u}(\xi) \in L^2(\mathbb{R}^d)$$

Show that $u \in H^{1/2}(\mathbb{R}^d)$ iff

$$\int\int \frac{|u(x+y)-u(y)|^2}{|y|^{d+1}} dydx < \infty$$

Proof. Notice by Plancheral and Tonelli that that

$$\int\int \frac{|u(x+y)-u(y)|^2}{|y|^{d+1}} dydx = \int |\hat{u}(\xi)|^2 \int \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dyd\xi$$

So it suffices to show there are $C_1, C_2 > 0$ such that

$$C_1(1+|\xi|^{1/2})^2 \leq \int \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy \leq C_2(1+|\xi|^{1/2})^2$$

But as $u \in L^2(\mathbb{R}^d)$ we know that $\hat{u} \in L^2(\mathbb{R}^d)$, so it suffices to prove

$$C_1|\xi| \leq \int \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy \leq C_2|\xi|$$

For the upper bound observe that

$$\int \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy = \int_{|y\xi|\leq 1} \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy + \int_{|y\xi|\geq 1} \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy$$

Using $e^x - 1 = \int_0^x e^x dx \Rightarrow |1-e^x| \leq |x||e^x|$ so

$$\leq \int_{|y\cdot\xi|\leq 1} \frac{4\pi^2|\xi|^2|y|^2}{|y|^{d+1}} + \int_{|y\cdot\xi|\geq 1} \frac{2|\xi|^{d+1}}{|y/\xi|^{d+1}}$$

$$\lesssim |\xi| + |\xi| = |\xi|$$

Now for the lower bound observe that, for any fixed ξ that there is an orthogonal matrix A such that $A(\xi/|\xi|) = e_n = (0,0,...,1)$. Also the FTC also tells us $|1-e^x| \geq |x|$.So we have

$$\int \frac{|1-e^{2\pi i\xi\cdot y}|^2}{|y|^{d+1}} dy \geq \int \frac{4\pi^2|\xi\cdot y|^2}{|y|^{d+1}} dy = \int \frac{4\pi^2|\xi|^2|y\cdot e_n|^2}{|y|^{d+1}}$$

$$\geq 4\pi^2 \int_{|y\xi|\leq 1\cap(|y\cdot e_n|\geq 1/2||y||)} \frac{|\xi|^2|y\cdot e_n|^2}{|y|^{d+1}} \geq \pi^2|\xi|^2 \int_{|y\xi|\leq 1\cap|y\cdot e_n|\geq 1/2||y||} \frac{|\xi|^2}{|y|^{d-1}} = C_n|\xi|$$

so we have the desired result ☐

Problem 7. Assume that $f(z)$ is analytic in $\{z : |z| < 1\}$ and continuous on $\{z : |z| \leq 1\}$. If $f(z) = f(1/z)$ when $|z| = 1$, prove that f is constant.

Proof. Note that $f(1/z)$ is analytic on $\{z : |z| \geq 1\}$ and extends continuously to $f(z)$ on $|z| = 1$. Therefore, by Morrera's Theorem we conclude that

$$g(z) := \begin{cases} f(z) \text{ if } z \in \{|z| \leq 1\} \\ f(1/z) \text{ else} \end{cases}$$

is an entire function. However, as f extends continuously to $\{|z| = 1\}$ $f(z)$ is bounded on $\overline{\mathbb{D}}$, which implies $f(1/z)$ is bounded on $\mathbb{C} \setminus \mathbb{D}$. Therefore, $g(z)$ is a bounded entire function, so by Liouville it is constant which implies f is constant.

☐

Problem 8. Assume that $f(z)$ is an entire function that is 2π-periodic in the sese that $f(z+2\pi) = f(z)$ and

$$|f(x+iy)| \le Ce^{\alpha|y|}$$

for some $C > 0$ and $0 < \alpha < 1$. Prove that f is constant.

Proof. Note that as $e^{\alpha|x|} > 1$ this implies that

$$|f(x+iy)| \le Ce^{\alpha|y|}e^{\alpha|x|} = Ce^{\alpha(|x|+|y|)}$$

that is f is an entire function of order $0 < \alpha < 1$. In particular, we also have $g(z) := f(z) - f(0)$ is an entire function of order α that is 2π-periodic. So by periodicity we have that if we denote Z as the set of zeros of g that

$$\{2n\pi\} \subset Z$$

Therefore, the zeros of g grow at least linearly. However, we know that if f is not identically 0 then by Jensen's formula that for large enough R that the number of zeros of g on $B_R(0)$ should be bounded by $C|R|^\alpha$ where $\alpha < 1$, but our zeros grow at least linearly, so which means g must be the zero function i.e. f is constant.

□

Problem 9. Let (f_j) be a sequence of entire functions such that

$$\int\int_{\mathbb{C}} |f_j(z)|^2 e^{-|z|^2} dxdy \le C$$

for some constant $C > 0$. Show that there is a sub-sequence $\{f_{j_j}\}$ and an entire function f such that

$$\int\int_{\mathbb{C}} |f_j(z) - f(z)|^2 e^{-2|z|^2} dxdy \to 0$$

Proof. We first claim that there is an entire function f such that $f_n \to f$ uniformly on every compact subset of $\mathbb{C}$. Indeed, fix an $R > 0$ then we have for any $z_0 \in B_R(0)$ that for any $0 < r < R$

$$f_j(z_0) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} f_j(z_0 + re^{i\theta})d\theta$$

so we have

$$\int_{r=0}^{R} rf_j(z)dr = \frac{1}{2\pi}\int_{r=0}^{R}\int_{\theta=0}^{2\pi} f_j(z_0 + re^{i\theta})rd\theta dr$$

$$f_j(z_0) = \frac{1}{\pi R^2}\int\int_{B_R(z_0)} f_j(z)dxdy$$

so Holders shows

$$|f_j(z)| \le \frac{1}{\sqrt{\pi}\sqrt{R}}\int\int_{B_R(z_0)} |f_j(z)|^2 \le \frac{1}{\sqrt{\pi}\sqrt{R}}\int\int_{B_{2R}(0)} |f_j(z)|^2 \le \frac{C(R)}{\sqrt{\pi}\sqrt{R}}\int\int_{\mathbb{C}} |f_j(z)|^2 e^{-|z|^2}$$

which implies by the given assumptions that $\{f_j(z)\}$ is a uniformly bounded family on every compact subset of $\mathbb{C}$. Which implies by Montel's theorem that on every compact subset we have a uniformly convergent sub-sequence. By taking a diagonal sub-sequence we can find a sub-sequence $\{f_{j_k}(z)\}$ that uniformly converges to some function f on every $B_n(0)$ where $n \in \mathbb{N}$. This implies thanks to the Morrera's Theorem that f is holomorphic on every $B_n(0)$ so it is an entire function. And uniform convergence on every compact subset implies that

$$C \ge \int\int_{\mathbb{C}} |f_{j_k}(z)|^2 e^{-|z|^2}\chi_{B_n(0)}dxdy \to \int\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\chi_{B_n(0)}dxdy$$

so it follows from Monotone Convergence Theorem that

$$C \ge \lim_{n\to\infty}\int\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\chi_{B_n(0)}dxdy = \int\int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}dxdy$$

Then we have for any $R > 0$ that

$$\int\int_{\mathbb{C}} |f_{j_k}(z)-f(z)|^2 e^{-2|z|^2} dxdy = \int\int_{B_R(0)} |f_{j_k}(z)-f(z)|^2 e^{-2|z|^2} dxdy + \int\int_{\mathbb{C}\setminus B_R(0)} |f_{j_k}(z)-f(z)|^2 e^{-2|z|^2} dxdy$$

and

$$\int\int_{\mathbb{C}\setminus B_R(0)} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2} dxdy \leq \exp(-R^2) \int\int_{\mathbb{C}\setminus B_R(0)} |f_{j_k}(z) - f(z)|^2 e^{-|z|^2} dxdy \leq K \exp(-R^2)$$

thanks to our previous computation. Therefore, if $\varepsilon > 0$ we can find an R so large such that $\int\int_{\mathbb{C}\setminus B_R(0)} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2} dxdy \leq \varepsilon/2$. And by uniform convergence on $B_R(0)$ we can choose a K so large such that for any $k \geq K$ we have

$$\int\int_{\mathbb{C}\setminus B_R(0)} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2} dxdy \leq \varepsilon/2$$

This means for any $k \geq K$ we have

$$\int\int_{\mathbb{C}} |f_{j_k}(z) - f(z)|^2 e^{-2|z|^2} dxdy \leq \varepsilon$$

so we have the desired result.

□

Problem 10. Use the residue theorem to prove that

$$\int_0^\infty e^{\cos(x)} \sin(\sin(x)) \frac{dx}{x} = \frac{\pi}{2}(e-1)$$

Use a large semi-circle as part of the contour.

Proof. Fix $R > 0$ large and $r > 0$ small. Let γ_R be the semi-circle centered at the origin of radius R i.e. $\gamma_R = Re^{i\theta}$ where $\theta \in [0, \pi]$ and $\gamma_{-R\to -r}$ be the line from $z = -R$ to $z = -r$ with counterclock wise orientation, and similarly for $\gamma_{r\to R}$. Let γ_r be the semi circle of radius r starting at $-r$ and ending at r i.e. $\gamma_r = re^{i\theta}$ for $\theta \in [\pi, 2\pi]$. Also notice that $\exp(\cos(x))\sin(\sin(x)) = \operatorname{Im}(e^{e^{ix}})$ so

$$\int_0^\infty \exp(\cos(x))\sin(\sin(x)) = \operatorname{Im}(e^{e^{ix}})\frac{dx}{x} = \operatorname{Im} \int_0^\infty \exp(\cos(x))\sin(\sin(x)) = \operatorname{Im}(e^{e^{ix}})\frac{dx}{x}$$

Let $\gamma := \gamma_R +_{-R\to -r} +\gamma_r + \gamma_{r\to R}$ then we know by the Residue Theorem since $e^{e^{iz}}/z$ has a residue of e at $z = 0$ that

$$\int_\gamma e^{e^{iz}} \frac{dz}{z} = 2\pi i e$$

and

$$\int_{\gamma_R} e^{e^{iz}} \frac{dz}{z} = \int_{\theta=0}^{\pi} i \exp(\exp(-R\sin(\theta))[\cos(R\cos(\theta)) + i\sin(R\cos(\theta))] d\theta \to i\pi$$

where the last convergence is due to the dominated convergence theorem. And we also have

$$\int_{\gamma_r} e^{e^{iz}} \frac{dz}{z} \to i\pi e$$

and notice on the real line our integrand is even, so we obtain by the Residue theorem that

$$2\int_0^\infty e^{\cos(x)} \sin(\sin(x)) \frac{dx}{x} + \pi e + \pi = 2\pi e \Rightarrow \int_0^\infty e^{\cos(x)} \sin(\sin(x)) \frac{dx}{x} = \frac{\pi}{2}(e-1)$$

□

Problem 11. Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$ and let u be subharmonic on Ω, continuous on $\overline{\Omega}$ such that

$$u(x,y) \leq |x+iy|$$

for large $(x,y) \in \Omega$. Assume that

$$u(x,0) \leq ax \quad u(0,y) \leq by \quad x,y \geq 0$$

for some $a,b > 0$. Show that

$$u(x,y) \leq ax + by \text{ for } (x,y) \in \Omega$$

Proof. This is a standard application of the Phragmén–Lindelöf method. In general, in the sector $\{z : \alpha \leq \text{Arg}(z) \leq \beta\}$ we should have for any $0 < k < \frac{\pi}{\beta-\alpha}$ that $|z|^k$ is a barrier function. Note for our domain $\alpha = 0, \beta = \pi/2$ so we should have for any $0 < k < 2$ that $|z|^k$ is a barrier. We want our barrier to grow much faster than u at infinity, so we want $1 < k < 2$ say $k = 3/2$. Now we notice that for any $\phi \in [0, 2\pi]$ that

$$\text{Re}(e^{i\phi} z^{3/2}) = |z|^{3/2} \cos(\frac{3}{2}\text{Arg}(z) + \phi)$$

now we want to choose ϕ such that $-\frac{\pi}{2} < \frac{3}{2}\text{Arg}(z) + \phi < \frac{\pi}{2}$ to make the phase term bounded above and below by a positive constant and recall $0 \leq \text{Arg}(z) \leq \frac{\pi}{2}$ so we take $\phi = -\frac{3\pi}{8}$ then we have the desired bounds. Therefore, $\phi(z) := |z|^{3/2}\cos(\frac{3}{2}\text{Arg}(z) - \frac{3\pi}{8})$ is harmonic so we have that for any $\varepsilon > 0$

$$v(x,y) := u(x,y) - ax - by - \varepsilon\phi(z)$$

is subharmonic and we have

$$v(x,0) \leq -by - \varepsilon|x|^{3/2}\cos(\frac{3}{2} - \frac{3\pi}{8}) \leq 0$$

and

$$v(0,y) \leq -ax - \varepsilon|y|^{3/2}\cos(\frac{3\pi}{4} - \frac{3\pi}{8}) \leq 0$$

Then there exists an $R(\varepsilon) > 0$ such that for any $r \geq R(\varepsilon)$ that

$$v(x,y) \leq 0 \text{ on } \Omega \cap \partial B_r(0)$$

since $u - ax - by$ grows at most linearly and our barrier function is super linear. This implies by the maximum principle that on $\Omega \cap B_r(0)$ that we have

$$v(x,y) \leq 0 \text{ on } B_r(0) \cap \Omega$$

and let $r \to \infty$ to conclude for any $\varepsilon > 0$ that

$$u(x,y) - ax - by - \varepsilon\phi(z) \leq 0 \text{ on } \Omega$$

let $\varepsilon \to 0$ to conclude

$$u(x,y) \leq ax + by$$

□

Problem 12. Find a function $u(x,y)$ harmonic in the region between the circles $|z| = 2$ and $|z-1| = 1$ which equals 1 on the outer circle and 0 on the inner circle (except at the point where the two circles are tangent to one another).

Proof. Note that the two circles are tangent at $z = 2$. We want to map this conformally onto a strip and solve the problem there then invert back. We recall that Mobius Transformations map circles to generalized circles i.e. circles and lines, so we choose a mobius transformation such that 2 is sent to infinity, to make the circles become lines. Indeed, consider $\phi(z) := \frac{1}{z-2}$ then this is a mobius transformation and

$$\phi(e^{i\theta}) = \frac{1}{2e^{i\theta} - 2} \Rightarrow \phi(e^{i\pi}) = -\frac{1}{4} \text{ and } \phi(e^{i\pi/2}) = -\frac{1}{4} - \frac{i}{4}$$

so $|z| = 2$ gets mapped to the line $\text{Re}(z) = -1/4$. And similarly $\phi(0) = -1/2$ and $\phi(1+i) = -1/2 - i/2$ so $|z-1|$ gets mapped to the line $\text{Re}(z) = -1/2$. And as ϕ is a continuous map on the interior, we know that it maps Ω to a connected set with boundary $\text{Re}(z) = -1/4$ and $\text{Re}(z) = -1/2$, so $\phi(\Omega) = \{z : -1/2 < \text{Re}(z) < -1/4\}$. Then we want to solve the Problem

$$\begin{cases} \Delta u = 0 \text{ on } \phi(\Omega) \\ u = 1 \text{ on } \text{Re}(z) = -1/2 \\ u = 0 \text{ on } \text{Re}(z) = -1/4 \end{cases}$$

so we make the guess $u = ax + by + c$ for constants a, b, c. Then the PDE becomes solving a 2×2 matrix, which implies that $a = -4, c = -1, b = 0$ so $u(x + iy) = -4x - 1 = \text{Re}(-4z - 1)$ solves that PDE. So $u \circ \phi = \text{Re}(-4/(z-2) - 1)$ is the desired harmonic function.

□

13. Spring 2016

Problem 1. Let

$$K_t(x) = (4\pi t)^{-3/2} e^{-|x|^2/4t}, \quad x \in \mathbb{R}^3, t > 0$$

where $|x|$ is the Euclidan norm of $x \in \mathbb{R}^3$.

(1) Show that the linear map

$$L^3(\mathbb{R}^3) \ni f \mapsto t^{1/2} K_t * f \in L^\infty(\mathbb{R}^3)$$

is bounded, uniformly in $t > 0$.

(2) Prove that $t^{1/2}||K_t * f||_{L^\infty} \to 0$ as $t \to 0$ for $f \in L^3(\mathbb{R}^3)$.

Proof. Notice that by Holder's Inequality that

$$|K_t * f| \leq ||f||_{L^3(\mathbb{R}^3)} ||K_t||_{L^{3/2}(\mathbb{R}^3)}$$

and

$$\int_{\mathbb{R}^3} |K_t|^{3/2} dx = C \int_{\mathbb{R}^3} t^{-9/4} \exp(-3|x|^2/8) dx = \tilde{C} \int_{\mathbb{R}^3} t^{-3/4} e^{-|u|^2} du = M t^{-3/4}$$

by the change of coordinates. So we have

$$||K_t||_{L^{3/2}} = \tilde{M} t^{-1/2}$$

So we have

$$t^{1/2} |K_t * f| \leq \tilde{M} ||f||_{L^3}$$

i.e. this functional is uniformly bounded in t.

For the second part, it suffices by the uniform boundness of the operator in t to show that the result is true for a dense subclass of L^3. In particular, it suffices to show it for simple functions and as the operator is linear, it suffices to show it for characteristics of measurable sets. Indeed, observe that

$$t^{1/2}\left(K_t * \chi_E\right) = \frac{1}{(4\pi)^{3/2} t} \int_E e^{-|x-y|^2/4t} dy \leq \frac{1}{(4\pi)^{3/2} t} \int_{\mathbb{R}^3} e^{-|x|^2/4t} dx$$
$$= \frac{t^{1/2}}{(4\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-|x|^2/4} dx = C t^{1/2} \to 0 \text{ as } t \to 0$$

so $t^{1/2}(K_t * f) \to 0$ is true for the dense subclass of simple functions since $(K_t * \chi_E) \geq 0$, so it is true by uniform boundness in L^3 for functions in L^3.

Alternative Proof Of Second Part: Heat Kernal Approach For the second part, notice that it suffices to prove the claim for $C_c^\infty(\mathbb{R}^3)$ since its a dense subclass of L^3 because we have if $g \in C_c^\infty(\mathbb{R}^3)$ and $f \in L^3$ then

$$t^{1/2}||K_t * f||_{L^\infty} \leq t^{1/2}||K_t * (f-g)||_{L^\infty} + t^{1/2}||K_t * g||_{L^\infty}$$
$$\leq C||f-g||_{L^3} + t^{1/2}||K_t * g||_{L^\infty}$$

and the first term can be made small by using density of test functions on L^3. Now we claim the following lemma: If $g \in C_c^\infty(\mathbb{R}^3)$ then as $t \to 0$ we have $K_t * g(x) \to g(x)$ uniformly. This implies the problem since then we have $t^{1/2}||K_t * g||_{L^\infty} \to 0$ due to uniform convergence. Now observe

$$\int_{\mathbb{R}^3} K_t(x) dx = 1$$

then

$$|K_t * g(x) - g(x)| \leq \int_{B_\delta(x)} K_t(x-y)|g(x) - g(y)| dy + \int_{|x-y| \geq \delta} K_t(x-y)|g(x) - g(y)|$$

by uniform continuity of g we can choose $\delta > 0$ so small such that if $\varepsilon > 0$ is given then

$$|g(x) - g(y)| \leq \varepsilon \Rightarrow \int_{B_\delta(x)} K_t(x-y)|g(x) - g(y)| dy \leq \varepsilon$$

since $K_t \geq 0$ and has mass 1. Now for the second integral, we have if $M = ||g||_{L^\infty}$ then

$$\int_{|x-y|\geq\delta} K_t(x-y)|g(x)-g(y)| \leq 2M \int_{|x|\geq\delta} K_t(x)dx$$

As $K_t \in L^1(\mathbb{R}^3)$ we can find a compact set K such that $\int_{x\notin K} K_t(x) < \varepsilon$ so

$$2M \int_{|x|\geq\delta} K_t(x)dx \leq 2M \int_{|x|\geq\delta\cap K} K_t(x) + 2M\varepsilon$$

Then observe on $|x| \geq \delta$ we have

$$K_t(x) \leq Ct^{-3/2}e^{-\delta^2/4t}$$

so

$$\int_{|x|\geq\delta} K_t(x)dx \leq Ct^{-3/2}e^{-\delta^2/4t}m(K)$$

and we know that as $t \to 0$ this term goes to 0 since exponential decays much faster than polynomials grow, so we obtain if $0 < t \ll 1$

$$|K_t * g(x) - g(x)| \leq \varepsilon + \varepsilon + 2M\varepsilon$$

and this bound is independent of x, so we conclude $K_t * g \to g$ uniformly as desired.

□

Problem 2. Let $f \in L^1(\mathbb{R})$. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f(x-\sqrt{n})$$

converges absolutely for almost all x.

Proof. Define $F(x) := f(-x)$ then it suffices to show that the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} F(x+\sqrt{n}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} f(-x-\sqrt{n})$$

converges for a.e. x. Indeed, fix a $k \in \mathbb{N}$ then WLOG by replacing F with $|F|$ if necessary we can assume that $F \geq 0$ to see

$$\int_{x=k}^{k+1} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} F(x+\sqrt{n}) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \int_{x=k}^{k+1} F(x+\sqrt{n}) = \sum_{n=1}^{\infty} \int_{k+\sqrt{n}}^{k+1+\sqrt{n}} \frac{1}{\sqrt{n}} F(x)dx$$

where the interswap of derivative is justified by Tonellis since $F \geq 0$. Now observe

$$= \sum_{j=2}^{\infty} \sum_{n=(j-1)^2}^{j^2} \int_{k+\sqrt{n}}^{k+1+\sqrt{n}} \frac{1}{\sqrt{n}} F(x)dx$$

and

$$\sum_{n=(j-1)^2}^{j^2} \int_{k+\sqrt{n}}^{k+1+\sqrt{n}} \frac{1}{\sqrt{n}} F(x)dx \leq \sum_{n=(j-1)^2}^{j^2} \int_{k+j-1}^{k+1+j} \frac{F(x)}{j-1} dx = \frac{(2j-1)}{j-1} \int_{k+j-1}^{k+1+j} F(x) \leq C \int_{k+j-1}^{k+1+j} F(x)dx$$

since $(2x-1)/(x-1)$ is bounded on $[2,\infty)$. Therefore,

$$\leq \sum_{j=2}^{\infty} C \int_{k+j-1}^{k+1+j} F(x)dx \leq 2C \int_{\mathbb{R}} F(x)dx$$

since each interval overlaps at most twice. Therefore, for a.e. $x \in [k, k+1]$ we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} F(x+\sqrt{n}) < \infty$. Then as these sets are a countable partiton of $\mathbb{R}$, it follows that for a.e. $x \in \mathbb{R}$ the sum is absolutely convergent.

□

Problem 3. Let $f \in L^1_{loc}(\mathbb{R})$ be real valued and assume for all integers $n > 0$, we have that
$$f(x + 1/n) \geq f(x)$$
for a.e. $x \in \mathbb{R}$. Show that for each real number $a \geq 0$ we have
$$f(x + a) \geq f(x)$$
for almost all $x \in \mathbb{R}$

Proof. For $m, n \in \mathbb{N}$ let
$$X_{m,n} := \{x \in \mathbb{R} : f(x + m/n) < f(x + (m-1)/n)\}$$
then $\bigcup_{m,n\in\mathbb{N}} X_{m,n}$ is a null set by the give assumptions. Therefore, $X := \bigcap_{m,n\in\mathbb{N}} X^c_{m,n}$ is a set of full measure i.e. for almost every $x \in \mathbb{R}$ we have
$$f(x + m/n) \geq f(x)$$
Then write E to be the Lebesgue points of f, and we have $m(E^c) = 0$ and let $Y := E \cap X$ then this is also a set of full measure. Finally if $g(x) := f(x + a)$ let E_2 be the Lebesgue points of g then this also a set of full measure, so $W := E \cap X \cap E_2$ is of full measure. And if $x \in W$ then we have that
$$f(x + a) = \lim_{r\to 0} \frac{1}{2r} \int_{x+a-r}^{x+a+r} f(y)dy \text{ and } f(x) = \lim_{r\to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(y)dy$$
and as $x \in W$ we know for any positive rational q that
$$f(x + q) \geq f(x)$$
Now fix $\varepsilon > 0$ and a positive rational such that $0 < a - q < \delta$ [we can assume $a \neq 0$ otherwise this is trivial] then we have
$$\frac{1}{2r} \int_{x+a-r}^{x+a+r} f(y)dy - \frac{1}{2r} \int_{x+q-r}^{x+q+r} f(y)dy = -\frac{1}{2r} \int_{x+q-r}^{x+a-r} f(y)dy + \frac{1}{2r} \int_{x+q+r}^{x+a+r} f(y)dy$$
Then as $f \in L^1_{loc}$ it is locally uniformly integrable, so we can make the above two integrals smaller than ε if $\delta > 0$ is sufficiently small i.e. choose $\delta > 0$ so small such that
$$\left| \frac{1}{2r} \int_{x+a-r}^{x+a+r} f(y)dy - \frac{1}{2r} \int_{x+q-r}^{x+q+r} f(y)dy \right| \leq \varepsilon$$
Therefore, as
$$\frac{1}{2r} \int_{x+q-r}^{x+q+r} f(y)dy - \frac{1}{2r} \int_{x-r}^{x+r} f(y)dy = \frac{1}{2r} \int_{x-r}^{x+r} f(y + q) - f(y)dy \geq 0$$
where for the above inequality we used that the set where $f(y + q) \geq f(y)$ is of full measure. So we have
$$\frac{1}{2r} \int_{x+a-r}^{x+a+r} f(y)dy - \frac{1}{2r} \int_{x-r}^{x+r} f(y) \geq -\varepsilon$$
and letting $\varepsilon \to 0$ and $r \to 0$ using these points are Lebesgue points yields
$$f(x + a) \geq f(x)$$
and as $x \in W$ is of full measure we are done.

□

Problem 4. Let V_1 be a finite-dimensional subspace of a Banach Space V. Show that there exists a continuous projection map $P : V \to V_1$.

Proof. Let $\{e_1, .., e_n\}$ be a basis of V_1 then define on $S := \text{span}(e_1, ..., e_n)$

$$f_j(x) = f_j(\sum_{i=1}^{n} a_i e_i) = a_j \text{ for } 1 \le j \le n$$

And as f_j is a linear functional on a finite dimensional vector space, it is automatically continuous. By Hanh-Banach, we can extend f_j to a continuous linear functional on V. Then write

$$P(x) := \sum_{i=1}^{n} f_j(x) e_j$$

then we have $\text{Im}(P) \subset V_1$. Also we have

$$P(x) = \sum_{i=1}^{n} \alpha_j e_j \Rightarrow P^2(x) = \sum_{i=1}^{n} \alpha_j e_j$$

where $\alpha_j = f_j(x)$. We also have $\text{Im}(P) = V_1$ by using $P(x)$ is the identity on V_1. So $P(x)$ is a projection map. □

Problem 5. For $f \in C_0^\infty(\mathbb{R}^2)$ define $u(x,t)$ by

$$u(x,t) = \int_{\mathbb{R}^2} e^{ix\cdot\xi} \frac{\sin(t|\xi|)}{|\xi|} f(\xi) d\xi, \quad x \in \mathbb{R}^2, t > 0$$

Show that $\lim_{t\to\infty} ||u(\cdot,t)||_{L^2} = \infty$ for a set of f that is dense in $L^2(\mathbb{R})$.

Proof. Define $f_t(x) := \sin(t|x|)/|x| f(x)$ then

$$u(x,t) = \hat{f}_t(x) \Rightarrow ||u||_{L^2} = ||\hat{\hat{f}}_t(x)||_{L^2} = ||f_t||_{L^2}$$

where we used Plancheral's and that $\hat{\hat{f}}(x) = f(-x)$. So we have

$$||u||_{L^2}^2 = \int_{\mathbb{R}^2} \left| \frac{\sin(t|\xi|)}{|\xi|} \right|^2 |f(\xi)|^2 d\xi$$

Notice that as

$$\lim_{(x,y)\to(0,0)} \frac{|\sin(|x|)|}{|x|} = 1$$

that we can find a $\delta > 0$ such that on $B_\delta(0)$ we have

$$|\sin(|x|)| \ge \frac{1}{2}|x|$$

this implies on $B_{\delta/t}(0)$ that

$$|\sin(t|x|)| \ge t/2|x|$$

so this gives us

$$\int_{\mathbb{R}^2} \left| \frac{\sin(t|\xi|)}{|\xi|} \right|^2 |f(\xi)|^2 d\xi \ge \int_{B_{\delta/t}(0)} \frac{t^2}{4} |f(\xi)|^2 \gtrsim \text{essinf}_{\xi \in B_{\delta/t}(0)} |f(\xi)|^2$$

Therefore, if $\liminf_{\varepsilon\to 0} \text{essinf}_{\xi\in B_\varepsilon(0)} |f(\xi)| = \infty$ then we will have

$$\lim_{t\to\infty} ||u(\cdot,t)||_{L^2} = \infty$$

and notice that such $f \in L^2$ is dense since for any $g \in L^2$ such that $\liminf_{\varepsilon\to 0} \text{essinf}_{\xi\in B_\varepsilon(0)} |g(\xi)|$ is finite we can consider $g + \varepsilon|x|^{-1/2}\chi_{B_1(0)} \in L^2$.

□

Problem 6. Suppose that $\{\phi_n\}$ is an orthonormal system of continuous functions in $L^2([0,1])$ and let S be the closure of the span of $\{\phi_n\}$. If $\sup_{f\in S\setminus\{0\}} ||f||_\infty/||f||_2$ is finite show that S is finite dimensional.

Proof. Let $K > 0$ be such that

$$||f||_\infty \leq K||f||_2$$

for $f \in S$. Then notice that $S \subset C([0,1])$ where f is endowed with the L^2 norm since if $f_n \to f$ in L^2 we have

$$||f_n - f||_\infty \leq K||f_n - f||_2 \to 0$$

and each $f_n \in S$ can be approximated arbitrarily well with finite linear combinations of $\phi_n \in C([0,1])$ which implies that f is continuous due to uniform convergence.

Now observe that this implies the evaluation functional for $y \in [0,1]$

$$L_y(f) = f(y)$$

is continuous in this norm since

$$|L_y(f)| \leq ||f||_\infty \leq K||f||_2$$

so by Risez Representation Theorem since S is a Hilbert Space, we know that there is a g_x such that

$$(f, g_x) = \int_0^1 f(y)\overline{g_x}(y)dy = f(x)$$

Now observe that

$$\sum_{n=1}^N |\phi_n(y)|^2 = \sum_{n=1}^N |(\phi_n, g_y)|^2 \leq ||g_y||_{L^2}$$

where the last inequality is due to Bessel's Inequality since ϕ_n are orthonormal. But we have that

$$||g_y||_{L^2}^2 = (g_y, g_y) = g(y) \leq ||f||_\infty \leq K||f_y||_{L^2}$$

i.e.

$$||g_y||_{L^2} \leq K$$

so we obtain

$$\sum_{n=1}^N |\phi_n(y)|^2 \leq K^2$$

so integrating and using ϕ_n are orthonormal implies

$$N \leq K^2$$

therefore, we can have at most $\lfloor K^2 \rfloor$ orthonormal vectors i.e. S is finite dimensional.

□

Problem 7. Determine

$$\int_0^\infty \frac{x^{a-1}}{x+z}dx$$

for $0 < a < 1$ and Re$z > 0$. Justify all computation.

Problem 8. Let $\mathbb{C}_+ := \{z \in \mathbb{C}; \text{Im} z > 0\}$ and let $f_n : \mathbb{C}_+ \to \mathbb{C}_+$ be a sequence of holomorphic functions. Show that unless $|f_n| \to \infty$ uniformly on compact subsets of $\mathbb{C}_+$, there exists a subsequence converging uniformly on compact subsets of $\mathbb{C}_+$.

Proof. **First proof via Conformal Maps and Montels** Define $\psi(z) := (z+i)/(z-i)$ then $\psi : \mathbb{C}_+ \to \mathbb{D}$ is a conformal map. Then notice that $g_n := \psi \circ f_n : \mathbb{C}_+ \to \mathbb{D}$, so the family is uniformly bounded, so by Montel's Theorem there exists a subsequence that converges locally uniformly to another holomorphic function g. We still denote this subsequence as g_n.

Case 1 If $g(z) \neq -1$ for any $z \in \mathbb{C}_+$ then we claim $f_n \to \psi^{-1} \circ g$ locally uniformly. Indeed, fix a compact set $K \subset \mathbb{C}_+$ and notice $\psi^{-1}(z) = i(1+z)/(z-1)$, so as $g(z) \neq 1$ on K and $g(K)$ is compact, there is a $\delta > 0$ such that $|g(z) + 1| > 2\delta$. By uniform convergence of g_n to g on K, we may by assuming n is large enough that $|g_n + 1| > \delta$. Then

$$|f_n - \psi^{-1}g| = |\psi^{-1} \circ \psi \circ f_n - \psi^{-1}g| \leq C(\delta)|\psi \circ f_n - g| \leq C(\delta)||\psi \circ f_n - g||_{L^\infty(K)}$$

where we used that ψ^{-1} is Lipschitz whenever we are a finite distance away from -1 since it is holomorphic. Therefore, $f_n \to \psi^{-1}g$ uniformly on K, so we have $f_n \to \psi^{-1}g$ locally uniformly on K.

Case 2 By the maximum principle, if $g(z) = -1$ for any $z \in \mathbb{C}_+$ then $g(z) \equiv -1$ on $\mathbb{C}_+$. Now notice we can reapply Montels theorem to any subsequence of $\{g_n\}$ to find a further subsequence that locally uniformly converges. If every subsequence has a further subsequence that locally uniformly converges to -1 then we have that g_n locally uniformly converges to -1, so $|f_n| \to \infty$ locally uniformly. But if there exists a subsequence of g_n that has a further subsequence that does not converge locally uniformly to -1, then we may reapply Case 1 to deduce that along a subsequence the subsequence locally uniformly converges. Hence, we are done.

Alternative Proof via Harmonic Function Theory Notice that if for all $w \in \mathbb{C}_+$ we have

$$\lim_{n\to\infty} \text{Im}(f_n(w)) = \infty$$

then Harnack's Inequality implies since $v_n := \text{Im}(f_n) \geq 0$ that for any compact set K

$$\sup_{z\in K} v_n(z) \leq C(K) \inf_{z\in K} v_n(z)$$

so we have local uniform convergence of the imaginary part to ∞. Now if there exists a point w such that

$$\liminf_{n\to\infty} v_n(w) = M < \infty$$

then we claim we have local uniform convergence to a harmonic function along a subsequence.

WLOG by looking at the subsequence $\text{Im}(f_{n_k})(w) \to \text{Im}(f)(w) = M < \infty$, we can assume $\text{Im}(f_n)(w) \to M$. By Harnack's Inequality we obtain for any compact set $K \subset \mathbb{C}_+$ with $w \in K$ that

$$\sup_{z\in K} v_n(z) \leq C(K)v_n(w) \leq M(K)$$

i.e. the family of harmonic functions is uniformly bounded. Then observe that for $z \in K$ there is a $\delta > 0$ such that $d(K, \partial\mathbb{C}_+) = 2\delta$ then for any $z \in K$ we have $B_\delta(z) \subset \mathbb{C}_+$ so

$$\partial_{x_i} v_n(z) = \frac{1}{\pi\delta^2} \int_{B_\delta(z)} \partial_{x_i} v_n(z) dA(z)$$

where dA is the lebesgue area measure. So by the divergence theorem we know

$$= \frac{1}{\pi\delta^2} \int_{\partial B_\delta(z)} v_n(z) n_i d\sigma(z)$$

where n_i is the ith component of the normal and σ is the surface area measure. So in particular, we have that

$$|\nabla v_n(z)| \leq C(K) \sup_{z\in K} |v_n(z)| \leq \tilde{C}(K)$$

where the last constant does not depend on n thanks to our earlier remarks. In particular, this implies the family is uniformly Lipschitz and Bounded. So by Arzela-Ascoli there is a uniformly convergent sub-sequence, which we denote by $v_{n_k} \to v$ uniformly on K. By taking a compact exhausation of K, and diagonal subsequence, we can find a subsequence which we denote by m such that $v_m \to v$ locally uniformly on K for any compact set $K \subset \mathbb{C}_+$.

Now this implies that by the mean value equivalence of harmonic functions, that v is harmonic on $\mathbb{C}_+$ which is simply connected, so we can find a u such that $u + iv$ is holomorphic on $\mathbb{C}_+$. Now we claim that $f_m \to u + C + iv := f$ where $C \in \mathbb{R}$ is some constant locally uniformly. Indeed, notice that on any compact set $K \subset \mathbb{C}_+$ we have

$$\sup_{z\in K} |\text{Im}(f_m(z)) - \text{Im}(f(z))| \to 0$$

so the imaginary part of the holomorphic function $f - f_m$ converges to 0. In particular, by the Cauchy Estimates we deduce that the gradients of the imaginary parts also go to zero uniformly, which imply that the gradients of the real parts of $f - f_m$ go to 0 uniformly. This means $f - f_m$ converges locally uniformly to a constant, and we choose C to make this constant 0 i.e. $f_m \to f$ locally uniformly on $\mathbb{C}_+$.

Therefore, we have shown if there is some z_0 such that

$$\liminf_{n\to\infty} v_n(z_0) = M < \infty$$

then f_n converges locally uniformly along a sub-sequence to a holomorphic function on $\mathbb{C}_+$. But if the alternative does not hold i.e.

$$\lim_{n\to\infty} v_n(z_0) = \infty$$

for every z_0 then $|f_n| \to \infty$ locally uniformly thanks to Harnack's Inequality. □

Problem 9. Let $f : \mathbb{C} \to \mathbb{C}$ be entire and assume $|f(z)| = 1$ when $|z| = 1$. Show that $f = Cz^m$ for some $m \in \mathbb{N}$ and some $C \in \mathbb{C}$ with $|C| = 1$.

Proof. As f is entire such that $|f(z)| = 1$ on $\mathbb{D}$, there are only finitely many zeros on $\overline{\mathbb{D}}$ and as each zero is isolated, there is some $\varepsilon > 0$ such that there are no zeros on $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$. Enumerate the zeros as $\{z_n\}_{n=1}^N$ then define the Blaschke products

$$\psi_n(z) := \frac{z - z_n}{1 - \overline{z_n} z}$$

then $|\psi_n(z)| = 1$ with poles on $1/\overline{z_n}$ on $\partial\mathbb{D}$ and $\psi_n(z) = 0$ iff $z = z_n$. Then define

$$g(z) := f(z)/\prod_{n=1}^N \psi_n(z)$$

then $g(z)$ is a holomorphic function on the disk such that $|g(z)| = 1$ on $\partial\mathbb{D}$. Then observe that for any $|z| = 1$ we have

$$z = \frac{1}{\overline{z}} \Rightarrow g(z) = 1/(\overline{g(1/\overline{z})}) := h(z) \text{ on } \partial\mathbb{D}$$

and g and h extends to be continuous on $\partial\mathbb{D}$ since f is entire and the blashcke factors do not have poles on $\{1 - \varepsilon < |z| < 1 + \varepsilon\}$. And similarly AS $h(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{D}$ and extends continuously to ∂D. Therefore, by Schwarz Reflection Principle, we know that $g(z)$ extends to be entire with

$$g(z) := \begin{cases} g(z) \text{ for } z \in \overline{\mathbb{D}} \\ h(z) \text{ for } z \in \mathbb{C} \setminus \mathbb{D} \end{cases}$$

Notice that $g(z)$ is a bounded entire function since g and h are bounded, so

$$g(z) = C \Rightarrow f(z) = C \prod_{n=1}^N \psi_n(z)$$

where $|C| = 1$ where the equality is due to analytic continuation since $f(z) = C \prod_{n=1}^N \psi_n(z)$ on $\mathbb{D}$. However, notice that $\psi_n(z)$ is entire if and only if $z_n = 0$ i.e. $\psi_n(z) = z$, so f is entire iff

$$f(z) = Cz^m$$

where m is the multiplicity of the zeros at the origin.

□

Problem 10. Does there exist a function $f(z)$ holomorphic on $\mathbb{D}$ such that $\lim_{|z|\to 1} |f(z)| = \infty$. Either find one or prove that it does not exist.

Proof. No. Suppose for the sake of contradiction that such a function existed, then for each $w \in \partial\mathbb{D}$ we have

$$\lim_{z \to w} |f(z)| = \infty$$

so for any $M \in \mathbb{N}$ we can find an $\varepsilon(w) > 0$ such that on $z \in B_{\varepsilon(w)}(w) \cap \partial\mathbb{D}$ we have $|f(z)| \geq M$. By compactness, there are $\{w_i\}_{i=1}^N \subset \partial\mathbb{D}$ such that

$$\partial\mathbb{D} \subset \bigcup_{i=1}^{N} B_{\varepsilon(w)}(w) \cap \partial\mathbb{D} := \Gamma_1$$

Define $\Omega_1 := \mathbb{D} \setminus \overline{\Gamma_1}$ then we the only zeros of f are in Ω_1 by construction, so as each zero is isolated there are only finitely many zeros on Ω_1. Enumerate the zeros as $\{z_i\}_{i=1}^N$ and define the Blaschke products

$$\psi_n(z) := \frac{z - z_n}{1 - \overline{z_n} z}$$

then $\psi_n(z) = 0$ iff $z = z_n$ and $|\psi_n(z)| = 1$ on $\partial\mathbb{D}$. In particular, define

$$g(z) := f(z) / \prod_{n=1}^{N} \psi_n(z)$$

then g is a holomorphic function on $\mathbb{D}$ with no zeros on $\mathbb{D}$ such that $\lim_{|z| \to 1} |g(z)| = \infty$ since the g extend continuously to $\partial\mathbb{D}$. Then by repeating the above argument, we can find for any $M \in \mathbb{N}$ a open and connected subset Ω_M of $\mathbb{D}$ such that on $\partial\Omega_M$ we have $|g(z)| \geq M$. Then as $1/g(z)$ is holomorphic, we see that $|1/g(z)| \leq 1/M$ on $\partial\Omega_M$ so we see this implies $1/|g(z)| \leq 1/M$ on Ω_M thanks to the maximum principle. Notice also by construction $\Omega_M \to \mathbb{D}$ as $M \to \infty$, so we conclude that

$$1/|g(z)| = 0$$

i.e. $|g(z)| = \infty$ for all z, which implies $|f(z)| = \infty$ everywhere which is our contradiction. □

Problem 11. Assume that $f(z)$ is holomorphic on $|z| < 2$. Show that

$$\max_{|z|=1} \left| f(z) - \frac{1}{z} \right| \geq 1$$

Proof. Notice that the residue of $f(z) - 1/z$ at $z = 0$ is -1. So in particular, as $f(z) - 1/z$ is holomorphic on $B_2(0) \setminus \{0\}$ and the curve $\gamma := e^{i\theta}$ where $\theta \in [0, 2\pi)$ with counterclock wise orientation is in the interior of $B_2(0) \setminus \{0\}$, we can apply the Residue Theorem. This gives

$$2\pi i = \int_\gamma f(z) - 1/z dz$$

but we also have the estimate

$$\left| \int_\gamma f(z) - 1/z dz \right| \leq 2\pi M$$

where $M := \max_{|z|=1} |f(z) - 1/z| \geq 1$ which exist since both functions are continuous on the compact set $\{|z| = 1\}$. So combining this we conclude

$$M \geq 1$$

as desired.

□

Problem 12. Find a real valued harmonic function v defined on the disk $\mathbb{D}$ such that $v(z) > 0$ and $\lim_{z \to 1} v(z) = \infty$.

Let u be a real valued harmonic function on the disk $\mathbb{D}$ such that $u(z) < M \leq \infty$ and $\limsup_{r \to 1} u(re^{i\theta}) \leq 0$ for all $\theta \in (0, 2\pi)$. Show that $u(z) \leq 0$.

Proof. Notice that as $\mathbb{D}$ is simply connected and $f(z) := z - 1$ is a holomorphic function on $\mathbb{D}$ that vanishes nowhere, we can find a branch of the log such that $g(z) := \log(z-1)$ is holomorphic on $\mathbb{D}$. This then implies that the real pet i.e. $\log|z-1|$ is a harmonic function on $\mathbb{D}$. Then observe that

$$h(z) := \log(2) - \log|z-1| = \log(\frac{2}{|z-1|}) \geq 0$$

and is harmonic such that $\lim_{z\to 1} h(z) = \infty$.

Observe that the usual ε-log trick does not work here since we only have radial limits, so we instead proceed via Poisson Integral Formula. Indeed, observe if $0 < t < 1$ and $\varphi \in [0, 2\pi)$ then for any $t < r < 1$

$$u(te^{i\varphi}) = \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} u(re^{i\theta})d\theta$$

and observe that

$$\left|\frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} u(re^{i\theta})\right| \leq M\frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} := A(r,\theta)$$

so by Fatou's Lemma we have

$$\int_{\theta=0}^{2\pi} \liminf_{r\to 1}\left(A(r,\theta) - \frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} u(re^{i\theta})d\theta\right) \leq \liminf_{r\to 1}\int_{\theta=0}^{2\pi}\left(A(r,\theta) - \frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} u(re^{i\theta})d\theta\right)$$

and by DCT we have

$$\lim_{r\to 1}\int_{\theta=0}^{2\pi} A(r,\theta)d\theta = \int_{\theta=0}^{2\pi} A(1,\theta)d\theta$$

so

$$u(te^{i\varphi}) = \limsup_{r\to 1}\int_{\theta=0}^{2\pi} \frac{r^2 - t^2}{|re^{i\theta} - te^{i\varphi}|^2} u(re^{i\theta})d\theta \leq \int_{\theta=0}^{2\pi} \frac{1 - t^2}{|e^{i\theta} - te^{i\varphi}|^2} \limsup_{r\to 1}(u(re^{i\theta}))d\theta \leq 0$$

where we used $\limsup_{r\to 1} u(re^{i\theta}) \leq 0$ and $1 - t^2 > 0$.

□

14. Fall 2016

Problem 1. We consider the space $L^1(\mu)$ of integrable functions on a measure space $(X, \mathcal{M}, \mu)$. Suppose that f and f_n are functions in $L^1(\mu)$ such that

(1) $f_n(x) \to f(x)$ for μ almost every $x \in X$ and
(2) $||f_n||_1 \to ||f||_1$

Show that then $||f_n - f||_1 \to 0$.

Proof. Observe by the triangle inequality that $|f| + |f_n| - |f - f_n| \geq 0$ so we may apply Fatou's Lemma to see

$$\int_{\mathbb{R}} \liminf_{n\to\infty} |f| + |f_n| - |f - f_n| d\mu \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f| + |f_n| - |f - f_n|$$

by pointwise convergence the left hand side converges to $\int_{\mathbb{R}} 2|f|$ while the right hand side by norm convergence and linearity of the integral becomes $\int_{\mathbb{R}} 2|f|d\mu - \limsup_{n\to\infty} \int_{\mathbb{R}} |f - f_n|d\mu$ which lets us conclude that since $f \in L^1(\mu)$ that

$$0 \geq \limsup_{n\to\infty} \int_{\mathbb{R}} |f - f_n|d\mu$$

i.e. $||f_n - f||_1 \to 0$ as desired. □

Problem 2. Let μ be a finite positive Borel measure on $\mathbb{R}$ that is singular to the Lebesgue measure. Show that

$$\lim_{r\to 0+} \frac{\mu([x - r, x + r])}{2r} = +\infty$$

for μ a.e. $x \in \mathbb{R}$.

Proof. Write the Lebesgue measure as m then notice that $2r = m([x - r, x + r])$. As $\mu \perp m$ there exists a set A such that $\mu(A^c) = m(A) = 0$. So now define for $k \in \mathbb{N}$

$$F_k := \{x \in A : \limsup_{r\to 0+} \frac{\mu([x - r, x + r])}{m([x - r, x + r])} < k\}$$

then we claim that $\mu(F_k) = 0$ for any k, which implies that as

$$F := \{x \in A : \limsup_{r\to 0+} \frac{\mu([x - r, x + r])}{m([x - r, x + r])} < \infty\} = \bigcup_{k\in\mathbb{N}} F_k$$

that $\mu(F) = 0$ since $F_k \subset F_{k+1}$ and $\lim_{k\to\infty} \mu(F_k) = \mu(\bigcup_{k\in\mathbb{N}} F_k) = \mu(F)$. Therefore, $F \cup A^c$ is a null set for μ, which implies the desired claim.

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$ then by outer regularity of the Lebesgue measure there exists an open set U_ε such that $A \subset U_\varepsilon$ and $m(U_\varepsilon) < \varepsilon$. Then for any $x \in F$ we have an $r_x > 0$ such that

$$\mu([x - r_x, x + r_x]) < km([x - r_x, x + r_x])$$

Now this implies F_k is covered by a collection of balls, so by Vitali's covering lemma, we can find countably many balls with radii x_i and center r_i such that $x_i \in F_k$ and the above inequality is true and the sub collection of balls is disjoint and

$$F_k \subset \bigcup_{k\in\mathbb{N}} B_{5r_i}(x_i)$$

and by choosing the radii even smaller if necessary we can assume each $B_{r_i}(x_i) \subset U_\varepsilon$ so in particular,

$$\mu(F_k) \leq \sum_{k=1}^{\infty} \mu(B_{5r_i}(x_i)) \leq k \sum_{k=1}^{\infty} m([x_i - 5r_i, x_i + 5r_i]) = 5k \sum_{k=1}^{\infty} m([x_i - r_i, x + i + r_i])$$
$$\leq 5km(U_\varepsilon) \leq 5k\varepsilon \to 0$$

where for the final inequality we used the balls are disjoint, so we conclude $\mu(F_k) = 0$ for all k. Therefore, the desired claim holds.

□

Problem 3. If X is a compact metric space, we denote by $\mathcal{P}(X)$ to be the set of postivie Borel measures μ on X such that $\mu(X) = 1$.

(1) Let $\varphi : X \to [0, \infty]$ be a lower-semi continuous function on a compact metric space X. Show that if μ and μ_n are in $\mathcal{P}(X)$ and $u_n \to \mu$ with respect to the weak-star topology on $\mathcal{P}(X)$, then
$$\int \varphi d\mu \leq \liminf_{n\to\infty} \int \varphi d\mu_n$$
(2) Let $K \subset \mathbb{R}^d$ be a compact set. For $\mu \in \mathcal{P}(K)$, we define
$$E(\mu) = \int_K \int_K \frac{1}{|x-y|} d\mu(x) d\mu(y)$$
Show that $E : \mathcal{P}(K) \to [0, \infty]$ attains its minimum on $\mathcal{P}(K)$ (which could possibly be ∞).

Proof. By Risez-Representation Theorem, we know that the dual space of the $C(X)$ (since X is compact we know all continuous functions are bounded) is the space of Radon Measures. So $\mu_n \to \mu$ in the weak star topology iff for every $f \in C(X)$ we have
$$\int f d\mu_n \to \int f d\mu$$
As φ is lower semi-continuous and bounded from below there exists a sequence of $\varphi_n \in C(X)$ such that $\varphi_n \leq \varphi$ and $\varphi_n \to \varphi$. We observe that as $\varphi - \varphi_n \geq 0$ that
$$\int \varphi d\mu_n \geq \int \varphi_m d\mu_n$$
so we have from weak* convergence
$$\liminf_{n\to\infty} \int \varphi d\mu_n \geq \liminf_{n\to\infty} \int \varphi_m d\mu_n = \int \varphi_m d\mu$$
and this holds for all m and from the Monotone Convergence Theorem (since φ_1 is bounded below since it is continuous and we are on a compact space) we have that
$$\int \varphi d\mu = \lim_{m\to\infty} \int \varphi_m d\mu$$
so it follows that
$$\liminf_{n\to\infty} \int \varphi d\mu_n \geq \int \varphi d\mu$$
For the second part assume the minimum isn't $+\infty$, otherwise every measure attains the minimum. Let $\{\mu_n\}$ be a minimizing sequence of $E(\mu)$ i.e. $E(\mu_n) \to \inf_{\mu \in \mathcal{P}(K)} E(\mu)$. By Banach-Alagou combined with $C(K)$ being separable , as μ_n are probability measures, we know that there is a weak* convergent subsequence say μ_{n_k} that converges to μ (since the weak* topology on $C(K)$ is metrizable sicne $C(K)$ is separable). μ is a Borel Probability Measure since 1 is continuous so $1 = \mu_n(X) \to \mu(X)$. By Stone-Weiestrass, functions of the form $f(x)g(y)$ are dense in $C(X \times X)$ $f \in C(X)$ and $g \in C(X)$. This implies that $\mu_{n_k} \otimes \mu_{n_k} \to \mu \otimes \mu$ in the weak* topology [Fubini is justified since μ is a probability measure and $f \in C(X)$ is bounded so $f \in L^1(X)$]. Notice that $1/|x-y|$ is lower semi-continuous, so by part (1) we know that
$$E(\mu) \leq \liminf_{n\to\infty} \int E(\mu_n) = \inf_{\nu \in \mathcal{P}(K)} E(\nu)$$
and since $\mu \in \mathcal{P}(K)$ we obtain that it is a minimizer.

□

Problem 4. Let $L^1 = L^1([0,1])$ and $L^2 = L^2([0,1])$. Show that L^2 is a meager subset of L^1.

Proof. First observe that

$$L^2 = \bigcup_{n=1}^{\infty} \{f \in L^2 \cap L^1 : \int_0^1 |f(x)|dx \leq n \text{ and } \int_0^1 |f(x)|^2 dx \leq n\} = \bigcup_{n=1}^{\infty} F_n$$

We claim that F_n is closed in L^1. Indeed, if $f_n \in F_n$ converge to $f \in L^1([0,1])$ then we have that along a subsequence we have $f_{n_k} \to f$ pointwise a.e. so by Fatou's Lemma

$$\int_0^1 |f(x)|^2 = \int_0^1 \liminf_{k\to\infty} |f_{n_k}(x)|^2 \leq \liminf_{k\to\infty} \int |f_{n_k}(x)|^2 \leq n$$

so $f \in F_n$ i.e. F_n is closed.

Now we claim F_n has empty interior, indeed fix $f \in F_n$. Then for $\varepsilon > 0$ consider

$$g_\varepsilon(x) := f(x) - \frac{\varepsilon}{2\sqrt{x}}$$

then we have that

$$\int_0^1 |f(x) - g_\varepsilon(x)|dx = \frac{\varepsilon}{2}\int_0^1 x^{-1/2}dx = \varepsilon$$

and $g_\varepsilon(x) \notin F_n$ since $|g_\varepsilon(x)|^2 = |f|^2 - \varepsilon x^{-1/2} + \varepsilon f^2/4x^{-1}$ and the first term is in L^1 by definition of F_n, the second term is in L^1 by Holder's Inequality, while the second term is not in L^1. In particular,

$$\int_0^1 |g_\varepsilon(x)|^2 dx = +\infty$$

since the first two terms are bounded and the last term is unbounded in L^1. Therefore, F_n has empty interior in L^1 since for any $f \in F_n$ we constructed a sequence $g_{1/n} \to f$ in L^1 and $g_{1/n} \in F_n^c$, so L^2 is meager. ☐

Problem 5. Let $X = C([0,1])$ be equipped with the norm $||f|| = \max_{x\in[0,1]} |f(x)|$. Let $\mathcal{A}$ be the borel σ-algebra on X. Show that $\mathcal{A}$ is the smallest σ-algebra on X that contains sets of the form

$$S(t,B) = \{f \in X : f(t) \in B\}$$

where $t \in [0,1]$ and B is a borel subset of $\mathbb{R}$.

Proof. Define the evaluation map for $t \in [0,1]$ via

$$\phi_t(f) = f(t)$$

then ϕ_t is continuous linear function on X since

$$|\phi_t(f)| = |f(t)| \leq ||f||$$

and observe $S(t,B) = \phi^{-1}(B)$ so by continuity, $S(t,B)$ is a Borel Subset of X i.e. $\mathcal{S} \subset \mathcal{A}$ where $\mathcal{S}$ is the σ-algebra generated by $S(t,B)$.

For the reverse inclusion observe that if $r > 0$ and $f \in X$ then

$$\{g : ||f-g|| \leq r\} = \bigcap_{q\in\mathbb{Q}\cap[0,1]} \phi_q^{-1}([f(t)-r, f(t)+r]) \in \mathcal{S}$$

since $g \in \bigcap_{q\in\mathbb{Q}\cap[0,1]} \phi_q^{-1}((f(t)-r, f(t)+r))$ iff for every q rational $|f(q)-g(q)| \leq r$, which by continuity implies $||f-g|| \leq r$. Then observe

$$B_r(f) = \bigcup_{m\in\mathbb{Q}:m<r} \bigcap_{q\in\mathbb{Q}\cap[0,1]} \phi_q^{-1}([f(t)-m, f(t)+m]) \in \mathcal{S}$$

since $g \in B_r(f)$ iff $||g-f|| < r$ so there is some rational q such that $||g-f|| \leq q < r$. So in particular, as $\mathcal{A}$ is the smallest σ-algebra that contains open sets of X we see that $\mathcal{A} \subset \mathcal{S}$ so combining all of this we obtain $\mathcal{S} = \mathcal{A}$ as desired.

$\square$

Problem 6. Show that there is no sequence $\{u_n\} \in \ell^1$ such that (i) $||u_n||_1 \geq 1$ for all $n \in \mathbb{N}$ and (ii) $\langle u_n, v\rangle \to 0$ for all $v \in \ell^\infty$.

Also show that every weakly convergent sequence in $\{u_n\}$ in ℓ^1 converges in the norm topology of ℓ^1.

Proof. **First Part Missing**

For the second part, assume $u_n \rightharpoonup u$. Assume for the sake of contradiction that there existed a sub-sequence and $\varepsilon > 0$ such that

$$||u_{n_k} - u||_1 > \varepsilon$$

then we know that

$$\langle \frac{1}{\varepsilon} u_{n_k} - u, v\rangle \to 0$$

since u_n weakly converges to 0, but $||2/\varepsilon(u_{n_k} - u)|| \geq 1$ for large enough k, which contradicts the first part, so $||u_n - v|| \to 0$ i.e. we have strong convergence to zero.

$\square$

Problem 7. Let $\mathcal{H}$ be the space of holomorphic functions on the unit disk that are in $L^2(\mathbb{D})$ with respect to the Lebesgue measure on $\mathbb{D}$. Endow $\mathcal{H}$ with the inner product

$$(f, g) = \int_{\mathbb{D}} f(z)\overline{g(z)}dA$$

Fix $z_0 \in \mathbb{D}$ and define $L_{z_0}(f) = f(z_0)$ for $f \in \mathcal{H}$.

(1) Show that $L_{z_0} : \mathcal{H} \to \mathbb{C}$ is a bounded linear functional on $\mathcal{H}$.
(2) Find an explicit $g_{z_0} \in \mathcal{H}$ such that

$$L_{z_0}(f) = f(z_0) = \langle f, g_{z_0}\rangle$$

for all $f \in \mathcal{H}$.

Proof. As $z_0 \in \mathbb{D}$ there is a $\delta > 0$ such that $\overline{B_\delta(z_0)} \subset \mathbb{D}$. Then for any $0 < r < \delta$ notice that by the mean value property that

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

so

$$\int_{r=0}^{\delta} rf(z_0) = \int_{r=0}^{\delta} r\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$
$$= \frac{1}{2\pi}\int_{r=0}^{\delta}\int_{\theta=0}^{2\pi} rf(z_0 + re^{i\theta})d\theta = \frac{1}{2\pi}\int_{B_\delta(z_0)} f(z)dA(z)$$

where all the computation is justified since $f \in L^2(\mathbb{D})$ which allows us to use Fubini since $\mathbb{D}$ is a finite measure space (i.e. $L^2(\mathbb{D}) \subset L^1(\mathbb{D})$). So in particular, we conclude that

$$f(z_0) = \frac{1}{\pi\delta^2}\int_{B_\delta(z_0)} f(z)dA(z)$$

so we have

$$|L_{z_0}(f)| = |f(z_0)| \leq \frac{1}{\pi\delta^2}\int_{\mathbb{D}} |f(z)|dA(z) \leq \frac{1}{\sqrt{\pi}\delta^2}\left(\int_{\mathbb{D}} |f(z)|^2\right)^{1/2} dA(z)$$

where the last inequality is due to Holder. So in particular, we conclude that $L_{z_0}(f)$ is a bounded linear functional on $\mathcal{H}$.
We know such a function exists by Risez Representation Theorem. Note that by our previous computation we have that if $z_0 = 0$ that for any $0 < \delta < 1$ that

$$L_0(f) = f(0) = \frac{1}{\pi\delta^2}\int_{B_\delta(0)} f(z)dA(z)$$

and since $\frac{1}{\delta} f\chi_{B_\delta(0)} \to f\chi_{B_1(0)}$ as $\delta \to 0$ and is dominated by $2f\chi_{B_\delta(0)}$ for $1/2 < \delta < 1$ we may apply the DCT to get that

$$L_0(f) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)dA(z) = (f, 1/\pi)$$

Now we observe that if we define the automorphism of the disk

$$\phi_{z_0}(z) := \frac{z_0 - z}{1 - \overline{z_0}z}$$

then $\phi_{z_0}(0) = z_0$ and $\phi_{z_0}(z_0) = 0$ so we have

$$f \circ \phi_{z_0}(0) = f(z_0)$$

so we have

$$L_{z_0}(f) = L_0(f \circ \phi_{z_0}) = \frac{1}{\pi}\int_{\mathbb{D}} f \circ \phi_{z_0}(z)dA(z)$$

So we know that by integration by substitution

$$= \frac{1}{\pi}\int_{\mathbb{D}} f(z)|(\phi_{z_0}^{-1})'|^2 dA(z) = (f\psi, \psi/\pi)$$

where $\psi = (\phi_{z_0}^{-1})'$. Therefore, we have shown that

$$f(z_0)\psi(z_0) = (f\psi, \overline{\psi(z_0)}\psi/\pi)$$

And as ψ is the derivative of a conformal map we know that $\psi \neq 0$ anywhere, so we have for any $F \in \mathcal{H}$ that

$$F(z_0) = F(z_0)/\psi(z_0)\psi(z_0) = ((F/\psi)\psi, \overline{\psi(z_0)}/\pi\psi) = (F, \frac{\overline{\psi}(z_0)}{\pi}\psi)$$

and $\frac{\overline{\psi}(z_0)}{\pi}\psi \in \mathcal{H}$, so this is the desired function. □

Problem 8. Let f be a continuous complex-valued function on the closed unit disk $\overline{\mathbb{D}}$ and f holomorphic on the open unit disk and $f(0) \neq 0$.

(1) Show that if $0 < r < 1$ and if $\inf_{|z|=r} |f(z)| > 0$, then

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \geq \log|f(0)|$$

(2) Show that $m(\{\theta \in [0, 2\pi] : f(e^{i\theta}) = 0)\}) = 0$.

Proof. Note that as $f(z)$ is holomorphic then whenever $f(z) \neq 0$ then in a small ball we know that $f(z) \neq 0$ since 0s are isolated, so whenever $f(z) \neq 0$ we can define a complex log in a small neighborhood of z to get that $\log|f(z)|$ is locally the real part of the holomorphic function $\log(f(z))$ whenever $z \neq 0$. Then if $f(z) = 0$ then $\log|f(re^{i\theta})| = -\infty$, so it follows that $\log|f(z)|$ is sub-harmonic since it is upper semi-continuous and locally satisfies the mean value inequality. So in particular,

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \geq \log|f(0)|$$

as desired.

For the second part, fix an $n \in \mathbb{N}$ and define $g_n(z) := \max(\log|f(z)|, -n)$. Then as $f(z)$ is continuous on $\overline{\mathbb{D}}$ we see that $g_n(z)$ is continuous on $\overline{\mathbb{D}}$ since

$$g_n(z) = \begin{cases} \log|f(z)| \text{ for } z \in |f|^{-1}(e^{-n}, \infty) \\ -n \text{ else} \end{cases}$$

and $\log|f(z)|$ is continuous on $|f|^{-1}(e^{-n}, \infty)$ since f is and they obtain the same value on the boundary. Therefore, from continuity and Fatou's Lemma we have

$$\frac{1}{2\pi}\int_0^{2\pi} |g_n(e^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \liminf_{r\to 1} |g_n(re^{i\theta}))|$$

$$\leq \liminf_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |g_n(re^{i\theta})|d\theta$$

where we used $g_n(e^{i\theta}) = \liminf_{r\to 1} g_n(re^{i\theta})$ by continuity. Now recall that from part 1 that

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta \geq \log|f(0)|$$

and since f is continuous then f is bounded, so $\log|f|$ is bounded from above, so this implies

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|f(re^{i\theta})||d\theta \leq C$$

where C is independent of r. And observe that $|g_n(re^{i\theta})| \leq |\log|f(re^{i\theta})||$ so we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |g_n(e^{i\theta})|d\theta \leq C$$

and since $|g_n(e^{\imath\theta})|$ increases to $|\log|f(e^{i\theta})||$ we conclude from the monotone convergence theorem that

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|(f(e^{i\theta}))|| \leq C$$

so $\log|(f(e^{i\theta}))| \in L^1([0, 2\pi])$ which implies the problem.

Remark: These inequalities can also be proved using that $|\log|f(z)||$ is lower semi-continuous combined with Fatou's Lemma. □

Problem 9. Let μ be a positive Borel measure on $[0, 1]$ with $\mu([0, 1]) = 1$.

(1) Show that the function f defined as

$$f(z) = \int_{[0,1]} e^{izt} d\mu(t)$$

for $z \in \mathbb{C}$ is holomorphic on $\mathbb{C}$.

(2) Suppose that there is an $n \in \mathbb{N}$ such that

$$\limsup_{|z|\to\infty} |f(z)|/|z|^n < \infty$$

Show that then μ is equal to the Dirac measure δ_0 at 0.

Proof. Note that on any compact set $K \subset \mathbb{C}$ that on $K \times [0, 1]$ we have that $|e^{izt}|$ is bounded due to continuity. Say on $K \times [0, 1]$ we have that $|e^{izt}| \leq M$ then by Tonelli since all the terms are non-negative we have

$$\int_K |f(z)|dA(z) \leq \int_K \int_{[0,1]} |e^{izt}|d\mu(t)dA(z) = \int_{K\times[0,1]} |e^{izt}|d\mu(t) \otimes dA(z)$$

$$\leq \int_{K\times[0,1]} M d\mu(t) \otimes dA(z) = MA(K) < \infty$$

where dA is the Lebesgue Area Measure on $\mathbb{R}^2 \cong \mathbb{C}$ and $A(K)$ is the Lebesgue measure of K. So $f(z) \in L^1_{loc}(\mathbb{C}, dA)$. So if R is a rectangle, then we know $f(z) \in L^1(R)$ so we have by Fubini that

$$\int_{\partial R} f(z)dz = \int_{[0,1]} \int_{\partial R} e^{izt} dz d\mu(t) = 0$$

since for any fixed t we have e^{izt} is holomorphic on the interior of R and its boundary. Also $f(z)$ is continuous due to DCT, so by Morrera's Theorem, we deduce that $f(z)$ is entire.

For any $m \geq n$, we know by Cauchy's Estimates that for large enough m that

$$|f^m(0)| \leq \frac{m!}{R^m} \sup_{z \in B_R(0)} |f(z)| \leq m! R^{n-M} \to 0$$

so it follows that $f(z)$ is a polynomial of degree at most n. Now we assume for the sake of contradiction that μ is not any scalar multiple of the Dirac measure at $z = 0$. Indeed, if not then there is a set $0 \notin A$ and that $\mu(A) > 0$. As $[0, 1]$ is compact and μ is finite, we know that it is inner regular, so $\mu(A) = \sup_{K \subset A} \mu(K)$ where K is compact, so we can find a compact K such that $0 \notin K$ and $\mu(K) > 0$. As K is compact there exists a min $0 < s \in K$. We will show that $f(z)$ blows up on the negative imaginary axis exponentially which contradicts that $f(z)$ is a polynomial. . Indeed, observe that for any $x \in (0, \infty)$

$$f(-ix) = \int_{[0,1]} e^{xt} d\mu(t) \geq \int_K e^{xt} d\mu(t) \geq e^{xs} \mu(K)$$

and $\mu(K) \neq 0$, so this implies on the negative real axis, that t blows up exponentially i.e. f cannot be a polynomial. Therefore, this is a contradiction so μ is a scalar multiple of the dirac measure δ_0 but from $\mu([0, 1]) = 1$ we see the scalar must be one. □

Problem 10. Consider the quadartic polynomial $f(z) = z^2 - 1$ on $\mathbb{C}$. We are interested in the iterates f^n of f defined to be $f^0 = id_{\mathbb{C}}$ for $n = 0$ and

$$f^n = f \circ .... \circ f \text{ n times}$$

(1) Find an explicit $M > 0$ such that the following dichotomy holds for each $z \in \mathbb{C}$: either (i) $|f^n(z)| \to \infty$ as $n \to \infty$ or (ii) $|f^n(z)| \leq M$ for all $n \in \mathbb{N}_0$.

(2) Let U be the set of all $z \in \mathbb{C}$ for which the first alternative (i) holds and K be the set of all $z \in \mathbb{C}$ for which the second alternative (ii) holds.
Show that U is an open set and K is a compact set without holes, i.e. $\mathbb{C} \setminus K$ has no bounded connected components.

Proof. For the first part, observe that

$$|f(z)| \geq |z|^2 - 1$$

so now we claim if $|f(z)| \geq M := 10$ then $|f^n(z)| \geq 10$ for all n. Indeed, $|f^2(z)| \geq |f(z)|^2 - 1 \geq 10^2 - 1 \geq 99$. Now by induction, assume it holds for m then $|f^{m+1}(z)| \geq |f^m(z)|^2 - 1 \geq 10^2 - 1 = 99$ as desired. It therefore,follows that for any m if $|f^m(z)| \geq 10$ then

$$|f^m(z)| \geq |f^{m-1}(z)|^2 - 1 \geq \frac{|f^{m-1}(z)|^2}{2}$$

so now we claim that if $|f^m(z)| \geq 10$ then

$$|f^{m+n}(z)| \geq \frac{|f^{m-1}(z)|^{2^n}}{2^{2^n - 1}}$$

Indeed, by induction, the base case is true so

$$|f^{m+n+1}(z)| \geq \frac{|f^{m+n}|^2}{2} \geq \frac{1}{2} |f^{m-1}(z)|^{(2^{n+1})} / 2^{2^{n+1}-2} = |f^{m-1}(z)|^{2^{n+1}} / 2^{2^{n+1}-1}$$

as desired. So if $|f^m(z)| \geq 10$ for some m then

$$|f^{m+n}(z)| \geq \frac{1}{2}\frac{(10)^{2^n}}{2^{2^n}} = \frac{1}{2}5^{2^n} \to \infty$$

so we have found such an M.

For the second part, notice that U is open since

$$U = \bigcup_{m \in \mathbb{N}_0} \{z : |f^m(z)| > M\}$$

and each set in its union is the preimage of a continuous function on an open set, so U is open. Notice that K is compact since

$$K = \bigcap_{m \in \mathbb{N}_0} \{z : |f^m(z)| \leq M\}$$

and each set is closed while the $m = 0$ set is bounded and closed, so the entire set is bounded and closed i.e. compact.

Assume for the sake of contradiction that $\mathbb{C} \setminus K$ has a bounded connected component. Write this component as S.Then we from $\mathbb{C} = K \cup (\mathbb{C} \setminus K)$ and S is a bounded connected component of $\mathbb{C} \setminus K$ that on ∂S we have for any m that $|f^m(z)| \leq M$, so by the maximum modulus principle we have $|f^m(z)| \leq M$ on S. However, as $S \subset U$ we have $|f^m(z)| \to \infty$, which is our desired contradiction.

□

Problem 11. Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic function such that $z \mapsto g(z) := f(z)f(1/z)$ is bounded on $\mathbb{C} \setminus \{0\}$.

(1) Show that if $f(0) \neq 0$, then f is constant.
(2) Show that if $f(0) = 0$, then there exists an $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $f(z) = az^n$ for all $z \in \mathbb{C}$.

Proof. Note that $g(z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and $f(z)$ is entire and $f(1/z)$ is holomorphic on $\mathbb{C} \setminus \{0\}$. As $g(z)$ is bounded near 0, we know by Riemann's Removable Singularity theorem that $g(z)$ extends to be an entire function on $\mathbb{C}$. It therefore, follows that $g(z)$ is a constant by Liouville theorem since $g(z)$ is a bounded entire function. So we have an $a \in \mathbb{C}$ such that

$$f(z)f(1/z) = a$$

As $f(0) \neq 0$ that

$$\lim_{z \to 0} f(1/z) = a/f(0)$$

so it follows that the singularity of $f(z)$ at ∞ is removable. Therefore, $f(z)$ has to be a constant since non-polynomials have an essential singularity at ∞ by just looking at the $f(1/z)$ Taylor Expansion at $z = 0$ and all non-constant polynomials have a pole at ∞.

Let $n \in \mathbb{N}$ be the multiplicity of the zero at $z = 0$ of $f(z)$. Then $f(z) = z^n g(z)$ where g and $g(0) \neq 0$ is entire. Therefore,

$$f(z)f(1/z) = g(z)g(1/z)$$

and $g(z)g(1/z)$ is bounded on $\mathbb{C} \setminus \{0\}$ with $g(0) \neq 0$, so we may reapply the first part to deduce $g(z) = a$ for some $a \in \mathbb{N}$ i.e. $f(z) = az^n$. □

15. Spring 2017

Problem 1. Let $K \subset \mathbb{R}$ be a compact set of finite measure and let $f \in L^\infty(\mathbb{R})$. Show that the function
$$F(x) := \frac{1}{m(K)} \int_K f(x+t)dt$$
is uniformly continuous on $\mathbb{R}$. Here $m(K)$ denotes the Lebesgue measure of K.

Proof. Fix $h \in \mathbb{R}$ then observe
$$F(x+h) = \frac{1}{m(K)} \int_K f(x+h+y)dy = \frac{1}{m(K)} \int_{K+h} f(x+t)dt$$
where $K + h := \{x + h : x \in K\}$. Therefore,
$$F(x+h) - F(x) = \frac{1}{m(K)} \int_{\mathbb{R}} f(x+y)\left(\chi_{K+h} - \chi_K\right)$$
so we have from the triangle inequality that
$$|F(x+h) - F(x)| \leq \frac{||f||_{L^\infty}}{m(K)} \int_{\mathbb{R}} |\chi_{K+h} - \chi_K| dx$$
Now observe that $\chi_{K+h} \to \chi_K$ pointwise a.e. as $h \to 0$. So as K is compact there is an $R > 0$ such that $K \subset B_R(0)$, so now observe for h small enough that $\chi_{K+h} \leq \chi_{2B_R(0)}$ so by the dominated convergence theorem
$$\int_{\mathbb{R}} |\chi_{K+h} - \chi_K| dx = m((K+h)\Delta K) \to 0 \text{ as } h \to 0$$
so we deduce that $F(x)$ is uniformly continuous since our above bounds are independent of x.

□

Problem 2. Let $f_n : [0,1] \to [0,\infty)$ be a sequence of functions, each of which is non-decreasing on $[0,1]$. Suppose that f_n is uniformly bounded in $L^2(\mathbb{R})$. Show that there exists a subsequence that converges in $L^2(\mathbb{R})$.

Proof. Observe that by assumption there is an $M > 0$ such that
$$\int_0^1 |f_n|^2 \leq M \Rightarrow (1-x)|f_n(x)|^2 \leq \int_x^1 |f_n|^2 \leq M$$
where the last implication is due to f_n being non-decreasing. So in particular, we deduce
$$|f_n(x)| \leq \frac{M}{\sqrt{1-x}} \in L^1([0,1])$$
so by the dominated convergence theorem it suffices to show that along a sub-sequence f_n converges pointwise a.e. Now observe for any $\varepsilon > 0$ that for $x \in [0, 1-\varepsilon]$ that the sequence $|f_n(x)|$ is uniformly bounded. Therefore, by doing a diagonal subsequence argument we know that there is a subsequence which we denote by n_k such that for $q \in \mathbb{Q} \cap [0,1)$ that
$$f_{n_k}(q) \to f(q)$$
Observe that if $q < q'$ then $f(q) \leq f(q')$ since f_{n_k} are non-decreasing.

Now for any $x \in [0,1)$ define
$$L_x := \inf_{q \leq x} f(q), R_x := \sup_{q \geq x} f(q), q \in \mathbb{Q} \cap [0,1)$$
Then observe that for $q_1 \leq x \leq q_2$ where $q_i \in \mathbb{Q} \cap [0,1)$ that
$$f_{n_k}(q_1) \leq f_{n_k}(x) \leq f_{n_k}(q_2)$$

so if $L_x = R_x$ we deduce from squeeze theorem that $f(x) := \lim_{k\to\infty} f_{n_k}(x)$ is well defined. So if $L_x \neq R_x$ we see that from non-decreasing of f_n that we have an interval (L_x, R_x) and each such interval is disjoint thanks to non-decreasing of f_n. There can only be countably many of these intervals since each of these intervals contain a rational point, therefore $L_x = R_x$ a.e., so $f_{n_k}(x) \to f(x) := L_x$ a.e., so we can find a pointwise sub-sequence on $[0, 1)$ by a diagonalization trick. Then by applying DCT on this subsequence on $[0, 1 - \varepsilon]$ for all $\varepsilon > 0$ shows the desired result.

□

Problem 3. Let $C([0, 1])$ denote the Banach Space of Continuous Functions on the interval $[0, 1]$ endowed with the sup-norm. Let $\mathcal{F}$ be a σ-algebra on $C([0, 1])$ such that the map defined by

$$L_x(f) := f(x)$$

is $\mathcal{F}$-measurable. Show that $\mathcal{F}$ contains all the open sets.

Proof. Observe that if $\varepsilon > 0$ and $f \in C([0, 1])$ then

$$B_\varepsilon(f) = \bigcup_{q\in\mathbb{Q}:q<\varepsilon} \{g \in C([0, 1]) : ||f - g||_\infty \leq q\}$$

and that

$$\{g \in C([0, 1]) : ||f - g||_\infty \leq \delta\} = \bigcap_{q\in\mathbb{Q}} L_q([f(q) - \delta, f(q) + \delta])$$

Indeed, observe that $g \in \bigcap_{q\in\mathbb{Q}} L_q((f(q) - \delta, f(q) + \delta))$ iff for all $q \in \mathbb{Q}$ we have

$$|g(q) - f(q)| \leq \delta \Rightarrow |g(x) - f(x)| \leq \delta$$

by density of rationals and continuity. Therefore, by definition as $L_q((f(q)-\delta, f(q)+\delta))$ is $\mathcal{F}$-measurable since closed intervals are borel in $\mathbb{R}$ so we deduce that $\{g \in C([0, 1]) : ||f - g||_\infty \leq \delta\} := A(\delta, f) \in \mathcal{F}$. So in particular, we have

$$B_\varepsilon(f) = \bigcup_{q\in\mathbb{Q}:q<\varepsilon} A(q, f) \in \mathcal{F}$$

Now as $C([0, 1])$ is separable (take polynomials by Stone Weiestrass), we deduce that every open set can be written as a countable union of balls, so every open set is in $\mathcal{F}$.

□

Problem 4. For $n \geq 1$, let $a_n : [0, 1) \to \{0, 1\}$ denote the nth digit in the binary expansion of x, so that

$$x = \sum_{n\geq 1} a_n(x)2^{-n} \text{ for all } x \in [0, 1)$$

(We remove any ambiguity from this definition by requiring that $\liminf a_n(x) = 0$ for all $x \in [0, 1)$.) Let $M([0, 1))$ denote the Banach space of finite complex Borel measures on $[0, 1)$ and define linear functionals L_n on $M([0, 1])$ via

$$L_n(\mu) := \int_0^1 a_n(x)d\mu(x)$$

Show that no subsequence of L_n converges in the weak* topology on $M([0, 1))^*$

Proof. Fix a subsequence n_k. Then define

$$x := \sum_{k \text{ even}} 2^{-n_k} \in [0, 1)$$

so we have

$$x = \sum_{n\geq 1} a_n(x)2^{-n}$$

where $a_n(x) = 0$ for $n \neq n_k$ for some k where k is even. Therefore, $\liminf_{n\to\infty} a_n(x) = 0$ for instance take k odd. So this is the digit binary expansion of x then define the Dirac Delta Measure at x δ_x i.e. $\delta_x(E) = 1$ if $x \in E$ and 0 else. Then

$$L_{n_k}(\delta_x) = a_{n_k}(x) = \begin{cases} 1 \text{ if } k \text{ is even} \\ 0 \text{ if } k \text{ is odd} \end{cases}$$

Therefore, L_{n_k} does not converge weak* in this subsequence and as this subsequence was arbitrary, we are done.

□

Problem 5. Let $d\mu$ be a finite complex Borel measure on $[0, 1]$ such that

$$\hat{\mu}(n) := \int_0^1 e^{2\pi inx} d\mu(x) \to 0 \text{ as } n \to \infty$$

Let $d\nu$ be a finite complex Borel measure on $[0, 1]$ that is absolutely continuous w.r.t. $d\mu$. Show that

$$\hat{\nu}(n) \to 0 \text{ as } n \to \infty$$

Proof. Notice by absolute continuity there is a $f \in L^1(d\mu)$ such that

$$\hat{\nu}(n) = \int_0^1 e^{2\pi inx} d\nu(x) = \int_0^1 e^{2\pi inx} f(x) d\mu(x)$$

Now observe that

$$\hat{\mu}(-n) = \int_0^1 \cos(2\pi nx) - i\sin(2\pi nx) d\mu(x) \to 0$$

where the convergence is due to $\hat{\mu}(n)$ converges to 0 so its real and imaginary parts converge to 0 i.e. $\int_0^1 \cos(2\pi nx)$ and $\int_0^1 \sin(2\pi nx)$ converge to 0 as $n \to \infty$.

Now by Stone Weiestrass, we know that trigonometric polynomials are dense in $C([0, 1])$ with the sup topology. Now observe that for $m \in \mathbb{Z}$

$$\int_0^1 e^{2\pi imx} e^{2\pi inx} d\mu(x) \to 0 \text{ as } n \to \infty$$

so by linearity we deduce that for each trignometirc polynomial $P(x)$ that

$$\int_0^1 P(x) e^{2\pi inx} d\mu(x) \to 0$$

Then as μ is a finite complex Borel measure on a compact set we know that it is regular, therefore, by Lusin's theorem $C([0, 1])$ is a dense subclass of $L^1([0, 1], d\mu)$ in the L^1 norm. So we have that we can find a trig polynomial $P(x)$ such that $||P - f||_{L^\infty([0,1])} \leq \varepsilon$ where $\varepsilon > 0$ is a fixed number. Then

$$|\hat{\nu}(n)| \leq \left| \int_0^1 (f(x) - P(x)) e^{2\pi inx} d\mu(x) \right| + \left| \int_0^1 P(x) e^{2\pi inx} d\mu(x) \right|$$

$$\leq \int_0^1 |f(x) - P(x)| d\mu(x) + \left| \int_0^1 P(x) e^{2\pi inx} d\mu(x) \right| \leq \varepsilon\mu([0, 1]) + \left| \int_0^1 P(x) e^{2\pi inx} d\mu(x) \right|$$

so we deduce as $\varepsilon > 0$ was arbitrary and μ is a finite measure that

$$|\hat{\nu}(n)| \to 0 \text{ as } n \to \infty$$

as desired. □

Problem 6. Let $\overline{\mathbb{D}}$ be the closed unit disc in the complex plane, let $\{p_n\}$ be distinct points in the open disc $\mathbb{D}$ and let $r_n > 0$ be such that $D_n = \{z : |z - p_n| \leq r_n\}$ satisfy

(1) $D_n \subset \mathbb{D}$;
(2) $D_n \cap D_m =$ if $n \neq m$
(3) $\sum r_n < \infty$

Prove $X = \overline{\mathbb{D}} \setminus \bigcup_n D_n$ has positive area.

Hint: For $-1 < x < 1$ consider $\#\{n : D_n \cap \{\text{Re}(z) = x\}\}$

Proof. Let $\pi(x, y) = x$ i.e. the projection map onto the x coordinate. Then we have that for $f(x) := \sum_n \chi_{\pi(D_n)}(x)$ that

$$\int_{-1}^{1} f(x)dx = \sum_n \int_{-1}^{1} \chi_\pi(D_n)dx = \sum_n 2r_n < \infty$$

where the second equality is due to the monotone convergence theorem. Now this implies $f(x) < \infty$ a.e. so if $f(x) < \infty$ then we must have

$$\#\{n : D_n \cap \text{Re}(z) = x\} < \infty$$

Then observe that $\bigcup_n (D_n \cap \{\text{Re}(z) = x\})$ is closed since this is a finite union of closed sets due to our earlier remark, so it cannot be all of $\mathbb{D} \cap \{\text{Re}(z) = x\}$. Indeed this follows from $\mathbb{D} \cap \{\text{Re}(z) = x\}$ being open. Now observe that for

$$E_x := (\mathbb{D} \setminus \bigcup_n D_n) \cap \{\text{Re}(z) = x\}\}$$

we have that

$$E_x \cup \bigcup_n (D_n \cap \{\text{Re}(z) = x\}) = \mathbb{D} \cap \{\text{Re}(z) = x\}$$

where the union is disjoint because each D_n is disjoint, so by countable additivity of the one dimensional Lebesgue measure m_1 we deduce $m_1(E_x) > 0$ for a.e. x since the second set is closed and their union is open. Now observe if m_2 is the two dimensional Lebesgue meausre then

$$m_2(X) = \int_{\mathbb{D}} \chi_X(x, y)dA(x, y) = \int_{x=-1}^{1} \int_{y=-\sqrt{1+x^2}}^{\sqrt{1+x^2}} \chi_{\overline{\mathbb{D}} \setminus (\bigcup_n D_n) \cap \text{Re}(z) = x}(y)dydx$$

$$= \int_{x=-1}^{1} \int_{y=-\sqrt{1+x^2}}^{\sqrt{1+x^2}} \chi_{E_x}(y)dydx = \int_{x=-1}^{1} m_1(E_x)dx > 0$$

since the measure is > 0 a.e.

□

Problem 7. Let $f(z)$ be a one-to-one continuous mapping from the closed annulus

$$\{1 \leq |z| \leq R\}$$

onto the closed annulus

$$\{1 \leq |z| \leq S\}$$

such that f is analytic on the open annulus $\{1 < |z| < r\}$. Prove that $S = R$.

Proof. Let f be such an analytic function. By the open mapping theorem and continuity of f, it follows that f maps boundary to boundary i.e. $|f(e^{i\theta})| = 1$ or S and by continuity the modulus must be constant on $\partial\mathbb{D}$. If necessary, by considering $f(z/R)$ we can assume that $|f(e^{i\theta})| = 1$ so that f maps $\partial\mathbb{D}$ to $\partial\mathbb{D}$. Now consider

$$g(z) := \overline{1/(f(1/\overline{z}))}$$

which is holomorphic since we took two conjugates and $1/f$ has no zeros. Then observe that $1/\overline{e^{i\theta}} = e^{i\theta}$

$$|g(e^{i\theta})| = \overline{1/f(e^{i\theta})} = f(e^{i\theta})$$

where for the last equality we used $|f(e^{i\theta})| = 1$. Therefore, by Schwarz Lemma, this implies that

$$\tilde{f}(z) := \begin{cases} f(z) \text{ for } z \in \{1 \leq |z| \leq R\} \\ g(z) \text{ for } z \in \{\frac{1}{R} \leq |z| \leq 1\} \end{cases}$$

is holomorphic such that $\tilde{f}(z) : \{1/R \leq |z| \leq R\} \to \{1/S \leq |z| \leq R\}$ such that $|\tilde{f}(1/Re^{i\theta})| = 1/S$. This map still is injective since $|f(z)| > 1$ for any $1 < |z| \leq R$ so $|f| \neq 1/|f|$. We rewrite $\tilde{f}$ as f with this extension.

Now as $|f(1/Re^{i\theta})| = 1/S$ we see by a similar agument that

$$f(z) = 1/S^2 \left(\overline{f}(1/(R^2\overline{z})\right) \text{ on } |z| = 1/R$$

so in particular, again f extends to a conformal map from $\{1/R^2 \leq |z| \leq R\} \to \{1/S^2 \leq |z| \leq S\}$ since as f is non-zero its maximum and minimum modulus must occur on the boundary and we have $|f(e^{i\theta})| = 1$. This map also has the property $|f(1/R^2e^{i\theta})| = 1/S^2$

Now by iterating this argument of Schwarz reflecting off on $\partial B_{1/R}(0)$ we can extend f to a holomorphic map on $\{0 < |z| \leq R\}$ to $\{0 < |z| \leq S\}$ such that $|f(e^{i\theta}/R^n)| = 1/S^n$, but then the singularity at 0 is removable since f is bounded by S. Then observe that by construction we have for any $n \in \mathbb{N}$

$$|f(e^{i\theta}/R^n)| = 1/S^n \to 0$$

so $f(0) = 0$. Therefore, f extends to a conformal map from $B_R(0)$ to $B_S(0)$. So as $f(z)$ is differentiable at $z = 0$ it is locally injective in a small neighborhood of 0, but we have

$$|f(e^{i\theta}/R^n) - f(0)| = 1/S^n \leq K_1/R^n$$

for some $K_1 > 0$ being the Lipschitz constant. But by considering the inverse of f and that $f^{-1}(0) = 0$ we have that for some constant K_2 that

$$|f^{-1}(e^{i\theta}/S^n) - f^{-1}(0)| = 1/R^n \leq K_2/S^n$$

so by letting $K := \max\{K_1, K_2\}$ we have that

$$(R/S)^n \leq K \text{ and } (S/R)^n \leq K$$

so if $S/R \neq 1$ one of these two sides will approach ∞ as $n \to \infty$. Therefore, $S = R$ as desired. □

Problem 8. Let $a_1, a_2, ..., a_n$ be $n \geq 1$ points in $\mathbb{D}$, so that

$$B(z) := \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j}z}$$

has n zeros in $\mathbb{D}$. Prove that $B'(z)$ has $n - 1$ zeros in $\mathbb{D}$.

Problem 9. Let $f(z)$ be an analytic function in the entire complex plane $\mathbb{C}$ and assume $f(0) \neq 0$. Let $\{a_n\}$ be the zeros of f, repeated according to their multiplicities.

(1) Let $R > 0$ such that $|f(z)| > 0$ on $|z| = R$. Prove

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta = \log|f(0)| + \sum_{|a_n|<R} \log\frac{R}{|a_n|}$$

(2) Prove that if there are constants C and λ such that $|f(z)| \leq Ce^{|z|^\lambda}$ for all z, then

$$\sum \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} < \infty$$

Proof. The first part is known as Jensen's formula. We will first prove the formula on $\mathbb{D}$. As each zero of $f(z)$ is discrete and f is entire there are only finitely many zeros on $\overline{\mathbb{D}}$. Then define

$$B(z) := \prod_{|a_n|<1} \frac{z-a_n}{1-\overline{a_n}z} = \prod_{|a_n|<1} B_{a_n}(z)$$

which is holomorphic on $\mathbb{D}$ since there are only finitely many zeros and the poles are at $1/\overline{a_n}$ which has magnitude bigger than 1. Now we see that

$$\log|B_{a_n}(z)| = \log|-a_n| = \log|a_n|$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \log|B_{a_n}(e^{i\theta})|d\theta = 0$$

since B_{a_n} is an automorphism of the unit disk so that $|B_{a_n}(e^{i\theta})| = 1$. And the only zero of B_{a_n} is at a_n so Jensen's formula is satisfied for each B_{a_n}. Then notice that $g = f/B$ is holomorphic on the unit disc such that g has no zeros. Therefore, there is a branch of the log such that $\log(g)$ is holomorphic since $\mathbb{D}$ is simply connected. There, $\log|g|$ is holomorphic, so Jensen's Formula holds for g due to the Mean Value Property. Now notice that Jensen's formula is multiplicative since $\log|xy| = \log|x| + \log|y|$ so we have that $f(z) = g(z)B(z)$ satisfies Jensen's formula since g and B satisfies the formula.

Now observe that

$$\int_{|a_n|}^{R} \frac{1}{x}dx = \log(R/|a_n|)$$

so

$$\sum_{|a_n|<R} \log\frac{R}{|a_n|} = \sum_{|a_n|<R} \int_{|a_n}^{R} \frac{1}{x}dx = \int_0^R \sum_{|a_n|<R} (\chi_{x\geq|a_n|})\frac{1}{x}dx$$

and observe $\sum_{|a_n|<R}(\chi_{x\geq|a_n|}(r)) = n(r)$ where $n(r)$ represents the number of zeros of f in $B_r(0)$. So

$$\int_0^R \frac{n(r)}{r}dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta - \log|f(0)|$$

So we have

$$\int_R^{2R} \frac{n(r)}{r} \leq \int_0^{2R} \frac{n(r)}{r}dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(2Re^{i\theta})|d\theta - \log|f(0)|$$

so as $n(r)$ is decreasign we deduce

$$n(R)\log(2) = n(R)\int_R^{2R} \frac{1}{r}dr \leq \int_R^{2R} \frac{n(r)}{r}dr$$

so we have

$$n(R)\log(2) \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(2Re^{i\theta})|d\theta - \log|f(0)|$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi} \log|Ce^{(2R)^\lambda}|d\theta - \log|f(0)| \leq \log|C| + 2^\lambda R^\lambda - \log|f(0)| \lesssim_f R^\lambda$$

so we have that

$$n(R) = O(R^\lambda)$$

for R large.

Now observe that we only need to control the tail of the sum to show convergence so we can take n so large such that if $n \geq N$ then $|a_n| \geq R$ where R is chosen to be so large such that $n(r) = O(r^\lambda)$ for any $r \geq R$. Now observe that if $2^M \leq R \leq 2^{M+1}$

$$\sum_{n\geq N}\left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon} \leq \sum_{n=M}^{\infty} \sum_{2^n\leq|a_m|\leq 2^{n+1}} \left(\frac{1}{|a_n|}\right)^{\lambda+\varepsilon}$$

$$\leq \sum_{n=M}^{\infty} 2^{-n(\lambda+\varepsilon)}n(2^{n+1}) \lesssim \sum_{n=M}^{\infty} 2^{-n(\lambda+\varepsilon)}(2^{\lambda n+\lambda}) = \sum_{n=M}^{\infty} 2^{-\varepsilon n+\lambda} < \infty$$

since $-\varepsilon < 0$.

□

Problem 10. Let $a_1, ..., a_n$ be $n \geq 1$ distinct points in $\mathbb{C}$ and let $\Omega := \mathbb{C} \setminus \{a_1, ..., a_n\}$. Let $H(\Omega)$ be the vector space of real-valued harmonic functions on Ω and let $R(\Omega) \subset H(\Omega)$ be the space of real parts of analytic functions on Ω. Prove the quotient space $\frac{H(\Omega)}{R(\Omega)}$ has dimension n, and find a basis for this space, and prove it is a basis.

Proof. Let $f \in H(\Omega)$ then by using the Cauchy Riemann equations we see that $g := f_x - if_y$ is holomorphic. Now g is an analytic function except for isolated singularities at $z = a_1, ..., a_n$. So for $1 \leq j \leq n$ define $c_j := \text{Res}(g, a_j)$. Now define

$$h(z) := g(z) - \sum_{j=1}^{n} \frac{c_j}{z - a_j}$$

so by the residue theorem over any closed curve γ in Ω this function integrates to 0. So this implies that $h(z)$ has a primitive, which we denote by $u(z)$. And the Cauchy Riemann Equation shows that $w(z)$ is the real part of $u(z)$ up to constants iff w is differentiable in the real sense and $u'(z) = w_x - iw_y$ and if we define

$$\tilde{w}(z) := f(z) - c_j \sum_{j=1}^{n} \log|z - a_j| \Rightarrow \tilde{w}_x - i\tilde{w}_y = h(z)$$

Therefore, $H(\Omega) \ni \tilde{w} = \text{Re}(u)$ up to constants, so in particular $\tilde{w} \in H(\Omega) \cap R(\Omega)$. So we have shown $\{\log|z - a_j|\}_{j=1}^n$ span $H(\Omega)/R(\Omega)$. Since the a_j are distinct we can find an $\varepsilon > 0$ such that $B_\varepsilon(a_j) \cap B_\varepsilon(a_k) = \emptyset$ for $j \neq k$ so if

$$\sum_{j=1}^{n} \alpha_j \log|z - a_j| = 0 \Rightarrow 0 = \int_{\partial B_\varepsilon(a_k)} \sum_{j=1}^{n} \alpha_j \log|z - a_j| = 2\pi i \alpha_j \Rightarrow \alpha_j = 0$$

so these functions are linearly independent and hence a basis of $H(\Omega)/R(\Omega)$, so this vector space has dimension n. □

Problem 11. Let $1 \leq p < \infty$ and let $U(z)$ be a harmonic function on the complex plane such that

$$\int\int_{\mathbb{C}} |U(x, y)|^p dxdy < \infty$$

Prove $U(z) = 0$ for all $z = x + iy \in \mathbb{C}$.

Proof. For any $r > 0$ and $z_0 \in \mathbb{C}$ the mean value property tells us that

$$U(z_0) = \frac{1}{\pi r^2} \int_{B_r(z_0)} U(x, y) dxdy$$

so we have from Holder's inequality that

$$|U(z_0)| \leq \frac{1}{\pi r^2} ||U||_{L^p(B_r(z_0), dxdy)} (\pi r^2)^{1-1/p} \lesssim r^{-2/p} ||U||_{L^p(\mathbb{C}, dxdy)} \to 0 \text{ as } r \to \infty$$

since $U \in L^p(\mathbb{C}, dxdy)$, so it follows that

$$|U(z_0)| = 0$$

so $U(z) = 0$ for all z.

□

Problem 12. Let $0 < \alpha < 1$ and $f(z)$ be an analytic function such that $f \in C^\alpha(\mathbb{D})$. Show there is a constant A such that

$$|f'(z)| \leq A(1 - |z|)^{\alpha - 1}$$

Proof. Notice by Cauchy's Theorem and the Residue Theorem that if $|z| = 1-2\delta$ then for any $0 < \varepsilon < 2\delta$

$$f'(z) = \frac{1}{2\pi i}\int_{|z-w|=\varepsilon} \frac{f(w)}{(w-z)^2}dw = \frac{1}{2\pi i}\int_{|z-w|=\varepsilon} \frac{f(w)-f(z)}{(w-z)^2}dw$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{f(z+\varepsilon e^{i\theta})-f(z)}{\varepsilon^2 e^{2i\theta}}\varepsilon e^{i\theta}$$

so we have

$$|f'(z)| \leq \sup_\theta |f(z+\varepsilon e^{i\theta}) - f(z)|/\varepsilon \leq C\varepsilon^{\alpha-1}$$

Now observe

$$1 - |z| = 2\delta$$

so we have by sending $\varepsilon \to 2\delta$ we see that

$$|f'(z)| \leq C(2\delta)^{\alpha-1} = C(1-|z|)^{\alpha-1}$$

as desired $\square$.

$\square$

16. Fall 2017

Problem 1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing. Show that if $A \subset \mathbb{R}$ is a Borel set, then so is $f(A)$.

Proof. Define the set $\mathcal{A} := \{A \subset \mathbb{R} \text{ is borel} : f(A) \text{ is Borel }\}$. We claim that $\mathcal{A}$ is a σ-algebra and that $\mathcal{A}$ contains the open sets, which implies the desired claim.

First we will show that $\mathcal{A}$ contains the open set. As $f(x)$ is monotone, it has only countably many discontinuities say $\{d_i\}_{i=1}^{\infty}$ and on each d_i we have that $f(d_i^-) := \lim_{x\to d_i^-} f(x) < \lim_{x\to d_i^+} f(x) := f(d_i^+)$ i.e. an upward jump discontinuity. Then for any open interval (a, b) we have that $f(a, b)$ is a countable union of the form $[f(a^-), f(q_i^-)) \cup [f(q_i^+), f(q_j^-)) \cup ...[f(q_k^+), f(b^+)]$ where the q_j are in $[a, b]$ and the end points at $f(a)$ and $f(b)$ may be included or not. In any case, it is always a countable union of a mixture of half open intervals, open intervals, or closed intervals, so $f(a, b)$ is Borel. Therefore, this implies $\mathcal{A}$ contains the open sets.

In particular, this implies $f(\mathbb{R})$ is Borel since $\mathbb{R}$ is open. Now if $\{A_i\}_{i=1}^{\infty} \in \mathcal{A}$ then

$$f(\bigcup_{i=1}^{\infty} A_i) = \bigcup_{i=1}^{\infty} f(A_i)$$

so $\mathcal{A}$ is closed under countable union. Then for any $A \in \mathcal{A}$ we have that

$$f(\mathbb{R}) = f(A) \cup f(A^c)$$

and due to monotocity $f(A)$ and $f(A^c)$ have at most a countable number of points common, so $f(A^c) = f(A) \setminus f(\mathbb{R}) \cup$ countably many points. Indeed, if $x \in A$ and $y \in A^c$ are such that $f(x) = f(y)$ then this implies there is an interval containing x and y such that f is constant on this interval. Then the collection of all intervals where f is constant is countable since on each of these intervals we can find a rational not in any of the other intervals since $x \neq y$. So this $f(A)$ and $f(A^c)$ differ by at most countably many points, so $(f(A))^c$ differs from $f(A^c)$ by at most countably many points, so $f(A^c)$ is Borel Measurable. Thus $\mathcal{A}$ contains the borel σ-algebra

□

Problem 2. Let $\{f_n\}$ denote a bounded sequence in $L^2([0, 1])$. Suppose that $\{f_n\}$ converges almost everywhere. Show that then $\{f_n\}$ converges in the weak topology on $L^2([0, 1])$

Proof. See Fall 2012 Problem 1.

□

Problem 3. Let $\{\mu_n\}$ denote a sequence of Borel probability measures on $\mathbb{R}$. For $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we define

$$F_n(x) := \mu_n((-\infty, x])$$

Suppose the sequence $\{F_n\}$ converges uniformly on $\mathbb{R}$. Show that for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$ the numbers

$$\int_{\mathbb{R}} f(x) d\mu_n(x)$$

converges as $n \to \infty$.

Proof. Note that $C(\mathbb{R})$ is not separable, so we are not guaranteed that weak* compactness is equivalent to weak* subsequential compactness. Indeed, let $\mu_n = \delta_n$ then μ_n does not weakly converge on any sub-sequence, but by Banach Alagou we know that $\{\mu_n\}$ is weak* compact. Therefore, we instead approximate μ by its restrictions on compact sets.

Fix an $R > 0$ then consider $\nu_n := \{\mu_n|_{[-R,R]}\}$ i.e. the measure restricted to $[-R, R]$. Then $C(K_R)$ is separable so by Banach Alagou, there is a subsequence such that ν_n weak* converges to ν (we still denote this subsequence as ν_n). Then as $F_n \to F$ uniformly, we know that $G_n(x) := \int_{(-\infty,x]} d\nu_n$ uniformly converges. Then by approximating intervals of the form $(a, b]$ with linear functions it is easy to see $G_n(x) \to G(x) := \int_{(-\infty,x]} d\nu$ uniformly. Therefore, this implies that the entire sequence $\nu_n \to \nu$.

Then let $\varepsilon > 0$ then by uniform convergence of F_n we can find an $N \in \mathbb{N}$ such that if $n, m \geq N$ then

$$\sup_{x\in\mathbb{R}} |F_n(x) - F_m(x)| \leq \varepsilon$$

Now as μ_N is a finite measure there is an $R > 0$ such that for $K_R = (-R, R]$ that $(1-(F_N(R)-F_N(-R)) = \mu_N((K_R^c) < \varepsilon$, then as

$$|F_n(x)| \leq |F_n(x) - F_N(x)| + |F_N(x)| \leq \varepsilon + |F_N(x)|$$

we deduce that $\mu_n(K_R^c) < 4\varepsilon$ for $n \geq N$.

Now for any bounded f we have

$$\left|\int_{\mathbb{R}} f(x)d(\mu_n - \mu_m)\right| = \left|\int_{K_R} f(x)d(\mu_n - \mu_m)\right| + \int_{K_R^c} |f(x)|d|\mu_n - \mu_m|$$

Notice the first term converges by our previous argument since $K_R \subset [-R, R]$ and we showed that $\mu_n|_{[-R,R]}$ weak* converges, so it is small for large n, m. And the second term is bounded by $8\sup_{x\in\mathbb{R}} |f(x)|\varepsilon$ since $|\mu_n(K_R) - \mu_m(K_R)| \leq mu_n(K_R) + \mu_m(K_R) = 8\varepsilon$. Hence, $\int_{\mathbb{R}} f(x)d\mu_n(x)$ converges for all $f \in C(\mathbb{R})$ that is bounded.

□

Problem 4. Consider the Banach Space $V = C([-1, 1])$ of all real-valued continuous functions on $[-1, 1]$ equipped with the sup norm. Let B be the closed unit ball in V.

Show that there exists a bounded linear functional $\Lambda : V \to \mathbb{R}$ such that $\Lambda(B)$ is an open subset of $\mathbb{R}$.

Proof. Define

$$\Lambda(f) := \int_0^1 f(x)dx - \int_{-1}^0 f(x)dx$$

then we claim $\Lambda(V) = (-2, 2)$. Indeed, fix $\varepsilon > 0$ and define

$$f_n(x) := \begin{cases} 1 \text{ on } [1/n, 1] \\ nx \\ -1 \text{ on } [-1/n, -1] \end{cases}$$

then

$$\Lambda(f) = 2(1 - 1/n) + \int_0^{1/n} nx - \int_{-1/n}^0 nx = 2 - 1/n$$

so $(-2, 2) \subset \Lambda(f)$. To see that it is equality, notice that $\int_0^1 f(x)$ and $\int_{-1}^0 f(x)dx$ are both bounded by 1 so to get $2 = \Lambda(f)$, we need both integrals the first integral to be 1 and the second to be -1, which means $f = 1$ a.e. on $[0, 1]$ and $f = -1$ a.e. on $[-1, 0]$ which contradicts continuity. And it's clear Λ is linear and we have

$$|\Lambda(f)| \leq 2||f||_V$$

so it is bounded.

□

Problem 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded measurable function satisfying $f(x+1) = f(x)$ and $f(2x) = f(x)$ for almost every $x \in \mathbb{R}$. Show that then there is a $c \in \mathbb{R}$ such that $f(x) = c$ almost everywhere.

Proof. Fix a representation of f such that $f(x+1) = f(x)$ and $f(2x) = f(x)$ everywhere. Then notice that $f(x) \in L^1_{loc}(\mathbb{R})$, and that as $f(x+1) = f(x)$ it suffices to show the theorem on $[0,1]$. Then denote the Lebesgue Points of f as E then $m(E \cap [0,1]) = 1$. Fix $x_0, y_0 \in E \cap [0,1]$

$$f(y_0) = \lim_{r \to 0} \frac{1}{2r} \int_{y_0 - r}^{y_0 + r} f(x)dx \text{ and } f(x_0) = \lim_{r \to 0} \frac{1}{2r} \int_{x_0 - r}^{x_0 + r} f(x)dx$$

Now notice for dyadic rationals of the form $m/2^n$ where $m, n \in \mathbb{Z}$ that

$$f(x + m/2^n) = f(2^n(x + m/2^n)) = f(2^n x + m) = f(2^n x) = f(x)$$

and we recall that dyadic rationals are dense in $\mathbb{R}$. So in particular, there exists a dyadic rational $m/2^n$ such that if $\varepsilon > 0$ then $x_0 \leq y_0 + m/2^n < x_0 + \varepsilon$ then

$$f(y_0) = f(y_0 + m/2^n) = \lim_{r \to 0} \frac{1}{2r} \int_{y_0 + m/2^n - r}^{y_0 + m/2^n + r} f(x)dx$$

so

$$\left| \frac{1}{2r} \left(\int_{y_0 + m/2^n - r}^{y_0 + m/2^n + r} f(x) \right) - \frac{1}{2r} \left(\int_{x_0 - r}^{x_0 + r} f(x) \right) \right| \leq \frac{1}{2r} \int_{x_0 - r}^{y_0 + m/2^n - r} |f(y)| + \frac{1}{2r} \int_{x_0 + r}^{y_0 + m/2^n + r} |f(y)|$$

$$\leq \frac{\varepsilon}{r} M$$

where M is a bound for f so by choosing $\varepsilon = r^2$ since ε was independent of r so we obtain the bound

$$|f(x_0) - f(y_0)| \leq \lim_{r \to 0} rM = 0$$

Therefore, $f(x_0) = f(y_0)$.

□

Problem 6. For $f \in L^2(\mathbb{C})$. For $z \in \mathbb{C}$ define

$$g(z) := \int_{B(1,z)} \frac{|f(w)|}{|w - z|} dA(w)$$

Show that then $|g|$ is finite a.e. and $g \in L^2(\mathbb{C})$

Proof. Observe that by Cauchy Schwarz that

$$|g(z)|^2 = \left(\int_{B(1,z)} \frac{|f(w)|}{|w - z|^{1/2}} \frac{1}{|w - z|^{1/2}} dA(w) \right)^2 \leq \left(\int_{B(1,z)} \frac{|f|^2}{|w - z|} dA(w) \right) \left(\int_{B(1,z)} \frac{1}{|w - z|} dA(w) \right)$$

and $1/|z| \in L^1(B_1(0))$ since $\mathbb{C} \cong \mathbb{R}^2$, so

$$|g(z)|^2 \leq C \left(\int_{\mathbb{C}} \chi_{B_1(z)}(w) \frac{|f(w)|^2}{|w - z|} dA(w) \right)$$

so we have

$$\int_{\mathbb{C}} |g(z)|^2 \leq C \int_{\mathbb{C}} \int_{\mathbb{C}} \chi_{B_1(z)}(w) \frac{|f(w)|^2}{|w - z|} dA(w) dz$$

So by Tonelli since the integrand is non-negative

$$= \int_{\mathbb{C}} |f(w)|^2 \int_{\mathbb{C}} \chi_{B_1(z)}(w) \frac{1}{|w - z|} dA(z) dA(w) = \int_{\mathbb{C}} |f(w)|^2 \int_{B_1(w)} \frac{1}{|w - z|} dA(z) dA(w)$$

$$= \int_{\mathbb{C}} |f(w)|^2 \int_{B_1(0)} \frac{1}{|z|} dA(z) dA(w) = C \int_{\mathbb{C}} |f(w)|^2 dA(w) < \infty$$

where we used $\int_{B_1(w)} \frac{1}{|w-z|} dA(w) = \int_{B_1(0)} \frac{1}{|w|} dA(w) < \infty$, so $g \in L^2$

$\square$

Problem 7. Prove that there exists a meromorphic function f on $\mathbb{C}$ with the following three properties:

(1) $f(z) = 0$ iff $z \in \mathbb{Z}$
(2) $f(z) = \infty$ iff $z - 1/3 \in \mathbb{Z}$
(3) $|f(x+iy)| \leq 1$ for all $x \in \mathbb{R}$ with $|y| \geq 1$

Proof. Define

$$g(z) := \frac{\sin(\pi z)}{\sin(\pi(z+1/3))}$$

then this a meromorphic function with the first 2 properties. Now we will show there is a $C > 0$ such that

$$\limsup_{z\to\infty \text{ with } |\text{Im}(z)|\geq 1} |g(z)| \leq C$$

We will deal with the case that $\text{Im}(z) \geq 1$ as the other case is a similar argument. Indeed, write $z = x+iy$ with $y \geq 1$ then by the triangle and reverse triangle inequalities

$$|g(z)| = \left| \frac{e^{i\pi z} - e^{-i\pi z}}{e^{i\pi(z+1/3)-e^{-i\pi(z+1/3)}}} \right| \leq \frac{|e^{i\pi z}| + |e^{-i\pi z}|}{||e^{i\pi z}| - |e^{-i\pi z}||} = \frac{e^{\pi y} + e^{-\pi y}}{e^{\pi y} - e^{-\pi y}}$$
$$= \frac{1+e^{-2\pi y}}{1-e^{-2\pi y}} \leq \frac{2}{1-e^{-2\pi}} := C_1 < \infty$$

and similarily one shows that when $y \leq 1$ $|g(x+iy)|$ is bounded. So by continuity on $\Omega := \mathbb{C}\backslash\{|\text{Im}(z)| \leq 1\}$, we deduce that $|g(z)|$ is bounded on Ω. Say this bound is M then

$$f(z) := g(z)/M$$

is the desired function $\square$.

$\square$

Problem 8. Show that a harmonic function $u : \mathbb{D} \to \mathbb{R}$ is uniformly continuous iff it admtis the representation

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta}) d\theta$$

where $f : \partial\mathbb{D} \to \mathbb{R}$ is continuous.

Proof. It is a basic fact that u is uniformly continuous on $\mathbb{D}$ iff u is continuous on $\overline{D}$. So define

$$v(z) := \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(e^{i\theta}) d\theta$$

Then observe that $v(z)$ is holomorphic on $\mathbb{D}$ since if $R \subset \mathbb{D}$ is a rectangle then by Fubini's which can be applied since $v(z) \in L^1_{loc}(\mathbb{D}, dA(z))$ (because the integrand is continuous on any compact subset of $\mathbb{D}$) we have that

$$\int_R v(z)dz = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_R \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(e^{i\theta}) dz d\theta = 0$$

where the last equality is due to $P(z,\theta) := \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)$ harmonic on $\mathbb{D}$. Then as $v(z)$ is continuous since $u(e^{i\theta})$ and $P(z,\theta)$ are, we deduce from Morrera's that v is holomorphic. Therefore, its real part is harmonic. Observe that

$$w(z) := \text{Re}(v(z)) = \frac{1}{2\pi} \int_0^{2\pi} \text{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(e^{i\theta}) d\theta$$

so by the Maximum Principle it suffices to show $w(e^{i\theta}) = u(e^{i\theta})$ to obtain $u = w$.

Note that $\int_0^{2\pi} P(z,\theta)d\theta = 2\pi$ which follows from the Fourier Series of P. By computation we have for $z \in \mathbb{D}$ that

$$\text{Re}(P(z,\theta)) = \frac{1-|z|^2}{1-2\text{Re}(ze^{-i\theta})+|z|^2} = \frac{1-|z|^2}{|e^{i\theta}-z|^2} \geq 0 \tag{16.1}$$

Therefore, we have

$$|w(re^{i\psi}) - u(e^{i\psi})| = \left|\frac{1}{2\pi}\int_0^{2\pi} \text{Re}(P(re^{i\psi},\theta))\left(u(e^{i\theta}) - u(e^{i\psi})\right) d\theta\right|$$

$$\leq \frac{1}{2\pi}\int_{|\theta-\psi|<\delta\cap\theta\in[0,2\pi]} \left(\text{Re}(P(re^{i\psi},\theta))\left|u(e^{i\theta}) - u(e^{i\psi})\right| d\theta\right) + \int_{|\theta-\psi|\geq\delta\cap\theta\in[0,2\pi]} \left(\text{Re}(P(re^{i\psi},\theta))\left|u(e^{i\theta}) - u(e^{i\psi})\right|\right) d\theta$$

$$= (I) + (II)$$

Note that we can bound (I) due to continuity continuity of u and and that $\text{Re}(P(z,\theta))$ has mass 1 to make (I) arbitrarily small for small δ. And notice that

$$\text{Re}(P(re^{i\psi},\theta)) = \frac{1-r^2}{|e^{i\theta}-re^{i\psi}|^2} \to 0 \text{ uniformly on } |\theta-\psi| \geq \delta \text{ as } r \to 1$$

this follows since the denominator is uniformly bounded on this set, while the numerator goes to zero. Therefore,

$$\lim_{r\to 1} w(re^{i\psi}) = u(e^{i\psi})$$

so it follows from continuity of w that $w = u$ on $\partial\mathbb{D}$, so by the maximum principle since both functions are harmonic we deduce that $w = u$ on $\overline{\mathbb{D}}$.

For the converse, note by the first part of our proof we showed that

$$u(z) := \frac{1}{2\pi}\int_0^{2\pi} \text{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) f(e^{i\theta})d\theta$$

is harmonic due to Morrera's and Fubini's Theorem. Also arguing like in the previous proof, we see that

$$u(e^{i\theta}) = f(e^{i\theta})$$

Therefore, u extends to a continuous function on $\mathbb{D}$ (since its clearly continuous in the interior since the integrand is continuous), so u is a uniformly continuous harmonic function.

□

Problem 9. Consider a map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ with the following properties:

(1) For each fixed $z \in \mathbb{C}$ the map $w \mapsto F(z,w)$ is injective
(2) For each fixed $w \in \mathbb{C}$ the map $z \mapsto F(z,w)$ is holomorphic
(3) $F(0,w) = w$ for $w \in \mathbb{C}$

Show that then

$$F(z,w) = a(z)w + b(z)$$

for $z, w \in \mathbb{C}$, where a and b are entire functions with $a(0) = 1$, $b(0) = 0$, and $a(z) \neq 0$ for $z \in \mathbb{C}$

Hint: Consider

$$\frac{F(z,w) - F(z,0)}{F(z,1) - F(z,0)}$$

Proof. Fix a $w \in \mathbb{C}$ such that $w \neq 0$ or 1 then define

$$f_w(z) := \frac{F(z,w) - F(z,0)}{F(z,1) - F(z,0)}$$

Then from condition (i) we see that the numerator is never 0 since as $w \neq 0$ we have $F(z,w) \neq F(z,0)$ by injectivity. Also as $w \neq 1$ we deduce from injectivity in w that $F(z,w) \neq F(z,1)$ so we have $f_w(z) \neq 1$. Therefore, since $f_w(z)$ is holomorphic that by Little Picard's Theorem, we deduce that $f_w(z)$ is constant for $w \in \mathbb{C} \setminus (\{0\} \cup \{1\})$ but we have $f_1(z) = 1$ and $f_0(z) = 0$, so we deduce that $f_w(z)$ is constant for any w.

In particular, this implies that for all $w \in \mathbb{C}$ that

$$f_w(z) = g(w)$$

but condition 3 implies that

$$g(w) = f_w(0) = \frac{F(0,w) - F(0,0)}{F(0,1) - F(0,0)} = w$$

we conclude that

$$f_w(z) = w$$

so we have

$$F(z,w) = (F(z,1) - F(z,0))w + F(z,0) := a(z)w + b(z)$$

and note that by the given assumptions we have all the desired properties for a and b.

□

Problem 10. Let $\{f_n\}$ be a sequence of holomorphic functions on $\mathbb{D}$ with the property that

$$F(z) := \sum_{n=1}^{\infty} |f_n(z)|^2 \leq 1$$

for all $z \in \mathbb{D}$. Show that the series defining $F(z)$ converges uniformly on compact subsets and that F is subharmonic.

Proof. We claim that if f is holomorphic then $|f(z)|^2$ is subharmonic. Indeed, if $f(z) = u + iv$ where u and v are the real and imaginary parts then

$$|f(z)|^2 = |u(z)|^2 + |v(z)|^2$$

so it suffices to show if u is a real valued harmonic function then $(u(z))^2$ is subharmonic. Indeed, observe

$$\Delta(u^2) = 2u\Delta u + 2\nabla u \cdot \nabla u = 2|\nabla u|^2 \geq 0$$

so u^2 is subharmonic. Therefore, each $|f_n(z)|^2$ is subharmonic. So in particular, as

$$F(z) = \sup_m \sum_{n=1}^{m} |f_n(z)|^2$$

i.e. F is the supremium of subharmonic functions, we have that F is subharmonic as long as $F(z)$ is upper semi continuous and not identically ∞ (the second part follows from $F(z) \leq 1$). We will show the last part by showing the sum converges locally uniformly to deduce that $F(z)$ is continuous, from which we get that its subharmonic.

Now we use Harnack's inequality sicne each $|f_n|^2 \geq 0$ is a non-negative subharmonic functions to for any $0 < R < 1/2$ that there is a constant $C = C(R)$ that depends only on R such that

$$|f_n(z)|^2 \leq C|f_n(0)|^2 \text{ for } z \in B_{2R}(0)$$

so we have for $z \in \overline{B_R(0)}$ that

$$\sum_{n=N}^{\infty} |f_n(z)|^2 \leq C \sum_{n=N}^{\infty} |f_n(0)|^2 \to 0$$

where the last implication is due to the sum at $z = 0$ being summable. Therefore, the series defining $F(z)$ is uniformly summable on any compact set, so $F(z)$ is the uniform limit of the partial sums, so $F(z)$ is continuous and hence by our earlier remarks we have $F(z)$ is subharmonic. □

Problem 11. Let $f : \mathbb{D} \to \mathbb{C}$ be an injective holomorphic function with $f(0) = 0$ and $f'(0) = 1$. Show that then

$$\inf\{|w| : w \notin f(\mathbb{D})\} \leq 1$$

with equality iff $f(z) = z$ for all $z \in \mathbb{D}$.

Proof. First assume for the sake of contradiction that

$$\inf\{|w| : w \notin f(\mathbb{D})\} > 1$$

Therefore, as $f(\mathbb{D})$ is an open map since injective functions are non-constant and $f(0) = 0 \in \mathbb{D}$, we deduce that there is an $\delta > 0$ such that $B_{1+\delta}(0) \subset f(\mathbb{D})$. Let $U \subset \mathbb{D}$ be open such that $f(U) = \mathbb{D}$. Note this implies $\overline{U} \subset \mathbb{D}$ since $B_{1+\delta}(0) \subset f(U)$ since conformal maps map boundary to boundary. So there is some $\varepsilon > 0$ such that $U \subset B_{1-\varepsilon}(0)$.

Now define

$$g := f^{-1}|_{\mathbb{D}}$$

then $g : \mathbb{D} \to U$ and

$$h(z) := \frac{g(z)}{1-\varepsilon} : \mathbb{D} \to \mathbb{D} \text{ with } h(0) = 0$$

so by Schwarz Lemma, we deduce that

$$|g'(0)| \leq 1 - \varepsilon \Rightarrow \frac{1}{|f'(0)|} \leq 1 - \varepsilon \Rightarrow 1 \leq 1 - \varepsilon$$

which is our contradiction. So we must have $\inf\{|w| : w \notin f(\mathbb{D})\} \leq 1$ for any injective holomorphic function with $f(0) = 0$ and $f'(0) = 1$.

Now we deal with the equality. Note that $f(z) = z$ trivially obeys the given bound and if

$$\inf\{|w| : w \notin f(\mathbb{D})\} = 1$$

implies $\mathbb{D} \subset f(\mathbb{D})$. Indeed, notice

$$\inf\{|w| : w \notin f(\mathbb{D})\} = 1 \Rightarrow \text{ for all } w \in f(\mathbb{D})^c, |w| \geq 1$$

so $f(\mathbb{D})^c \subset \mathbb{D}^c$, so by $\mathbb{D} \subset f(\mathbb{D})$. So it follows that $f^{-1}(\mathbb{D}) \subset \mathbb{D}$ with $f^{-1}(0) = 0$ and $(f^{-1}(0))' = 1$, and this is a map from the unit disk to the unit disk, so by Schwarz Lemma we deduce that there is some θ such that

$$f^{-1}(z) = e^{i\theta} z \Rightarrow f(z) = e^{-i\theta} z$$

so $f'(0) = 1$ implies $\theta = 0$ so $f(z) = z$.

□

Problem 12. Let f, g, h be complex valued functions on $\mathbb{C}$ with

$$f = g \circ h$$

Show that if h is continuous and both f, g are non-constant holomorphic functions then h is holomorphic too.

Proof. As g is non-constant, then $Z := \{z \in \mathbb{C} : g'(z) = 0\}$ is discrete. So for each $z_0 \in \mathbb{C} \setminus Z$ there is an $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset \mathbb{C} \setminus Z$. So by the inverse mapping theorem by taking ε smaller if necessary, we can find an inverse of g on $B_\varepsilon(z_0)$ and g^{-1} on this neighborhood is holomorphic since $g'(z) \neq 0$ on $B_\varepsilon(z_0)$. Therefore, we have for any $z_0 \notin h^{-1}(Z)$ we can locally invert thanks to continuity which allows us to avoid all the other zeros since the zeros are discrete

$$g^{-1} \circ f(z) = h(z) \text{ for } z \in B_\delta(z_0)$$

for $\delta > 0$ small enough that $h(z) \notin Z$ and small enough to apply Inverse Function Theorem. In particular, $h(z)$ is holomorphic on $\mathbb{C} \setminus Z$ as it is the composition of two holomorphic functions.

Now fix $R > 0$ then $\overline{B_R(0)} \cap Z = \{z_1, .., z_N\}$ for some finite collection z_i since the zeros are isolated. Now it suffices by Riemann's Removable Singularities theorem to show that h is bounded on $\overline{B_R(0)}$ to show that h extends to a holomorphic function on $B_R(0)$. But this follows from h being continuous. Therefore, h is holomorphic on $B_R(0)$ for any $R > 0$ so its holomorphic on $\mathbb{C}$. □

17. Spring 2018

Problem 1. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{h\to 0} \int_{\mathbb{R}} \left| \frac{f(x+h)-f(x)}{h} \right| dx = 0$$

Show that $f = 0$ a.e.

Proof. Define

$$F(x) := \int_0^x f(y)dy$$

Let $h > 0$ and $x \in E$ where E is the lebesgue points of f then

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_x^{x+h} f(y)dy\right) \to f(x)$$

and now observe that if $x, y \in E$ where $y < x$ then

$$F(x+h) - F(y+h) + F(y) - F(x) = \int_{y+h}^{x+h} F(z)dz - \int_y^x F(z)dz = \int_y^x F(z+h) - F(z)dz$$

so it follows by the given assumputions that

$$\frac{|F(x+h)-F(y+h)+F(y)-F(x)|}{h} \leq \frac{1}{h}\int_{\mathbb{R}} |F(z+h)-F(z)|dz \to 0$$

but as x and y are Lebesgue points $\frac{|F(x+h)-F(y+h)+F(y)-F(x)|}{h} \to |f(x)-f(y)|$. Therefore, we conclude if $x, y \in E$ then $f(x) = f(y)$ i.e. f is constant a.e. and the only constant in L^1 is 0 so $f = 0$ a.e.

□

Problem 2. Given $f \in L^2(\mathbb{R})$ and $h > 0$ define

$$h^2 Q(f,h) := \int_{\mathbb{R}} (2f(x) - f(x+h) - f(x-h))f(x)dx$$

(1) Show that

$$Q(f,h) \geq 0 \text{ for all } f \in L^2(\mathbb{R}) \text{ and all } h > 0$$

(2) Show that the set

$$E := \{f \in L^2(\mathbb{R}) : \limsup_{h\to 0} Q(f,h) \leq 1\}$$

is closed in $L^2(\mathbb{R})$

Proof. For thje first part observe that

$$h^2 Q(f,h) = 2(f,f) - (f_h, f) - (f_{-h}, f) \geq 2||f||^2_{L^2} - ||f||^2_{L^2} - ||f||^2_{L^2} = 0$$

where $(\cdot,\cdot)$ is the L^2 inner product, $f_h(x) := f(x+h)$, and we used Cauchy-Schwarz.

For the second part observe that formally we expect that

$$\limsup_{h\to 0} Q(f,h) = \int_{\mathbb{R}} -f''(x)f(x)dx = \int_{\mathbb{R}} |f'(x)|^2$$

where the last equality is due to integration by parts, but we do not have that f is smooth or that the operator is uniformly bounded in h, so we remedy this by recalling that the Fourier Transform converts differentiation into multiplication by polynomials. So by taking the fourier transform, we expect that E

will be a space where $(\hat{f}(\xi)\cdot\xi) \in L^2(\mathbb{R})$ with $||\hat{f}(\xi)\cdot\xi||_{L^2(\mathbb{R})} \leq 1$ which will be a closed subspace. Hence, we are motivated to use Plancheral to rewrite $Q(f,h)$ as

$$Q(f,h) = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (\frac{2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}}{h^2})$$

so if $f \in E$ we have by Fatou's Lemma that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 4\pi^2\xi^2 d\xi \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (\frac{2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}}{h^2})$$

$$\leq \limsup_{n\to\infty} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (\frac{2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}}{h^2}) \leq 1$$

where the first inequality is due to recognizing the exponential terms as a second finite difference scheme for its negative second derivative.

Now let $f_n \in E$ such that $f_n \to f$ in L^2 then by passing along subsequences we can assume $f_n \to f$ and $\hat{f}_n \to \hat{f}$. Therefore, by Fatou's Lemma we have that

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 4\pi^2\xi^2 \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |\hat{f_n}(\xi)|^2 4\pi^2\xi^2 \leq 1$$

So now it suffices to show

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 4\pi^2\xi^2 d\xi = \lim_{h\to 0} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 (\frac{2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}}{h^2}) d\xi$$

to deduce that $f \in E$. But this follows from the Dominated Convergence Theorem since

$$(\frac{2 - e^{2\pi i h\xi} - e^{-2\pi i h\xi}}{h^2}) \lesssim \xi^2$$

for h small enough where the implies constant does not depend on h.

□

Problem 3. Suppose $f \in L^1(\mathbb{R})$ satisfies

$$\limsup_{\varepsilon\to 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2 + \varepsilon^2} dxdy < \infty$$

Show that $f = 0$ almost everywhere.

Proof. By the monotone convergence theorem, we deduce that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2} < \infty$$

Now we assume for the sake of contradiction that $f \neq 0$ a.e., so there is a $\delta > 0$ such that $m(\{x : |f(x)| > \delta\}) \neq 0$ where m is the Lebesgue measure on $\mathbb{R}^1$. Write $A(\delta) := \{x : |f(x)| > \delta\}$ then as $\chi_{A(\delta)} \in L^1_{loc}(\mathbb{R})$ we deduce by Lebesgue differentiation theorem that there is a $z \in A(\delta)$ such that for any $\varepsilon > 0$ we can find a $\eta > 0$ such that

$$m(B(\eta, z) \cap A(\delta)) \geq (1-\varepsilon) m(B(\eta, z)) = 2(1-\varepsilon)\eta$$

Now observe that

$$B(\eta, z) = \bigcup_{n=0}^{\infty} B(\eta 2^{-n}, z) \setminus \overline{B(\eta 2^{-n-1}, z)} := \bigcup_{n=0}^{\infty} B(n)$$

so we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x)f(y)|}{|x-y|^2} dxdy \geq \int_{A(\delta)\cap B(\eta,z)} \int_{A(\delta)\cap B(\eta,z)} \frac{|f(x)f(y)|}{|x-y|^2} dxdy$$

$$\geq \delta \sum_{m=0}^{\infty} \int_{A(\delta)\cap B(n)} \int_{A(\delta)\cap B(n)} \frac{1}{|x-y|^2} \geq \delta \sum_{m=0}^{\infty} \frac{(m(A(\delta)\cap B(n)))^2}{8\eta^2 2^{-2(n+1)}}$$

Now notice that most of the measure of this annular partition of $B(\eta, z)$ lives in the first few annuli, so this implies there is an $N(\varepsilon)$ such that if $n \leq N(\varepsilon)$ then $m(A(\delta) \cap B(n)) \geq 1/2m(B(n))$ which follows from $m(B(\eta, z) \cap A(\delta)) \geq (1-\varepsilon)m(B(\eta, z))$ and $N(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Therefore,

$$\geq \frac{\delta}{2} \sum_{m=0}^{N(\varepsilon)} \frac{\eta^2 2^{-2(n+1)}}{8\eta^2 2^{-2(n+1)}} = \frac{\delta}{2} \sum_{m=0}^{N(\varepsilon)} \frac{1}{8} \to \infty$$

as $\varepsilon \to 0$, so we have arrived at a contradiction.

Alternative Proof via Lebesgue Differentiation Theorem As $f \in L^1(\mathbb{R})$ we have that the set of Lebesgue Points is a set of full measure. Let x_0 be a Lebesgue point of f then

$$f(x_0) = \lim_{h \to 0} \frac{1}{2h} \int_{-h}^{h} f(x + x_0) dx$$

and we have for any $h > 0$

$$\left(\frac{1}{h} \int_{x_0-h}^{x_0+h} |f(x)| dx \right)^2 = \left(\int_{x_0-h}^{x_0+h} \frac{|f(x)|}{h} dx \right) \left(\int_{x_0-h}^{x_0+h} \frac{|f(y)|}{h} dy \right) = \int_{x_0-h}^{x_0+h} \int_{x_0-h}^{x_0+h} \frac{|f(x)||f(y)|}{h^2} dxdy$$

where the interswap of integrand is justified by Fubini since the integrand is non-negative.

$$\leq 4 \int_{x_0-h}^{x_0+h} \int_{x_0-h}^{x_0+h} \frac{|f(x)||f(y)|}{|x-y|^2} dxdy \to 0$$

where the convergence to zero as $h \to 0$ is due to the itegrand being in L^1 thanks to the give assumption combined with the monotone convergence theorem. Hence, for every lebesgue point of f we have $f = 0$, so $f = 0$ a.e.

□

Problem 4. Fix $1 < p < \infty$. Show that

$$f \mapsto [Mf](x, y) := \sup_{r > 0, \rho > 0} \frac{1}{4r\rho} \int_{-r}^{r} \int_{-\rho}^{\rho} f(x + h, y + \ell) dh d\ell$$

is bounded on $L^p(\mathbb{R}^2)$.

Show that

$$[A_r f](x, y) := \frac{1}{4r^3} \int_{-r}^{r} \int_{-r^2}^{r^2} f(x + h, y + \ell) dh\ell$$

converges to f a.e. as $r \to 0$.

Proof. For the first part, we need the following result: If $1 < p < \infty$ then for

$$Tf(x) := \sup_{r > 0} \int_{-r}^{r} \frac{1}{2r} |f(y + x)| dx$$

we have

$$\int_{\mathbb{R}} |Tf(x)|^p \lesssim \int_{\mathbb{R}} |f(x)|^p$$

So observe that by Layer Cake Decomposition that

$$\int_{\mathbb{R}} |Tf(x)|^p dx = \int_0^\infty p s^{p-1} m(\{x : |Tf(x)| \geq s\}) ds$$

So as we have

$$f(x) = f(x)\chi_{|f(x)| \leq s/2} + f(x)\chi_{|f(x)| \geq s/2} := g(x) + h(x)$$

we deduce that from $||Tf||_{L^\infty} \leq ||f||_{L^\infty}$ if

$$s \leq |Tf(x)| = |T(g(x) + h(x))| \leq |Tg(x)| + |Th(x)| \leq s/2 + |Tg(x)|$$

i.e.

$$m(\{x : |Tf(x)| \geq s\}) \leq m(\{x : |Th(x)| \geq s/2\})$$

so in particular, we have that

$$\int_{\mathbb{R}} |Tf(x)|^p \leq \int_0^\infty ps^{p-1} m(\{x : |Th(x)| \geq s/2\}) \lesssim \int_0^\infty s^{p-2} (\int_{\mathbb{R}} |f(x)| \chi_{|f(x)| \geq s/2})$$

where the last inequality is due to Hardy Littlewood Maximal Inequality. Then Tonelli gives

$$\leq \int_{\mathbb{R}} |f(x)| \int_0^{2|f(x)|} s^{p-2} ds dx \lesssim \int_{\mathbb{R}} |f(x)|^p$$

as desired.

Now we use this Lemma on our problem. Observe for $f_x(y) := f(x, y)$ that

$$\frac{1}{4r\rho} \int_{-r}^{r} \int_{-\rho}^{\rho} |f(x + h, y + \ell)| dh d\ell \leq \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} \sup_{\rho>0} \frac{1}{2\rho} \int_{-\rho}^{\rho} |f_{y+\ell}(x + h)| dh d\ell$$

Then observe $\sup_{\rho>0} \frac{1}{2\rho} \int_{-\rho}^{\rho} |f_{y+\ell}(x + h)| dh d\ell = T(f_{y+\ell}(x))$ where T was defined earlier as the Hardy Littlewood Maximal Operator. So we have

$$|Mf(x, y)| \leq T(T(f_{y+\ell}(x)))$$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |Mf(x, y)|^p dy dx \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |T(f_{y+\ell}(x))|^p dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}} |T(f_{y+\ell}(x))|^p dx dy$$

$$\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x, y + \ell)|^p dx dy = ||f||_{L^p(\mathbb{R})}^p$$

so Mf is a bounded linear operator from L^p to L^p.

For the second part, it'll be identical to the usual proof of Hardy Little Maximal Inequality implies Lebesgue Differentiation Theorem since thanks to the first part lets us obtain a similar estimate as the Hardy Little Maximal Inequality via Chebyshev's inequality. □

Problem 5. Let μ be real valued Borel measure on $[0, 1]$ such that

$$\int_0^1 \frac{1}{x + t} d\mu(t) = 0 \text{ for all } x > 1$$

Show that $\mu = 0$.

Proof. We will show that if $f(t)$ is continuous then

$$\int_0^1 f(t) d\mu(t) = 0$$

which will imply the claim. Observe that μ is finite since it is real valued everywhere and $\mu([0, 1]) \in \mathbb{R}$. Observe that for $h > 0$ we have

$$0 = \frac{1}{h} \int_0^1 \frac{1}{x + t} - \frac{1}{x + h + t} d\mu(t) = \int_0^1 \frac{1}{(x + t)(x + h + t)} d\mu(t)$$

So letting $h \to 0$ with the DCT shows that (since $1/(x + t)(x + h + t) \leq 1/x^2 \leq 1 \in L^1(d\mu, [0, 1])$)

$$\int_0^1 \frac{1}{(x + t)^2} d\mu(t) = 0$$

and repeating this argument n times shows for any $n \in \mathbb{N}$

$$\int_0^1 \frac{1}{(x + t)^n} d\mu(t) = 0$$

so in particular

$$\int_0^1 \sum_{n=1}^{N} \frac{a_n}{(x + t)^n} d\mu(t) = 0$$

where $a_n \in \mathbb{N}$. Observe that $\mathcal{A}$ defined to be the set of finite linear combinations of $1/(2+t), 1/(2+t)^2, ..., 1/(2+t)^n$ forms an algebra. It vanishes nowhere since $1/(2+t) \neq 0$ for $t \in [0,1]$ and it separates points since if $t_1 \neq t_2$ then $1/(2+t_1) \neq 1/(2+t_2)$ since $1/(2+t)$ is injective on $t \in [0,1]$. Therefore, by Stone Weiestrass, $\mathcal{A}$ is dense in $C([0,1])$ with the sup-norm, so it follows by uniform convergence that if $f(t)$ is continuous then

$$\int_0^1 f(t)d\mu(t) = 0$$

which implies $\mu = 0$

□

Problem 6. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$ and let $\mathcal{P}(\mathbb{T})$ denote the space of Borel Probability Measures on $\mathbb{T}$ and $\mathcal{P}(\mathbb{T}\times\mathbb{T})$ denote the space of Borel Probability Measures on $\mathbb{T}\times\mathbb{T}$. Fix $\mu, \nu \in \mathcal{P}(\mathbb{T})$ and define

$$\mathcal{M} := \{\gamma \in \mathcal{P}(\mathbb{T}\times\mathbb{T}) : \int\int_{\mathbb{T}\times\mathbb{T}} f(x)g(y)d\gamma(x,y) = \int_{\mathbb{T}} f(x)d\mu(x) \cdot \int_{\mathbb{T}} g(y)d\mu(y) \text{ for all } f, g \in C(\mathbb{T})\}$$

Show that $F : \mathcal{M} \to \mathbb{R}$ defined by

$$F(\gamma) := \int\int_{\mathbb{T}\times\mathbb{T}} \sin^2(\frac{\theta-\phi}{2})d\gamma(e^{i\theta}, e^{i\phi})$$

achieves its minimum on $\mathcal{M}$.

Proof. Let $\{\gamma_n\} \subset \mathcal{P}(\mathbb{T}\times\mathbb{T})$ be a minimizing sequence i.e. $F(\gamma_n) \to \inf_\gamma F(\gamma)$ then by Banach Alagou and Risez Representation Theorem (since $C(\mathbb{T})$ is separable, we can upgrade weak* compactness to weak* subsequential compactness on bounded subsets) we can find a subsequence which we still denote by $\{\gamma_n\}$ such that γ_n weak star converges to γ. That is if $f(x,y) \in C(\mathbb{T}\times\mathbb{T})$ then

$$\int\int f(x,y)d\gamma_n \to \int\int f(x,y)d\gamma$$

Taking $f \equiv 1$ implies $\gamma \in \mathcal{P}(\mathbb{T}\times\mathbb{T})$. Also observe that if $f, g \in C(\mathbb{T}) \Rightarrow f(x)g(y) \in C(\mathbb{T}\times\mathbb{T})$ so we have

$$\int\int_{\mathbb{T}\times\mathbb{T}} f(x)g(y)d\gamma(x,y) = \lim_{n\to\infty} \int\int_{\mathbb{T}\times\mathbb{T}} f(x)g(y)d\gamma_n(x,y) = \int_{\mathbb{T}} f(x)d\mu(x) \cdot \int_{\mathbb{T}} g(y)d\mu(y)$$

so $\gamma \in \mathcal{M}$. Also observe that $\sin^2(\frac{\theta-\phi}{2}) \in C(\mathbb{T}\times\mathbb{T})$ so weak* convergence implies

$$F(\gamma_n) \to F(\gamma)$$

but as γ_n is a minimizing sequence this implies γ is a minimizer.

□

Problem 7. Let $F : \mathbb{C}\times\mathbb{C} \to \mathbb{C}$ be (jointly) continuous and holomorphic in each variable. Show that $z \mapsto F(z,z)$ is holomorphic.

Proof. Fix an $R > 0$ then if $|w| < R$ then we have from the Residue Theorem that

$$F(z,w) = \frac{1}{2\pi i}\int_{|\xi|=R} \frac{F(\xi,w)}{(\xi - z)}d\xi$$

Now using the residue formula again gives

$$F(\xi,w) = \frac{1}{2\pi i}\int_{|\eta|=R} \frac{F(\xi,\eta)}{(\eta - w)}d\eta$$

so it follows

$$F(z,w) = -\frac{1}{4\pi^2}\int_{|\xi|=R}\int_{|\eta|=R} \frac{F(\xi,\eta)}{(\xi - z)(\eta - w)}d\xi d\eta$$

i.e.

$$F(z,z) = -\frac{1}{4\pi^2}\int_{|\xi|=R}\int_{|\eta|=R}\frac{F(\xi,\eta)}{(\xi-z)(\eta-z)}d\xi d\eta$$

Now let $R \subset B_R(0)$ be a reactangle, then

$$\int_{\partial R} F(z,z)dz = \frac{1}{4\pi^2}\int_{\partial R}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi}\frac{F(Re^{i\varphi},Re^{i\theta})}{(Re^{i\varphi}-z)(Re^{i\theta}-z)}R^2e^{i(\varphi+\theta)}d\varphi d\theta dz$$

$$= \frac{1}{4\pi^2}\int_{\theta=0}^{2\pi}\int_{\varphi=0}^{2\pi}\int_{\partial R}\frac{F(Re^{i\varphi},Re^{i\theta})}{(Re^{i\varphi}-z)(Re^{i\theta}-z)}R^2e^{i(\varphi+\theta)}d\varphi d\theta dz = 0$$

since the integrand is holomorphic in z and the swap of integrals is justified by continuity of the integrand which on $z \in B_R(0)$ (just parametrize ∂R and convert the contour integral into a regular integral then undo the parametrization on the last step for the rectangle). And as $F(z,z)$ is continuous, we deduce from Morrera's that $F(z,z)$ is holomorphic on $B_R(0)$. And as R was arbitrary we conclude $F(z,z)$ is entire. □

Problem 8. Determine the supremum of

$$|\frac{\partial u}{\partial x}(0,0)|$$

among all harmonic functions $u : \mathbb{D} \to [0,1]$. Prove that your answer is correct.

Proof. Fix $0 < R < 1$ and define $u_R(z) := u(Rz) \in C(\overline{D})$ and is harmonic on $\mathbb{D}$. So by Poisson Integral Formula we have for $0 < r < 1$ that

$$u_R(re^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{|e^{i\theta}-re^{i\varphi}|^2}u(Re^{i\theta})d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}u(Re^{i\theta})d\theta$$

so now we have by identifying $re^{i\varphi} = r\cos(\varphi) + ir\sin(\varphi) = x + iy$ that for $x > 0$

$$u_R(x,0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-x^2}{1-2x\cos(\theta)+x^2}u(Re^{i\theta})d\theta$$

Hence

$$\frac{\partial u_R}{\partial x}(x,0) = \frac{1}{2\pi}\int_0^{2\pi}\frac{-2x(1-2x\cos(\theta)+x^2)-(-2\cos(\theta)+2x)(1-x^2)}{(1-2x\cos(\theta)+x^2)^2}u(Re^{i\theta})d\theta$$

where differentiating under the integral sign is fine since $R < 1$ so the Poisson Kernal is smooth in $\overline{B_R(0)}$. In particular,

$$\frac{\partial u_R}{\partial x}(0,0) = \frac{1}{2\pi}\int_0^{2\pi}2\cos(\theta)u_R(Re^{i\theta})d\theta$$

so we have from $0 \le u \le 1$ that

$$\frac{\partial u_R}{\partial x}(0,0) \le \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}2\cos(\theta)d\theta = \frac{2}{\pi}$$

and

$$\frac{\partial u_R}{\partial x} = R\frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x}(0,0) \le \frac{2}{\pi}$$

so define $f : S^1 \to [0,1]$ via

$$f = \begin{cases} 1 \text{ on } \theta \in [-\pi/2,\pi/2] \\ 0 \text{ else} \end{cases}$$

then

$$u(z) := \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|}f(e^{i\theta})d\theta$$

is the desired harmonic function with these properties. Note that the usual proof of if $f \in C(\partial\mathbb{D})$ then the usual proof of

$$\lim_{r\to 1} u(re^{i\theta}) := \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{i\theta}-z|}f(e^{i\theta})d\theta = f(e^{i\theta})$$

extends to bounded functions at every point of continuity of f. Therefore, $u(re^{i\theta}) \to \chi_{[-\pi/2,\pi/2]}$ everywhere except at $\theta = \pm\pi/2$ i.e. u obtains the boundary data f a.e., so by our above computation our function u obtains the boundary data.

Alternative Proof By the mean value property we have for any $0 < r < 1$

$$\frac{\partial u}{\partial x}(0,0) = \frac{1}{\pi r^2}\int_{B_r(0)} \frac{\partial u}{\partial x}(z)dA(z) = \frac{1}{\pi r^2}\int_{\partial B_r(0)} un_1 d\sigma$$

where n_1 is the first component of the unit normal on $\partial B_r(0)$ i.e. x/r.

$$= \frac{1}{\pi r^2}\int_{\theta=0}^{2\pi} ru(re^{i\theta})\cos(\theta)d\theta = \frac{1}{\pi r}\int_{\theta=0}^{2\pi} u(re^{i\theta})\cos(\theta)d\theta$$

Since $0 \le u \le 1$ we deduce

$$\frac{\partial u}{\partial x}(0,0) \le \frac{1}{\pi r}\int_{\theta=-\pi/2}^{\pi/2} \cos(\theta)d\theta = \frac{2}{\pi r}$$

So sending $r \to 1$ we deduce that

$$\frac{\partial u}{\partial x}(0,0) \le \frac{2}{\pi}$$

And from construction we see that the max must be 1 on $[-\pi/2, \pi/2]$ and 0 else, so we use the same function as before. ☐

Problem 9. Consider the formal product

$$\prod_{n=1}^{\infty}(1+\frac{1}{n})^z(1-\frac{z}{n})$$

(1) Show that the product converges for any $z \in (-\infty, 0)$

(2) Show that this resulting function extends to an entire function.

Proof. Notice if $a_n := (1+\frac{1}{n})^z(1-\frac{z}{n})$ then $a_n \to 1$ as $n \to \infty$, so by looking at the tail of the product if necessary, we can assume $a_n \in B_{1/2}(1)$. This means we can define the standard complex logarithim i.e. $\log(z)$ where $\theta \in (-\pi, \pi]$ and $\log(a_n)$ is well defined. So now by taking logs we see that by taking limits

$$\log(\prod_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty}\log(a_n) = \sum_{n=1}^{\infty}\log(a_n - 1 + 1) = \sum_{n=1}^{\infty}(a_n - 1) + O((a_n-1)^2)$$

so the product converges iff

$$\sum_{n=1}^{\infty}(a_n - 1)$$

converges. So we need to show the following sum converges

$$\sum_{n=1}^{\infty}|(1+\frac{1}{n})^z(1-\frac{z}{n}) - 1|$$

Observe that

$$\frac{\partial}{\partial x}(1+x)^z = z(1+x)^{z-1}$$

so Taylor Expansion at $x = 0$ gives

$$(1+x)^z = 1 + zx + O(x^2)$$

so we have

$$(1+\frac{1}{n})^z(1-\frac{z}{n}) = (1-\frac{z}{n})(1+\frac{z}{n}) + O_{|z|}(1/n^2) = 1 - \frac{z^2}{n^2} + O_{|z|}(1/n^2)$$

Therefore,

$$\sum_{n=1}^{\infty}|(1+\frac{1}{n})^z(1-\frac{z}{n} - 1)| = \sum_{n=1}^{\infty}\frac{|z|^2}{n^2} + O_{|z|}(1/n^2)$$

which converges.

For the second part, notice that for any $z \in \mathbb{C}$ we can define the complex log for a_n so we have $(1+1/n)^z = \exp(z\log(1+1/n))$ is a well defined holomorphic function (where we are using the standard log branch). And by our earlier computations the sum converges locally uniformly on compact subsets of $\mathbb{C}$ so

$$\prod_{n=N}^{M} a_n = \exp(\sum_{n=N}^{M}(a_n-1)+O((a_n-1)^2)) = \exp(\sum_{n=N}^{M} O(\frac{|z|}{n^2}))$$

so the product converges locally uniformly, hence the limit is holomorphic.

□

Problem 10. Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann Sphere and $\Omega := \mathbb{C}^* \setminus \{0,1\}$. Let $f : \Omega \to \Omega$ be a holomorphic function.

(1) Prove that if f is injective then $f(\Omega) = \Omega$

(2) Make a list of all such injective functions.

Proof. Let

$$\varphi(z) := \frac{z}{z-1}$$

then φ is a Mobius Transformation such that $\varphi(0) = 0$, $\varphi(1) = \infty$, and $\varphi(\infty) = 1$. In particular, if $U := \mathbb{C} \setminus \{0\}$ then $\varphi(U) = \Omega$ so

$$g(z) := \varphi^{-1} \circ f \circ \varphi : U \to U$$

And since φ is conformal it suffices to show that if g is injective then it is surjective.

Now g is a holomorphic function on $\mathbb{C} \setminus \{0\}$. By Great Picard Theorem the singularity at 0 is either removable or a pole.

Removable Singularity Case If $g(z)$ has a removable singularity at 0 then g extends to be an injective entire function. Indeed, if $g(0) = g(w)$ for some $w \neq 0$, the open mapping theorem implies that g is not injective on $\mathbb{C}\setminus\{0\}$ (since a small ball around 0 and around w maps to a ball around $g(0) = g(w)$). Therefore, $g(z)$ is an injective entire function so g is a linear function i.e. $g(z) = az + b$, so it follows that g is surjective. And as $g(z) \neq 0$ for $z \neq 0$ it follows that $b = 0$ so $g(z) = az$.

Pole Case If $g(z)$ has a pole at $z = 0$, then $1/g$ has a removable singularity at $z = 0$ and since g does not map to 0, so we see $1/g$ is an injective holomorphic function on $\mathbb{C} \setminus \{0\}$ that has a removable singularity at $z = 0$. By applying our previous case, we deduce that $1/g(z) = az + b \Rightarrow g(z) = \frac{1}{az+b}$. As $g(z)$ has a pole at $z = 0$ we must have $b = 0$ and $a \neq 0$, so $g(z) = 1/(az)$ which is a Mobius Transformation and $g(0) = \infty$ with $g(\infty) = 0$, so it follows from Mobius Transformations being conformal that g is surjective on U.

By our previous considerations we must have

$$g(z) = 1/(az) \text{ or } az$$

for some $a \in \mathbb{C}$ that is not 0. So by undoing our mobius transformations we deduce that

$$f(z) = \varphi \circ (1/az) \circ \varphi^{-1} \text{ or } \varphi \circ (az) \circ \varphi^{-1}$$

□

Problem 11. For $R > 1$ let A_R be the annulus $\{1 < |z| < R\}$. Assume there is a conformal map F from A_{R_1} onto A_{R_2}. Prove that $R_1 = R_2$.

Proof. See Spring 2017 Number 7. ☐

Problem 12. Let $f(z)$ be a bounded and holomorphic on $\mathbb{D}$. Prove that for any $w \in \mathbb{D}$ that we have

$$f(z) = \frac{1}{\pi}\int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}w)^2}dz$$

Proof. Let $w \in \mathbb{C}$ then define

$$\psi_w(z) := \frac{w-z}{1-\overline{z}w}$$

then ψ_w is an auotmorphism of the disk such that $\psi_w(w) = 0$ and $\psi_w(0) = w$. Therefore, notice if $f \circ \psi_w : \mathbb{D} \to \mathbb{C}$ so the mean value property tells us that for $0 < r < 1$

$$f(w) = f(\psi_w(0)) = \frac{1}{\pi r^2}\int_{B_r(0)} f \circ \psi_w$$

and as f is bounded we have $f \in L^1(\mathbb{D})$ so we can take limits to get

$$f(\psi_w(0)) = \frac{1}{\pi}\int_{\mathbb{D}} f \circ \psi_w dz = \frac{1}{\pi}\int_{\mathbb{D}} f(z)|(\psi_w^{-1}(z))'|^2 dz$$

where the last equality is due to change of variables and that ψ_w is an automorphism so its inverse derivative is well defined by the inverse function theorem. Note that

$$\psi_w^{-1}(z) = \frac{z-w}{1-\overline{w}z} \Rightarrow \partial_z(\psi_w^{-1}(z)) = \frac{1-|w|^2}{(1-\overline{w}z)^2}$$

so in particular,

$$f(w) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)(\frac{1-|w|^2}{(1-\overline{w}z)^2})(\frac{1-|w|^2}{(1-w\overline{z})^2})dz = \frac{1}{\pi}(f(z)\psi_w'(z), \psi_w'(z))$$

so as $\psi_w' \neq 0$ anywhere, we have that if F is holomorphic then

$$\frac{1}{\pi}(F(z), \psi_w'(z)) = \frac{1}{\pi}(\psi_w'\frac{F}{\psi_w'}, \psi_w') = F(w)/\psi_w'(w)$$

It therefore, follows that

$$\frac{1}{\pi}(F(z), \overline{\psi_w'(w)}\psi_w'(z)) = F(w)$$

and

$$\psi_w'(w) = \frac{1-|w|^2}{(1-|w|^2)^2} = \frac{1}{1-|w|^2}$$

i.e.

$$f(w) = \frac{1}{\pi}\int_{\mathbb{D}} f(z)\frac{1}{(1-\overline{z}w)^2}dz$$

as desired. ☐

18. Fall 2018

Problem 1. Let $\{f_n\}$ be a sequence of real-valued Lebesgue measurable functions on $\mathbb{R}$, and let f be another such function. Assume that

(1) $f_n \to f$ Lebesgue a.e.
(2) $\int_{\mathbb{R}} |x||f_n|dx \leq 100$, for all n, and
(3) $\int_{\mathbb{R}} |f_n(x)|^2 dx \leq 100$

Prove that $f_n \in L^1$ for all n, that $f \in L^1$, and that $||f_n - f||_{L^1} \to 0$. Also show that neither assumptions (2) nor (3) can be omitted while making these deductions.

Proof. Notice that for any n we have

$$\int_{\mathbb{R}} |f_n|dx = \int_{|x|\leq 1} |f_n|dx + \int_{|x|\geq 1} |f_n|dx \leq \sqrt{2}\left(\int_{|x|\leq 1} |f_n|^2\right)^{1/2} + \int_{|x|>1} |x||f_n|$$
$$\leq \sqrt{2}||f_n||_{L^2(\mathbb{R})} + 100 \leq 10\sqrt{2} + 100 =: C$$

where we used Holder's inequality and the given inequalities. Therefore, $f_n \in L^1(\mathbb{R})$ and by Fatou's Lemma

$$\int_{\mathbb{R}} |f|dx = \int_{\mathbb{R}} \liminf_{n\to\infty} |f_n|dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f_n|dx \leq C$$

so $f \in L^1(\mathbb{R})$ and similarly Fatou implies that

$$\int_{\mathbb{R}} |x||f_n| \leq 100 \text{ and } \int_{\mathbb{R}} |f_n|^2 dx \leq 100$$

Now observe that for any $M > 0$ that

$$\int_{\mathbb{R}} |f - f_n|dx = \int_{|x|\leq M} |f - f_n|dx + \int_{|x|\geq M} \frac{|x|}{|x|}|f - f_n|dx$$
$$\leq \int_{|x|\leq M} |f - f_n|dx + \frac{1}{M}\int_{|x|\geq m} |x||f - f_n|dx$$
$$\leq \int_{|x|\leq M} |f - f_n|dx + \frac{200}{M}$$

so if $\varepsilon > 0$ choose M so large such that $200/M < \varepsilon/3/$ Also as $\overline{B_M(0)}$ is a set of finite measure, we know by Egorov's Theorem that for any $\delta > 0$ we can find a set $K \subset \overline{B_M(0)}$ such that $f_n \to f$ uniformly on K. So we choose $N \in \mathbb{N}$ so large such that if $n \geq N$ then $\sup_{x\in K} |f(x) - f_n(x)| \leq \varepsilon/(6M)$ then

$$\leq \int_K |f - f_n|dx + \int_{\overline{B_M(0)}\setminus K} |f - f_n| + \varepsilon/3$$
$$\leq \varepsilon/3 + \left(\int_{\overline{B_M(0)}\setminus K} |f - f_n|^2\right)^{1/2} \delta^{1/2} + \varepsilon/3$$
$$\leq 2\varepsilon/3 + 200\delta^{1/2}$$

so choosing $\delta > 0$ such that $200\delta^{1/2} < \varepsilon/3$ gives

$$\leq \varepsilon$$

so $f_n \to f$ in L^1.

Now if (2) is satisfied but (3) this means we do not necessarily have control of the function near the origin. Indeed, consider

$$f_n(x) := n\chi_{[0,1/n]}(x) \Rightarrow f_n \to 0 \text{ a.e.}$$

with

$$\int_{\mathbb{R}} |f_n|dx = 1, \int_{\mathbb{R}} |f_n|^2 = n \to \infty, \int_{\mathbb{R}} |x||f_n| \leq \int_{\mathbb{R}} |f_n| = 1$$

and f_n clearly does not converge to 0 in L^1 since each f_n has mass 1.

Now if (3) is satisfied but (2) is not, then we do not have control of the tail. Indeed consider

$$f_n(x) := \frac{1}{n}\chi_{[0,n]}(x) \Rightarrow \int_{\mathbb{R}} |f_n|^2 = \frac{1}{n}, \int_{\mathbb{R}} |x||f_n| = \int_0^n x/n = n/2 \to \infty$$

and f_n pointwise goes to 0 but clearly does not go to 0 in L^1 since each f_n has mass 1.

□

Problem 2. Let (X, ρ) be a compact metric space which has at least two points, and let $C(X)$ be the space of continuous functions on X with the sup norm. Let D be a dense subset of X and for each $y \in D$ define $f_y \in C(X)$ by

$$f_y(x) = \rho(x, y)$$

Let A be the sub-algebra generated by the collection f_y (with pointwise addition and multiplication of functions).

(1) Prove that A is dense in $C(X)$ under the uniform norm.
(2) Prove that $C(X)$ is separable.

Proof. For the first part, we will use Stone-Weiestrass. First notice that since X has at least two points so does D since the two points are a non-zero distance away from one another. We claim A seprates points. Indeed, let $x, y \in X$ with $x \neq y$ then $d(x, y) = \delta > 0$. As D is dense there is an $x_n \in D$ such that $d(x, x_n) < \delta/2$ then we have

$$0 \leq f_{x_n}(x) < \delta/2$$

and we also have from the triangle inequality that

$$f_{x_n}(y) = d(x_n, y) \geq d(x, y) - d(x_n, y) \geq \delta - \delta/2 = \delta/2 > f_{x_n}(x)$$

Therefore, $f_{x_n}(x) \neq f_{x_n}(y)$, so A separates points.

Now it vanishes nowhere since if $x \in X$ then as X has at least two points there is a $y \in X$ such that $x \neq y \iff d(x, y) = \delta > 0$. then there is a $y_n \in D$ such that $d(y, y_n) < \delta/2$ which implies

$$f_{y_n}(x) = d(x, y_n) \geq d(x, y) - d(y, y_n) \geq \delta/2 > 0$$

so as X is a compact metric space, Stone Weiestrass implies that A is dense in $C(X)$.

For the second part, we know that as X is compact, it is totally bounded. Therefore, X is separable so we can find a countable dense set E and we can similarly define A_E to be the sub-algebra generated by f_y for $y \in E$. Then $\overline{A_E} = C(X)$ and let

$$S := \{ \text{ finite rational polynomial combinations of } f_y \}$$

i.e. $f \in S$ if $f = \sum_{j=1}^{M} c_j \prod_{i=1}^{N(j)} f_{y_i}$ where $c_j \in \mathbb{Q}$ then S is countable and $\overline{S} = A_E$, so S is a countable dense set in $C(X)$ so $C(X)$ is separable.

□

Problem 3. Let (X, ρ) be a compact metric space and let $P(X)$ be the set of all positive probability measures on the Borel σ-algebra. Assume that μ_n weakly converges to μ. Prove that $\mu_n(E) \to \mu(E)$ for all E with $\mu(\overline{E}) = \mu(\text{Int}(E))$.

Proof. If F is a closed set then we know that $\chi_F(x)$ is upper semi-continuous so there exists a sequence of $f_n \in C(X)$ such that $f_n \geq \chi_F$ and $f_n \to \chi_F$ pointwise. Then we have for any m

$$\limsup_{n\to\infty} \int_X \chi_F d\mu_n \leq \limsup_{n\to\infty} \int_X f_m d\mu_n = \int_X f_m d\mu$$

so then as $f_m \to \chi_F$ and $f_1 \geq f_m$ for any m with $f_1 \in L^1(X, d\mu)$ (this follows from X is compact so f_1 is bounded and μ being a probability measure), so by DCT we know $\int_X f_m d\mu \to \int_X \chi_F d\mu$ i.e.

$$\limsup_{n\to\infty} \int_X \chi_F d\mu_n \leq \mu(F)$$

Now if G is an open set, we know that χ_G is lower semi continuous so we can find $f_n \in C(X)$ such that $\chi_G \geq f_n$ and $f_n \to \chi_G$. Therefore,

$$\liminf_{n\to\infty} \int_X \chi_G d\mu_n \geq \liminf_{n\to\infty} \int_X f_m d\mu_n = \int_X f_m d\mu$$

so applying the monotone convergence theorem gives

$$\liminf_{n\to\infty} \int_X \chi_G d\mu_n \geq \int_X \chi_G d\mu = \mu(G)$$

Now by applying these two lemmas we see that for any set E with the given conditions that

$$\limsup_{n\to\infty} \mu_n(E) \leq \limsup_{n\to\infty} \mu_n(\overline{E}) \leq \mu(\overline{E}) = \mu(E) = \mu(\text{Int}(E)) \leq \liminf_{n\to\infty} \mu_n(\text{Int}(E)) \leq \liminf_{n\to\infty} \mu_n(E)$$

which implies

$$\mu(E) = \lim_{n\to\infty} \mu_n(E)$$

as desired.

□

Problem 4. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and for each $\alpha \in \mathbb{T}$ define the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ by $R_\alpha(z) = \alpha z$. A Borel probabiltiy measure μ on $\mathbb{T}$ is called α-invariant if $\mu(R_\alpha(E)) = \mu(E)$ for any Borel set $E \subset \mathbb{T}$.

(1) Let m be the Lebesgue measure defined on $\mathbb{T}$ (defined, for instance, by identifying $\mathbb{T}$ with $[0, 1)$ through the exponential map). Show that for each $\alpha \in \mathbb{T}$ that m is also α-invariant.
(2) Prove that if α is not a root of unity, then the set of powers $\{\alpha^n : n \in \mathbb{Z}\}$ is dense in $\mathbb{T}$.
(3) Prove that if we fix $\alpha \in \mathbb{T}$ that is not a root of unity then the only α-invariant Borel probability measures is the Lebesgue measure.

Proof. For the first part if $\alpha \in \mathbb{T}$ then $\alpha = e^{i\psi}$ for some $\psi \in [0, 2\pi)$. So if $E = \{e^{i\theta} : \theta \in [\theta_0, \theta_1]\}$ then we have $R_\alpha(E) = \{e^{i(\theta+\psi)} : \theta \in [\theta_0, \theta_1]\} \cong [\theta_0 + \psi, \theta_1 + \psi]$ so

$$m(R_\alpha(E)) = m([\theta_0 + \psi, \theta_1 + \psi]) = m([\theta_0, \theta_1]) = m(E)$$

where in the last inequality we used that the Lebesgue measure is translation invariant for intervals. Then as E generates the Borel σ-algebra on $\mathbb{T}$, we see that for any Borel Set E that $m(R_\alpha(E)) = m(E)$.

For the second part, let α be a non root of unity, then as $\mathbb{T}$ is compact in $\mathbb{C}$ there is a subsequence which we still denote by α^n such that α^n is convergent in $\mathbb{T}$. We define $||e^{i\theta}||_{\mathbb{T}} := \inf\{\psi \in [0, 2\pi) : e^{i\psi} = e^{i\theta}\}$ i.e. we want the angle in $[0, 2\pi)$. Let the limit point be denote by β. Now if $\varepsilon > 0$ we know that evenetually for large enough n, m with $m > n$ that

$$0 < ||\alpha^n - \alpha^m||_{\mathbb{T}} < \varepsilon$$

where we have the lower bound since α is not a root of unity. This means α^{m-n} corresponds to a non trivial rotation that rotates us at most by ε degrees. Since $\varepsilon > 0$ is arbitrarily small and the rotation is non-trivial, we can rotate our fixed point β to be arbitrarily close to any other $\gamma \in \mathbb{T}$ with α^n. This means $\{\alpha^n\}$ is dense in $\mathbb{T}$.

For the third part, we claim that the Lebesgue measure m is the unique borel measure on $\mathbb{R}$ such that $m([0, 1]) = 1$ and is translation invariant. Recall by Stone Weiestrass that trignometric polynomials are dense in $C([0,1])$ under the sup-norm. So we first show that if μ is another translation invariant measure such that $\mu([0,1]) = 1$ then

$$\int_0^1 e^{2\pi ikx} d\mu(x) = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ else} \end{cases}$$

Note that the $k = 0$ case is trivial since we imposed $\mu([0,1]) = 1$. Indeed observe that

$$\int_0^1 \cos(2\pi x) d\mu(x) = \int_0^{1/2} \cos(2\pi x) d\mu(x) + \int_{1/2}^1 \cos(2\pi x) d\mu(x) = \int_0^{1/2} \cos(2\pi x) d\mu(x) - \int_0^{1/2} \cos(2\pi x) d\mu(x) = 0$$

where for the third equality we used the translation invariance of μ to change the integration to 1/2 to 1 combined with $\cos(2\pi(x + 1/2)) = -\cos(2\pi x)$. The same trick works for $e^{2\pi ikx}$ of splitting the integral to where $\cos(2\pi kx)$ or $\sin(2\pi kx)$ changes sign and using translation invariance lets us deduce that

$$\int_0^1 e^{2\pi ikx} d\mu(x) = \begin{cases} 1 \text{ if } k = 0 \\ 0 \text{ else} \end{cases}$$

Therefore, we see that for any trignometric polynomial P that

$$\int_0^1 P(x) d\mu(x) = \int_0^1 P(x) dm(x)$$

so Stone Weiestrass implies that for all $f \in C([0,1])$ that

$$\int_0^1 f(x) d\mu(x) = \int_0^1 f(x) dm(x)$$

Therefore, $\mu = m$.

So by this lemma it suffices to show that if μ is an $\alpha-$invariant Borel probability measure on $[0,1]$ then it is translation invariant. Indeed, observe if $E \subset \mathbb{T}$ is Borel then observe for all $n \in \mathbb{N}$

$$\mu(R_\alpha^n(E)) = \mu(R_\alpha^{n-1}(E)) = ... = \mu(E)$$

where the power to n indicates we do do the rotation n times. Then by part b the set $\{\alpha^n\}$ is dense in $\mathbb{T}$, so if $\beta \in S^1$ we can find a sequence n_k such that $\alpha^{n_k} \to \beta$. And we have

$$\mu(R_\alpha^{n_k} E) = \mu(E) \text{ for all } k$$

Then to take the limit observe that

$$\mu(R_\alpha^{n_k} E) = \int_{\mathbb{T}} \chi_{\alpha^{n_k} E}(x) d\mu(x)$$

and this is dominated by $\chi_{\mathbb{T}}(x)$ lets us use the DCT combined with $\chi_{\alpha^{n_k} E}(x) \to \chi_{\beta E}(x)$ to conclude

$$\mu(R_\beta E) = \lim_{k\to\infty} \mu(R_\alpha^{n_k} E) = \mu(E)$$

so μ is a translation invariant measure such that $\mu([0,1]) = 1$ so it follows that μ is the Lebesgue measure.

□

Problem 5. Let $\{f_n\}$ be a sequence of continuous real valued functions such that $f_n \to f$ pointwise everywhere on $[0,1]$.

(1) Prove that for every $\varepsilon > 0$ there is a dense set D_ε such that if $x \in D_\varepsilon$ then there is an open interval $I \ni x$ and a positive integer N_x such that if $n \geq N_x$ then

$$\sup_{y \in I} |f_n(y) - f(y)| < \varepsilon$$

(2) Prove that f cannot be the characteristic function of $\chi_{\mathbb{Q}\cap[0,1]}$

Proof. For each $N \in \mathbb{N}$ and $\varepsilon > 0$ define

$$F_{N,\varepsilon} := \bigcap_{n,m \geq N} \{x \in [0,1] : |f_n(x) - f_m(x)| \leq \varepsilon\}$$

then $F_{N,\varepsilon}$ is closed since each f_n is continuous. And pointwise convergence everywhere implies that

$$[0,1] = \bigcup_{N=1}^{\infty} F_{N,\varepsilon}$$

so by Baire Categroy Theorem, we know that

$$D_\varepsilon := \bigcup_{N=1}^{\infty} \text{Int}(F_{N,\varepsilon})$$

is open and non-empty. Now we claim D_ε is dense; indeed, if (a,b) is an open interval then

$$[a,b] = \bigcup_{n=1}^{\infty} [a,b] \cap F_{n,\varepsilon}$$

so again by Baire we now that there is some M such that $\text{Int}([a,b] \cap F_{M,\varepsilon}) \neq \emptyset$ this implies that

$$(a,b) \cap F_{M,\varepsilon} \neq \emptyset$$

so D_ε is dense. And for each $x \in D_\varepsilon$ by definition there is some N_x such that $x \in \text{Int}(F_{N_x,\varepsilon})$ so there is some $\delta > 0$ such that $B_\delta(x) \in D_\varepsilon$ and

$$\sup_{y \in B_\delta(x)} |f_n(y) - f_m(y)| \leq \varepsilon \text{ for all } n,m \geq N_x$$

using pointwise convergence implies that

$$\sup_{y \in B_\delta(x)} |f_n(y) - f(y)| \leq \varepsilon \text{ for all } n,m \geq N_x$$

as desired.

For the second part, suppose $C([0,1]) \ni f_n \to \chi_{\mathbb{Q}\cap[0,1]}$ everywhere. Then let $x \in [0,1]$, so by part 1 there is some interval I and integer N_x such that if $n \geq N_x$ then

$$\sup_{y \in I} |f_n(y) - f(y)| \leq 1/3$$

so by density of $\mathbb{Q}$ there is some $r \in \mathbb{R} \setminus \mathbb{Q} \cap I$ so $f(q) = 0$ this implies

$$\sup_{q \in I \cap \mathbb{Q}} |f_n(r)| \leq 1/3$$

so by density of $\mathbb{R} \setminus \mathbb{Q}$ in I and f_n being continuous we deduce that

$$\sup_{y \in I} |f_n(r)| \leq 1/3$$

so if $q \in \mathbb{Q} \cap I$ we have

$$|f_n(q) - f(q)| \geq |f(q)| - |f_n(q)| \geq 1 - 1/3 = 2/3$$

so we cannot have $f_n \to f$ everywhere.

□

Problem 6. Let f in $L^2(\mathbb{R})$ and assume that the fourier transform satisfies $|\hat{f}(\xi)| > 0$ a.e. Prove that the set of finite linear combinations of translates $f_a(x) := f(x-a)$ is norm dense in $L^2(\mathbb{R})$

Proof. See Spring 2012 Problem 6. □

Problem 7. Let $f(z)$ be an analytic function on the entire plane such that $U(z) := \log |f(z)|$ is in $L^1(\mathbb{C}, dA)$. Prove that f is constant.

Proof. See Spring 2013 Problem 7. □

Problem 8. Let $\mathcal{D}$ be the space of analytic functions on $\mathbb{D}$ such that $f(0) = 0$ and $f' \in L^2(\mathbb{D}, dA)$

(1) Prove that $\mathcal{D}$ is complete in the norm
$$||f|| := \left(\int_{\mathbb{D}} |f'(z)|^2\right)^{1/2}$$

(2) Give a necessary and sufficient condition on the coefficients a_n for the function
$$f(z) := \sum_{n\geq 1}^{\infty} a_n z^n$$
to belong to $\mathcal{D}$.

Proof. Let $f_n \subset \mathcal{D}$ be a Cauchy Sequence in $||\cdot||$. Now we use that for any compact set $K \subset \mathbb{D}$ that
$$||f||_{L^\infty(K)} \leq C||f||_{L^2(\mathbb{D})}$$
where $C = C(K)$ is a constant that depends only on K and not on f. This is shown by Mean Value Property combined with integration (for instance see Fall 2013 Number 5 for a proof of this lemma). Therefore, we have for $f \in \mathcal{D}$ that
$$||f'||_{L^\infty(K)} \leq C||f||$$
so we see that
$$||f_n' - f_m'||_{L^\infty(K)} \leq C(K)||f_n - f_m||$$
so as the space $C(K)$ is complete we see there exists a function $g \in C(\mathbb{D})$ such that $f_n' \to g$ locally uniformly on $\mathbb{D}$. In particular, g is holomorphic. And as $\{f_n'\}$ is a Cauchy Sequence in L^2, which is complete we know that there is a limit $h \in L^2$, so it follows from uniqueness of limits that $g = h$. So in particular, $g \in L^2(\mathbb{D})$ and is holomorphic on $\mathbb{D}$.

As $\mathbb{D}$ is simply connected by Cauchy's Theorem, there exists a primitive of g which we denote by f such that $f(0) = 0$ (since primitives are determined uniquely up to constants). In particular as $f' = g$ we deduce that f is holomorphic with $f(0) = 0$ and $f' \in L^2(\mathbb{D}, dA)$ so $f \in \mathcal{D}$. And we have
$$||f_n - f|| = ||f_n' - g||_{L^2(\mathbb{D})} \to 0$$
where the last line is due to our definition of g and $f \in \mathcal{D}$. Therefore, $\mathcal{D}$ is complete in the norm $||\cdot||$.

For the second part, notice $f(re^{i\theta}) = \sum_{n\geq 1} a_n r^n e^{in\theta}$ and we have $f(0) = 0$ so it suffices to find equivalent conditions for the derivative of the sum being in $L^2(\mathbb{D})$. By local uniform convergence, we know that we can differentiate term by term to see
$$f'(re^{i\theta}) = \sum_{n\geq 1} na_n r^{n-1} e^{i(n-1)\theta}$$
so we have by local uniform convergence and $f' \in L^2(\mathbb{C}, dA)$ that
$$\int_{\mathbb{D}} |f'(z)|^2 dA(z) = \int_{r=0}^1 \int_{\theta=0}^{2\pi} r|f'(re^{i\theta})|^2 d\theta dr = \sum_{n,m\geq 1} \int_{r=0}^1 \int_{\theta=0}^{2\pi} r(na_n r^{n-1} e^{i(n-1)\theta})(m\overline{a_m} r^{n-1} e^{-i(n-1)\theta})$$
$$= 2\pi \sum_{n\geq 1} \int_{r=0}^1 r^{2n-1} n^2 |a_n|^2 dr$$
where for the final equality we used $e^{in\theta}$ is an orthogonal family in $[0, 2\pi]$. So
$$||f'||_{L^2(\mathbb{D})}^2 = \pi \sum_{n\geq 1} n|a_n|^2$$
so we have
$$f' \in L^2(\mathbb{D}) \iff \sum_{n\geq 1} n|a_n|^2 < \infty$$

which is the desired equivalence

□

Problem 9. Consider the meromoprhic function $g(z) = -\pi z \cot(\pi z)$ on the entire plane $\mathbb{C}$.

(1) Find all poles of g and determine the residues of g at each pole.
(2) In the Taylor series representation $\sum_{k=0}^{\infty} a_k z^k$ of $g(z)$ about $z = 0$, show that for each $k \geq 1$
$$a_{2k} = \sum_{n \geq 1} \frac{2}{n^{2k}}$$

Proof. The poles are at $n \in \mathbb{Z} \setminus \{0\}$ and are of order 1. Now fix $n \in \mathbb{Z} \setminus \{0\}$ then we have

$$\text{Res}(g, n) = \lim_{z \to n} -\pi(z-n)z \frac{\cos(\pi z)}{\sin(\pi z)} = -n$$

For the second part, recall that

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{k \geq 1} \frac{2z}{z^2 - k^2} \Rightarrow g(z) = -1 - \sum_{k \geq 1} \frac{2z^2}{z^2 - k^2}$$

so if we define $h(z) := -1 - \sum_{k \geq 1} \frac{2z}{z-k^2}$ then $h(z^2) = g(z)$ and h is holomorphic since the sum converges locally uniformly away from $n \in \mathbb{Z} \setminus \{0\}$. In particular, observe that the $2k$th taylor coefficient of g is the kth taylor coefficient of $h(z)$. Now observe

$$h(z) = -1 - z \sum_{k \geq 1} \frac{2}{z^2 - k^2} \Rightarrow h^{(n)}(z) = -\sum_{j=0}^{n} \binom{n}{x} (\frac{d^j}{dz^j}(z)) \frac{d^{n-j}}{dz^{n-j}} f(z)$$

where $f(z) := \sum_{k \geq 1} \frac{2}{z^2 - k^2}$. So in particular,

$$h^{(n)}(0) = -2n \frac{d^{n-1}}{dz^{n-1}} f(z)$$

since the sum for $f(z)$ converges locally uniformly we can differentiate term by term to see that

$$f^{(n)}(z) = (-1)^n n! \sum_{k \geq 1} \frac{2}{(z-k^2)^{n+1}}$$

so

$$h^{(n)}(0) = n! \sum_{k \geq 1} \frac{2}{k^{2n}}$$

so we see that

$$a_{2k} = \sum_{k \geq 1} \frac{2}{k^{2n}}$$

as desired.

□

Problem 10. For $-1 < \beta < 1$ evaluate
$$\int_0^\infty \frac{x^\beta}{1+x^2} dx$$

Proof. See Spring 2014 Number 11.

□

Problem 11. An analytic Jordan Curve is a set of the form

$$\Gamma = f(\{|z| = 1\})$$

where f is analytic and $1 - 1$ on an annulus $\{r < |z| < 1/r\}$, $0 < r < 1$.

Let $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, let $N < \infty$, and let $\Omega \subset \mathbb{C}^*$ be a domain for which $\partial\Omega$ has N connected components, none of which are single points. Prove there is a conformal mapping from Ω onto a domain bounded by N pairwise disjoint analytic Jordan Curves.

Proof. □

Problem 12. If $\alpha \in \mathbb{C}$ satisfies $0 < |\alpha| < 1$ and if $n \in \mathbb{N}$ show that the equation

$$e^z(z-1)^n = \alpha$$

has exactly n simple roots in the half plane $\{z : \text{Re}z > 0\}$.

Proof. Let us first show all the roots are simple of

$$f(z) := e^z(z-1)^n - \alpha$$

So if z_0 is a repeated root we know that $f(z_0) = f'(z_0) = 0$, so by differentiation,

$$e^{z_0}(z_0 - 1)^n + e^{z_0}(n)(z_0 - 1)^{n-1} = 0$$

and as $0 < |\alpha|$ we know that $z_0 - 1 \neq 0$ so

$$(z_0 - 1) + n = 0 \Rightarrow z_0 = 1 - n$$

Now we claim that

$$|e^{z_0}(z_0 - 1)^n| \geq 1$$

which will imply all the roots are simple since $0 < |\alpha| < 1$. Indeed, observe that

$$|e^{1-n}(-n)^n| \geq 1 \iff \left(\frac{n}{e}\right)^n \geq \frac{1}{e}$$

we note the equality is true for $n = 1$ and by differentiation in n we see the left hand inequality is increasing in n, so we have the desired inequality. Therefore, every root is simple since $f(z_0) \neq 0$.

Now it suffices to show there are n roots in the half plane $\{z : \text{Re}(z) > 0\}$. To do this we parameterize γ_R as a square of length R with center R for $R > 1$. It is easy to see that on the boundary of this contour that $|e^z(z-1)^n| \geq 1 > |\alpha|$ so by Rouche's theorem,

$$e^z(z-1)^n - \alpha$$

has the same number of zeros as $e^z(z-1)$ in this square. Therefore, by taking $R \to \infty$ we see in this half plane there are n roots in this half plane.

□

19. Spring 2019

Problem 1. Let $f \in C^2(\mathbb{R})$ be a real valued function that is uniformly bounded on $\mathbb{R}$. Prove that there exists a point $c \in \mathbb{R}$ such that $f''(c) = 0$.

Proof. Assume for the sake of contradiction that $f''(x) \neq 0$ for any $x \in \mathbb{R}$, so as $f \in C^2$ we know that either $f''(x) > 0$ or $f''(x) < 0$ for all $x \in \mathbb{R}$. Without loss of generarlity, by looking at $-f$ if necessary we can assume that $f''(x) > 0$ for all $x \in \mathbb{R}$. Then $f(x)$ is convex and as $f''(x) > 0$ we know that $f'(x)$ is strictly increasing, so there is some $x_0 \in \mathbb{R}$ such that $f'(x_0) \neq 0$. Then by a first order taylor series with remainder term expansion we see that

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \text{ for all } x \in \mathbb{R}$$

i.e. convex functions lie above their tangent lines. Now if $f'(x_0) > 0$ then taking $x \to \infty$ gives $\lim_{x\to\infty} f(x) \geq \infty$ so f is not uniformly bounded. And if $f'(x_0) < 0$ then taking $x \to -\infty$ implies $\lim_{x\to-\infty} f(x) \geq \infty$ so f is unbounded in either case, so we have arrived at a contradiction. So $f''(c) = 0$ for some $c \in \mathbb{R}$.

□

Problem 2. Let μ be a Borel probability measure on $[0, 1]$ that has no atoms. Let also $\mu_1, \mu_2, ..$ be Borel probability measures on $[0, 1]$ such that μ_n weak* converges to μ. Denote $F(t) := \mu([0, t])$ and $F_n(t) := \mu_n([0, t])$ for each $n \geq 1$ and $t \in [0, 1]$. Prove that F_n converges uniformly to F.

Proof. Note that for each fixed $t \in [0, 1]$ that we can find a sequence of continuous functions $f_n, g_n \in C([0, 1])$ such that

$$f_n(x) \leq \chi_{[0,t]}(x) \leq g_n(x)$$

with $f_n \to \chi_{[0,t]}(x)$ and $g_n \to \chi_{[0,t]}(x)$ pointwise, so in particular, we have

$$\int_0^1 f_n(x) d\mu_m(x) \leq F_m(t) \leq \int_0^1 g_n(x) d\mu_m$$

so taking $m \to \infty$ along with μ_n weak* converges to μ gives

$$\begin{cases} \limsup_{n\to\infty} F_n(t) \leq \int_0^1 g_n(x) d\mu \\ \int_0^1 f_n(x) d\mu \leq \liminf_{n\to\infty} F_n(t) \end{cases}$$

so now using DCT and MCT we see that

$$\lim_{n\to\infty} \int_0^1 f_n(x) d\mu = \lim_{n\to\infty} \int_0^1 g_n(x) d\mu = F(t)$$

so we deduce that

$$\lim_{n\to\infty} F_n(t) = F(t)$$

so we have pointwise convergence everywhere. A small modification of the above proof shows that $\mu_n([a, b]) \to \mu([a, b])$ for all $a \leq b$.

Notice each $F_n(t)$ is a monotone function that converges to a continuous function $F(t)$ (it is continuous since it has no atoms), so we have $F_n \to F$ uniformly since $[0, 1]$ is compact. Indeed, as F is continuous on $[0, 1]$ if $\varepsilon > 0$ there is a $\delta > 0$ such that if $x, y \in [0, 1]$ and $|x - y| < \delta$ then $|F(x) - F(y)| < \varepsilon/2$. Now as fix $q \in \mathbb{Q} \cap [0, 1]$ then by pointwise convergence we can find an $N(q)$ such that if $n \geq N(q)$ then

$$|f_n(q) - F(q)| \leq \varepsilon/2$$

and as $\mathbb{Q} \cap [0, 1]$ is dense in $[0, 1]$ we know that $[0, 1] \subset \bigcup_{n=1}^{\infty} B_{\delta/2}(q_n)$ where q_n is an enumeration of $\mathbb{Q}$. Therefore, we may find a finite subcover $\bigcup_{n=1}^{N} B_{\delta/2}(q_n)$ and let $M := \max_{1\leq j\leq N} N(q_n)$. Then for $m \geq M$, if $x \in B_{\delta/2}(q_n)$ such that if $x \leq q_n$ we have from F_n being monotone increasing that (where we reorder q_n such that $q_n < q_{n+1}$)

$$F_m(x) - F(q_n) \leq F_m(q_n) - F(q_n) \leq \varepsilon/2$$

but we also have

$$F_m(x) - F(q_n) \geq F_m(q_{n-1}) - F(q_n) \geq -\varepsilon/2$$

so it follows that

$$|F_m(x) - F(q_n)| \leq \varepsilon/2$$

for all $m \geq M$. Now observe if $n \geq M$ that

$$|F_n(x) - F(x)| \leq |F_n(x) - F(q_m)| + |F(q_m) - F(x)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon$$

where $x \in B_\delta(q_m)$. The case of $x \geq q_n$ is also similar. Therefore, we have shown $F_n \to F$ uniformly.

□

Problem 3. Let $f(t)$ be a positive continuous function such that $\lim_{|t|\to\infty} f(t) = 0$. Show that the set $\{hf : h \in L^1(\mathbb{R}, m), ||h||_1 \leq K\}$ is a closed nowhere dense set in $L^1(\mathbb{R}, m)$, for any $K \geq 1$.

Let $\{f_n\}$ be a sequence of positive continuous function on $\mathbb{R}$ such that for each n we have $\lim_{|t|\to\infty} f(t) = 0$. Show that there exists $g \in L^1(\mathbb{R}, m)$ such that $g/f_n \notin L^1(\mathbb{R}, m)$ for all n.

Proof. Denote $E := \{hf : h \in L^1(\mathbb{R}, m), ||h||_1 \leq K\}$ then if $E \ni f_n \to f$ in L^1 then we have for each f_n that $f_n = g_n h$ where $g_n \in L^1(\mathbb{R}, m)$ with $||g_n||_1 \leq K$. So by looking at a subsequence if necessary, we have that $g_n \to f/h$ pointwise, so Fatou's Lemma gives

$$\int_{\mathbb{R}} |f/h| dx = \int_{\mathbb{R}} \liminf_{n\to\infty} |g_n| dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |g_n| dx \leq K$$

so it follows that $f \in E$. To see that it has empty interior, observe that if $f \in E$ and $\varepsilon > 0$ that for any $N \in \mathbb{N}$ we have that for $g_M(x) := f(x) + (\varepsilon/2)\chi_{[M,M+1]}(x)$

$$||f(x) - g_M(x)||_1 = \int_M^{M+1} \varepsilon/2 dx = \varepsilon/2$$

i.e. $g_M(x) \in B_\varepsilon(f)$. Now observe that

$$\int_{\mathbb{R}} |g_M(x)/h(x)| dx = \int_M^{M+1} \frac{\varepsilon}{2|h(x)|} dx$$

so now as $\lim_{|t|\to\infty} f(t) = 0$, we can find an $M \in \mathbb{N}$ such that if $\delta > 0$ then on $[M, M+1]$ we have $|f(t)| < \delta$. So in particular, we conclude

$$\int_{\mathbb{R}} |g_M(x)/h(x)| dx \geq \frac{\varepsilon}{2\delta}$$

now choose δ so small such that $\varepsilon/(2\delta) > K + 1$. Thus it follows that $g_M(x) \notin E$. So E has empty interior in L^1.

For the second part define $E_{n,M} := \{hf_n : h \in L^1(\mathbb{R}, m), ||h||_1 \leq M\}$ where $n, M \in \mathbb{N}$. Then $E_{n,M}$ is a closed set with empty interior. Then as $L^1(\mathbb{R}, m)$ is complete we know that L^1 is not meager i.e.

$$L^1(\mathbb{R}, m) \neq \bigcup_{n,M\in\mathbb{N}} E_{n,M}$$

so in particular there is some $g \in L^1(\mathbb{R}, m)$ such that $G \in \bigcap_{m,M\in\mathbb{N}} E^c_{n,M}$ i.e. for all $n, M \in \mathbb{N}$ $||g/f_n|| \geq M$ i.e. $g/f_n \notin L^1(\mathbb{R}, m)$ for any $n \in \mathbb{N}$.

□

Problem 4. Let $\mathcal{V}$ be the subspace of $L^\infty([0,1],\mu)$ (where μ is the Lebesgue measure on $[0,1]$) defined by

$$\mathcal{V} := \{f \in L^\infty([0,1],\mu) : \lim_{n\to\infty} n \int_{[0,1/n]} f(x)d\mu \text{ exists } \}$$

Prove that there is a continuous linear functional $\varphi \in L^\infty([0,1],\mu)^*$ such that $\varphi(f) = \lim_{n\to\infty} \int_{[0,1/n]} f(x)d\mu$ for every $f \in \mathcal{V}$.

Show that, given any $\varphi \in L^\infty([0,1],\mu)^*$ satisfying the condition above that there is no $g \in L^1([0,1],\mu)$ such that $\varphi(f) = \int_0^1 f(x)g(x)d\mu$ for all $f \in L^\infty([0,1],\mu)$

Proof. Define the linear functional $\varphi : \mathcal{V} \to \mathbb{R}$ via

$$\varphi(f) = \lim_{n\to\infty} n \int_{[0,1/n]} f(x)d\mu$$

and this is linear and well defined since $f \in \mathcal{V}$. It is also continuous since

$$\left| n \int_{[0,1/n]} f(x)d\mu \right| \leq n \int_{[0,1/n]} |f(x)|d\mu \leq ||f||_{L^\infty}$$

hence by taking limits

$$|\varphi(f)| \leq ||f||_{L^\infty}$$

so $\varphi \in \mathcal{V}^*$. So by Hanh-Banach we can extend φ to a linear functional ϕ on $L^\infty([0,1],\mu)$ such that the $||\phi|| = ||\varphi||$ where the operator norm on the left is taken over $L^\infty([0,1],\mu)$ and the right is over $\mathcal{V}$. So this extension is continuous and this proves the first part.

For the second part note that over $C([0,1]) \subset L^\infty([0,1],\mu)$ that $\varphi|_{C([0,1])} = \delta_0$ where δ_0 means the dirac delta at zero since if $f \in C([0,1])$

$$f(0) = \lim_{n\to\infty} \int_{[0,1/n]} f(x)d\mu$$

So now if we assume for the sake of contradiction that there is some $g \in L^1([0,1],\mu)$ such that $\varphi(f) = \int_0^1 f(x)g(x)d\mu$. Then for any $y \in (0,1)$ we have for $h > 0$ small enough that $(y-h, y+h) \subset (0,1)$ and we can find a continuous function $f_n \geq \chi_{[y-h,y+h]}$ such that $f_n \to \chi_{[y-h,y+h]}$ pointwise with $f_n \in C([0,1])$ with $||f_n||_{L^\infty} \leq 2$ so we have from the dominated convergence theorem

$$f_n(0) = \int_0^1 f_n(x)g(x)dx \to \int_{y-h}^{y+h} g(x)dx$$

where the first equality is due to $\varphi(f) = f(0)$ for all $f \in C([0,1])$. In particular, we have

$$0 = \int_{y-h}^{y+h} g(x)dx \Rightarrow \lim_{h\to 0} \frac{1}{2h} \int_{y-h}^{y+h} g(x)dx = 0$$

so by Lebesgue differentiation Theorem we have $g(x) = 0$ a.e., but

$$1 = \varphi(1) = \int_0^1 g(x)dx = 0$$

which is our contradiction.

□

Problem 5. Prove that $L^p([0,1],\mu)$ are separable Banach spaces for $1 \leq p < \infty$ but $L^\infty([0,1],\mu)$ is not (μ is the Lebesgue measure on $[0,1]$).

Also prove that there is no linear bounded surjective map $T : L^p([0,1],\mu) \to L^1([0,1],\mu)$, if $p > 1$.

Proof. We claim that for $1 \leq p < \infty$ that step functions with rational coefficients over rational intervals are dense in $L^p([0,1],\mu)$ for $1 \leq p < \infty$. Indeed, first fix $E \subset [0,1]$ μ-measurable, then as μ is outer regular, we know

$$\int_0^1 \chi_E d\mu(x) = \mu(E) = \inf\{\mu(G) : E \subset G \text{ and } G \text{ is open}\}$$

Now let $E \subset G$ where G is open. Then since $[0,1]$ is separable, we know that $G = \bigcup_{i=1}^{\infty} I_i$ where I_i are disjoint open intervals. So then we have $\mu(G) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(I_j)$, so if $\varepsilon > 0$ we can find some $N \in \mathbb{N}$ such that if $n \geq N$ $|\mu(G) - \sum_{j=1}^{n} \mu(I_j)| < \varepsilon$. Now each I_i we can thanks to density of rationals on $[0,1]$ a rational interval J_j such that $I_j \subset J_j$ and $\mu(J_j \setminus I_j) \leq \frac{\varepsilon}{2^j}$, so it follows that And

$$\begin{aligned}\left(\int_0^1 |\chi_G(x) - \sum_{i=1}^{n} \chi_{J_i}|^p d\mu(x)\right)^{1/p} &\leq \left(\int_0^1 \sum_{i=1}^{n} |\chi_{I_i} - \chi_{J_i}|^p\right)^{1/p} + \left(\int_0^1 \sum_{i=n+1}^{\infty} |\chi_{I_i}|^p\right)^{1/p} \\ &\leq \left(\sum_{i=1}^{n} \mu(J_i \setminus I_i)\right)^{1/p} + \left(\sum_{i=n}^{\infty} \mu(I_i)\right)^{1/p} \\ &\leq 2\varepsilon^{1/p}\end{aligned}$$

So now it follows that we can approximate simple functions arbitrarily well with finite rational combinations of step functions on rational intervals in L^p for $p \in [1,\infty)$ and as this set of functions is countable, we deduce $L^p([0,1],\mu)$ is separable for $p \in [1,\infty)$.

$L^\infty([0,1])$ is not separable since for every irrational $p \in [1/4, 3/4] \setminus \mathbb{Q}$ we can define for $0 < \varepsilon < 1/8$ $f_p(x) := \chi_{[p-\varepsilon, p+\varepsilon]}(x)$. Then we have for p, q distinct irrationals in $[1/4, 3/4]$ that

$$||f_p(x) - f_q(x)||_{L^\infty([0,1])} = 1$$

But there are uncountably many such functions, which implies $L^\infty([0,1])$ is uncountable.

For the second part, assume for the sake of contradiction that such a T existed, then we define its adjoint $T^* : (L^1([0,1],\mu))^* \cong L^\infty([0,1],\mu) \to (L^p([0,1],\mu))^* \cong L^q([0,1],\mu)$ where $1/p + 1/q = 1$ via

$$T^*(f) := f(T) : L^p([0,1],\mu) \to \mathbb{R}$$

Now as T is surjective we claim that T^* is injective. Indeed, as the adjoint is linear it suffices to show if $T^*(g) = 0$ then $g = 0$. Indeed, observe $T^*(g) = g(T)$ so as T is surjective, we know this means for all $f \in L^1([0,1],\mu)$ that $g(f) = 0$ so by Hanh-Banach, we must have that $g \in (L^1([0,1],\mu))^*$ is the zero functional. So T^* is injective. So we have that $L^\infty([0,1],\mu)$ is a isomorphic to a subset of $L^q([0,1],\mu)$ i.e. $T^*(L^\infty([0,1],\mu))$ (note T^* is bounded). Notice that the first space is separable, while the second is not. This contradicts the isomorphism, so no such maps exist.

□

Problem 6. Let $\mathcal{H}$ be a Hilbert space and $\{\xi_n\}_n$ a sequence of vectors in $\mathcal{H}$ such that $||\xi_n|| = 1$ for all n.

Show that if $\{\xi_n\}$ converges weakly to a vector $\xi \in \mathcal{H}$ with $||\xi|| = 1$, then $\lim_{n\to\infty} ||\xi_n - \xi|| = 0$.

Show that if $\lim_{n,m\to\infty} ||\xi_n + \xi_m|| = 2$,then there exists a vector $\xi \in \mathcal{H}$ such that

$$\lim_{n\to\infty} ||\xi_n - \xi|| = 0$$

Proof. For the first part, by the Parallelogram law, we have that

$$||\xi_n + \xi||^2 + ||\xi_n - \xi||^2 = 2||\xi_n||^2 + 2||\xi||^2 = 4$$

Now observe that

$$||\xi_n + \xi||^2 = (\xi_n + \xi, \xi_n + \xi) = ||\xi_n||^2 + ||\xi||^2 + (\xi_n, \xi) + (\xi, \xi_n) = 2 + (\xi_n, \xi) + (\xi, \xi_n)$$

Now as $\xi_n \rightharpoonup \xi$ we know that for any $g \in \mathcal{H}$ that $(\xi_n, g) \to (\xi, g)$. Taking $g = \xi$ gives that

$$||\xi_n + \xi||^2 \to 4 \text{ as } n \to \infty$$

that is

$$||\xi_n - \xi||^2 \to 0 \text{ as } n \to \infty$$

For the second part, notice again by Parallelogram Law that

$$||\xi_n + \xi_m||^2 + ||\xi_n - \xi||^2 = 2||\xi_n||^2 + 2||\xi_m||^2 = 4$$

so

$$||\xi_n - \xi_m||^2 = \lim_{n,m\to\infty} 4 - ||\xi_n + \xi_m||^2 = 0$$

where the last equality is by the given assumption. Now that $\mathcal{H}$ is complete we conclude since $\{\xi_n\}$ forms a Cauchy Sequence.

□

Problem 7. Let $f : \mathbb{C} \to \mathbb{C}$ be entire non-constant, and let us set

$$T(r) := \frac{1}{2\pi} \int_0^{2\pi} \log_+ |f(re^{i\varphi})| d\varphi$$

Here $\log_+(s) = \max(\log(s), 0)$. Show that $T(r) \to \infty$ as $r \to \infty$

Proof. By Jensen's formula if we use ρ to denote the zeros of f and 0 is not a root of f we have

$$\log |f(0)| + \sum_{|\rho| \le r} \log |\frac{r}{\rho}| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| d\varphi \le T(r)$$

So in particular, if f has at least one zero denoted by ξ then for r large enough we have

$$T(r) \ge \log |f(0)| + \log \left|\frac{r}{\xi}\right| \to \infty \text{ as } r \to \infty$$

so this proves the theorem when $f(0) \ne 0$ and f has at least one zero.

Now if f has no zeros, we know there is an entire function $g(z)$ such that $f(z) = \exp(g(z))$. Then computation gives $\log |f(z)| = \text{Re}(g(z))$ which is a harmonic function. So it suffices to show if $u : \mathbb{C} \to \mathbb{R}$ is harmonic then

$$\frac{1}{2\pi} \int_0^{2\pi} u_+(re^{i\varphi}) d\varphi \to \infty \text{ as } r \to \infty$$

WLOG $u(0) \ne 0$ by adding a constant if necessary. Then by the mean value theorem we know that for $r > 0$

$$|u(0)| \le \frac{1}{\pi r^2} \int_{B_r(0)} |u(z)| dA(z)$$

so it follows that if $0 < \varepsilon \ll 1$ that for R large that

$$\int_{B_R(0)} |u(z)| dA(z) \ge R^{2-\varepsilon}$$

otherwise we could find a sequence $\{R_n\}$ such that $R_n \to \infty$ and

$$|u(0)| \le \frac{1}{\pi R_n^{\varepsilon}} \to 0$$

So now we have two cases either R large or there is an $\varepsilon > 0$ such that $\int_{B_R(0)} |u(z)| dA(z) \gtrsim R^{2+\varepsilon}$ for R large or no such ε exist i.e. $\int_{B_r(0)} |u(z)| dA(z) \lesssim R^2$. For the first case, observe that

$$\int_{B_R(0)} u_+(z) dA(z) = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} r(u_+(re^{i\theta})) dr d\theta = \int_{r=0}^{R} rT(r) dr \gtrsim R^{2+\varepsilon}$$

where for the last equality follows from

$$u(0) = \int_{B_R(0)} (u_+ - u_-) dA(z)$$

where $u_- := -\min\{u, 0\}$ and $|u| = u_+ + u_-$ to deduce the above inequality. From which it follows that $T(r) \to \infty$ as $R \to \infty$ since if $T(r)$ is bounded on some subsequence $R_n \to \infty$ the above inequality will be invalid. Now if

$$\int_{B_R(0)} |u(z)| dA(z) \lesssim R^2$$

this implies for any $0 < r < R$ that

$$|u(z)| \lesssim_r R^2$$

where $z \in B_r(0)$, so it follows that u is at most a polynomial of degree 2 at most, from which computation implies the result.

Now if $f(z)$ has a zero at the origin, we know that $f(z)/z^m$ for some m is entire with no-zeros at the origin (take m to be the order of the zero). Then we can repeat the computation above to get

$$\frac{1}{2\pi} \int_0^m \log_+ |f(re^{i\theta})/r^m| d\varphi \to \infty$$

so in particular, as $\log |f(re^i)/r^m| = \log |f(re^{i\theta})| - \log |r^m| \leq \log |f(re^{i\theta})|$ for $r \geq 1$ so it follows that

$$\frac{1}{2\pi} \int_0^m \log_+ |f(re^{i\theta})| d\varphi \to \infty$$

□

Problem 8. Show that

$$\sin(z) - z\cos(z) = \frac{z^3}{3} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right), z \in \mathbb{C}$$

where λ_n is a sequence in $\mathbb{C}$ such that $\lambda_n \neq 0$ or all n and

$$\sum_{n \in \mathbb{N}} |\lambda_n|^{-2} < \infty$$

Proof. Observe that $f(z) := \sin(z) - z\cos(z)$ is an entire function of order 1, so by Hadamard's factorization theorem, we know if $\{\lambda_n\}$ are the zeros of $f(z)$ with $\lambda_n \neq 0$ that

$$f(z) = \exp(az + b) z^3 \prod_{n=1}^{\infty} (1 - \frac{z}{\lambda_n}) e^{z/\lambda_n}$$

since f is an entire function of order 1 with a zero of order 3 at $z = 0$ where the product converges locally uniformly on $\mathbb{C}$. We also know there are infinitely many roots since if $f(z)$ had finitely many roots say $z_1, .., z_M$ then $f(z) = \prod_{j=1}^{M} (z - z_i) \exp(g(z))$ where $g(z)$ is an entire function and $f(z)$ is not of this form. Now that $f(z) = -f(-z)$ combined with λ_n being the zeros of $f(z)$ (with $\lambda_n \neq 0$) shows there is an m such that $\lambda_n = -\lambda_m$, so using the local uniform convergence let us rearrange terms in the sum to get

$$f(z) = \exp(az + b) z^3 \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n^2})$$

since $(1 - z/\lambda_n) e^{z/\lambda_n} (1 - z/\lambda_m) e^{z/\lambda_m} = (1 - z^2/\lambda_n^2)$ because $\lambda_m = -\lambda_n$. Now using oddness of f again gives

$$\exp(az + b) z^3 \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n^2}) = \exp(-az + b) z^3 \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n^2})$$

so it follows that $a = 0$. So now it suffices to show that $b = \log(1/3)$ since Jensen's formula gives for any $\varepsilon > 0$ that $\sum_{n\in\mathbb{N}} |\lambda_n|^{-1-\varepsilon} < \infty$ and specifying $\varepsilon = 1$ gives the desired sum bound. Now we differentiate $f(z)$ to see that

$$f'''(z) = 6\exp(b) + zh(z)$$

where $h(z)$ is a holomorphic function. Now using $f'''(0) = 2$ gives $\exp(b) = 1/3$ i.e.

$$f(z) = z^3/3\prod_{n=1}^{\infty}(1 - \frac{z^2}{\lambda_n^2})$$

as desired and the sum properties hold.

□

Problem 9. Show that if $A(\mathbb{D})$ is the space of holomorphic functions on $\mathbb{D}$ and

$$U := \{f \in A(D) : |f| = 1 \text{ on } \partial\mathbb{D}\}$$

then show $f \in U$ if and only if

$$f(z) = \lambda\prod_{j=1}^{N}\frac{z - a_j}{1 - \overline{a_j}z}$$

for some $a_j \in \mathbb{D}$ and $1 \le j \le N < \infty$ and $|\lambda| = 1$

Proof. Let $a_j \in \mathbb{D}$ then for $|z| = 1$ then $z = e^{i\theta}$ for some θ so

$$|z - a_j| = |z||1 - \frac{a_j}{z}| = |1 - \overline{z}a_j| = |1 - z\overline{a_j}|$$

i.e. $|z - a_j|/|1 - \overline{a_j}z| = 1$ on ∂D and this is holomorphic since $1/\overline{a_j} \notin \mathbb{D}$ because $1/|\overline{a_j}| > 1$. Therefore,

$$\lambda\prod_{j=1}^{N}\frac{z - a_j}{1 - \overline{a_j}z}$$

with $a_k \in \mathbb{D}$ and $|\lambda| = 1$ with finite N is in U.

For the converse, let $f \in U$, then notice as f is holomorphic its zeros are isolated and it has no zeros on $\partial\mathbb{D}$, so there are only finitely many zeros of f on $\mathbb{D}$. Enumerate them as $\{a_j\}_{j=1}^N$ then $g(z) := f(z)/\prod_{j=1}^N \frac{z-a_j}{1-\overline{a_j}z}$ is holomorphic on $\mathbb{D}$ with no zeros and is norm 1 on the boundary of the disk. Therefore, by applying the maximum modulus principle on $1/g$ combined with the maximum modulus principle on g we see that $|g| = 1$ everywhere on $\mathbb{D}$. In particular, the Cauchy Riemann Equations then imply g is constant so $g = \lambda$ where $||= 1$ i.e.

$$f(z) = \lambda\prod_{j=1}^{N}\frac{z - a_j}{1 - \overline{a_j}z}$$

for some $a_j \in \mathbb{D}$ with $1 \le j \le N < \infty$ and $|\lambda| = 1$.

□

Problem 10. For $a > 0, b > 0$, evaluate the integral

$$\int_0^\infty \frac{\log x}{(x + a)^2 + b^2}dx$$

Problem 11. Let $u \in C^\infty(\mathbb{R})$ be a smooth 2π-periodic function. Show that there exists a bounded holomorphic function f_+ in the upper half-plane $\text{Im}z > 0$ and a bounded holomorphic function f_- in the lower half plane $\text{Im}z < 0$, such that

$$u(x) = \lim_{\varepsilon\to 0+}(f_+(x + i\varepsilon) - f_-(x - i\varepsilon))$$

Proof. As u is C^∞ and 2π-periodic we know that

$$u(x) = \sum_{n\in\mathbb{Z}} a_n e^{inx} \text{ where } a_n = \int_0^1 u(x)e^{-inx}$$

where the sum converges absolutely i.e. the Fourier Series of u agrees with u. Therefore, we can define

$$f_+(z) := \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{inz} \text{ and } f_-(z) := -\frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{-inz}$$

then f_+ is a bounded holomorphic function in the upper half plane and f_- is a bounded function on the lower half plane. (Note they are holomorphic since the sums are locally uniformly convergent in the upper and lower half plane respectively). Then observe that

$$f_+(x+i\varepsilon) - f_-(x-i\varepsilon) = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n(e^{in(x+i\varepsilon)} + e^{in(x-i\varepsilon)}) = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n e^{inx}(e^{-n\varepsilon} + e^{n\varepsilon})$$

Now we recall that the sum is absolutely convergent on $\mathbb{R}$ so f_+ converges absolutely on $0 < \text{Im}(z) < M$ for any $M > 0$ and similarily for f_-, so we can take the limit $\varepsilon \to 0$ inside the sum to get

$$\lim_{\varepsilon\to 0} f_+(x+i\varepsilon) - f_-(x-i\varepsilon) = f(x)$$

as desired.

□

Problem 12. Let $\mathcal{H}$ be the vector space of entire functions $f : \mathbb{C} \to \mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 d\mu(z) < \infty$$

Here $d\mu(z) = e^{-|z|^2} d\lambda(z)$ where λ is the Lebesgue measure on $\mathbb{C}$.

(1) Show that $\mathcal{H}$ is a closed subspace of $L^2(\mathbb{C}, d\mu)$.
(2) Show that for all $f \in \mathcal{H}$ we have

$$f(z) = \frac{1}{\pi}\int_{\mathbb{C}} f(w)e^{z\overline{w}} d\mu(w)$$

Proof. Refer to Fall 2014 Number 10 for a solution of (1).

Observe that if $(f(z), g(z)) = f(w)$ where $(,)$ is the inner product on $L^2(d\mu)$ then

$$(f(z), g(z)) = \sum_{n=0}^{\infty} a_n w^n$$

where $\sum a_n z^n$ is the Taylor Expansion of $f(z)$ (which is possible since f is entire). As $g(z)$ is entire we can write $g(z) = \sum b_n z^n$ then we have

$$(f,g) = \lim_{R\to\infty}\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r(\sum a_n r^n e^{in\theta})(\sum \overline{b_n} r^n e^{-in\theta})e^{-|r|^2} d\theta dr$$

and by uniform convergence of the sums we have

$$\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r(\sum a_n r^n e^{in\theta})(\sum \overline{b_n} r^n e^{-in\theta})e^{-|r|^2} d\theta dr = \sum_{n,m=0}^{\infty}\int_{r=0}^{R}\int_{\theta=0}^{2\pi} r^{2n+1} a_n \overline{b_m} e^{i(n-m)\theta} e^{-|r|^2} d\theta dr$$

now orthogonality of $e^{in\theta}$ tells us

$$= \sum_{n=0}^{\infty} 2\pi \int_{r=0}^{R} r^{2n+1} a_n \overline{b_n} e^{-|r|^2} dr = \sum_{n=0}^{\infty} \pi a_n \overline{b_n} \int_{r=0}^{R^2} r^n e^{-r} dr \to \sum_{n=0}^{\infty} \pi a_n \overline{b_n} n!$$

so

$$\sum_{n=0}^{\infty} a_n w^n = (f,g) = \sum_{n=0}^{\infty} \pi a_n \overline{b_n} n!$$

so $b_n = \frac{1}{\pi}\frac{\overline{w}^n}{n!}$ i.e.

$$g(z) = \sum_{n=0}^{\infty} \frac{1}{\pi}\frac{\overline{w}^n}{n!} z^n = \frac{1}{\pi} e^{z\overline{w}}$$

as desired. Note that all of the above formal computation is justified since our $g(z)$ is entire so the convergence of the sums is uniform.

$\square$

20. Fall 2019

Problem 1. Given σ -finite measures $\mu_1, \mu_2, \nu_1, \nu_2$ on a measurable space $(X, \mathcal{X})$, suppose that $\mu_i \ll \nu_i$ for $i = 1, 2$. Prove that the product measure $\mu_1 \otimes \mu_2$ and $\nu_1 \otimes \nu_2$ on $(X \times X, \mathcal{X} \otimes \mathcal{X})$ satisfy $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and the Random-Nikodym derivatives obey

$$\frac{d(\mu_1 \otimes \mu_2)}{d(\nu_1 \otimes \nu_2)}(x, y) = \frac{d\mu_1}{d\nu_1}(x)\frac{d\mu_2}{d\nu_2}(y)$$

for $\nu_1 \otimes \nu_2$ a.e. $(x, y) \in X \times X$.

Proof. By Radom-Nikodym as $\mu_i \ll \nu_i$ and all the measures are σ-finite that there exists $f_i \in L^1(X, d\nu_i)$ such that for any $E \in \mathcal{X}$ we have for $i = 1, 2$

$$\mu_i(E) = \int_E f_i(x) d\nu_i(x)$$

So it follows that if $E_i \in \mathcal{X}$ that by the definition of the product measure that

$$\mu_1 \otimes \mu_2(E_1 \times E_2) = \mu_1(E_1) \cdot \mu_2(E_2) = \left(\int_{E_1} f_1(x) d\nu_1(x)\right)\left(\int_{E_2} f_2(y) d\nu_2(y)\right)$$

$$= \int_{E_1 \times E_2} f_1(x) f_2(y) d(\nu_1 \otimes \nu_2)(x, y) = (f_1(x) f_2(y)) d(\nu_1 \otimes \nu_2)(E_1 \times E_2)$$

where the last equality is justified by Fubini since $f_1(x)f_2(y) \in L^1(d(\nu_1 \otimes \nu_2))$ and the measures being σ-finite. So as $\mu_1 \otimes \mu_2 = (f_1(x)f_2(y))d(\nu_1 \otimes \nu_2)$ for rectangles and rectangles generate the $\mathcal{X} \otimes \mathcal{X}$ it follows that

$$d(\mu_1 \otimes \mu_2) = (f_1(x) f_2(y)) d(\nu_1 \otimes \nu_2)$$

so in particular, we see that $\mu_1 \otimes \mu_2 \ll \nu_1 \otimes \nu_2$ and the Random-Nikodym derivative of $\mu_1 \otimes \mu_2$ is $f_1(x)f_2(y)$ for $\nu_1 \otimes \nu_2$ a.e. in $(x, y) \in X \times X$ as desired. ☐

Problem 2. Let μ be a fintie Borel measure on $\mathbb{R}$ with $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$ and let $\varphi(t) = \int_{\mathbb{R}} e^{itx} d\mu(x)$. Prove that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = 0$$

Proof. Observe that

$$\int_{-T}^{T} |\varphi(t)|^2 dt = \int_{-T}^{T} \left(\int_{\mathbb{R}} e^{itx} d\mu(x)\right)\left(\int_{\mathbb{R}} e^{-ity} d\mu(y)\right) dt = \int_{-T}^{T} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{it(x-y)} d\mu(x) d\mu(y) dt$$

where the last equality is due to Fubini's Theorem which can be applied since μ is a finite measure so $e^{it(x-y)} \in L^1(d\mu(x) \otimes d\mu(y) \otimes dt, \mathbb{R} \times \mathbb{R} \times [-T, T])$ since the integrand is bounded. Applying Fubini again gives

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-T}^{T} e^{it(x-y)} dt d\mu(x) d\mu(y) = 2 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(T(x-y))}{(x-y)} d\mu(x) d\mu(y)$$

so we have

$$\frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\sin(T(x-y))}{T(x-y)} d\mu(x) d\mu(y)$$

where we define

$$\frac{\sin(T(x-y))}{T(x-y)} := 1 \text{ for } x = y$$

and this definition makes the integrand continuous. So as the integrand is bounded by a constant and μ is finite we may apply DCT to see that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|^2 dt = \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_{\{x=y\}} d\mu(x) d\mu(y) = \int_{\mathbb{R}} \mu(\{y\}) d\mu(y) = 0$$

where the last equality is due μ having no atoms. $\square$

Problem 3. Consider a measure space $(X, \mathcal{X})$ with σ-finite measure μ and let $p \in (1, \infty)$. Let $L^{p,\infty}$ be the set of measurable $f : X \to \mathbb{R}$ with $[f]_p := \sup_{t>0} t\mu(|f| > t)^{1/p}$ finite. Let

$$||f||_{p,\infty} := \sup_{E \in \mathcal{X}, \mu(E) \in (0,\infty)} \frac{1}{\mu(E)^{1-1/p}} \int_E |f| d\mu$$

prove that there exists $c_1, c_2 \in (0, \infty)$- which may depend on p and μ - such that

$$\forall f \in L^{p,\infty} : c_1[f]_p \leq ||f||_{p,\infty} \leq c_2[f]_p$$

Proof. Observe that if $\mu(|f| > t) > 0$ then

$$||f||_{p,\infty} \geq \frac{1}{\mu(\{|f| > t\})^{1-1/p}} \int_{\{|f|>t\}} |f| d\mu \geq t\mu(|f| > t)^{1/p}$$

so taking the supremium implies

$$||f||_{p,\infty} \geq [f]_p$$

For the reverse inequality fix a $E \in \mathcal{X}$ with $\mu(E) \in (0, \infty)$. Then from the Layer Cake Decomposition

$$\frac{1}{\mu(E)^{1-1/p}} \int_E |f| d\mu = \frac{1}{\mu(E)^{1-1/p}} \int_{t=0}^{\infty} \mu(\{x \in E : |f(x)| > t\}) dt \leq \frac{1}{\mu(E)^{1-1/p}} \int_{t=0}^{\infty} \min\{\mu(E), \mu(\{|f| > t\})\}$$

$$\leq \frac{1}{\mu(E)^{1-1/p}} \int_{t=0}^{\infty} \min\{\mu(E), [f]_p^p/t^p\} = \frac{1}{\mu(E)^{1-1/p}} \int_{t=0}^{\alpha} \mu(E) dt + \frac{1}{\mu(E)^{1-1/p}} \int_{\alpha}^{\infty} [f]_p^p/t^p dt$$

where $\alpha = [f]_p/\mu(E)^{1/p}$ so

$$= [f]_p + \frac{1}{p-1}[f]_p = C_p([f]_p)$$

as desired $\square$

Problem 4. Let $A \subset \mathbb{R}$ be measurable with positive Lebesgue measure. Prove that the set $A - A := \{z - y : z, y \in A\}$ has non-empty interior. Hint: Consider the function $\varphi(x) = \int \chi_A(x + y)\chi_A(y)$.

Proof. Assume for the sake of contradiction that $A - A$ has empty interior. Notice that $0 \in A - A$, so this means there exists a sequence $x_n \to 0$ such that $x_n \notin A - A$. Notice by translation continuity of the Lebesgue Integral that

$$\lim_{n\to\infty} \int \chi_A(y + x_n)\chi_A(y) dy = \int |\chi_A(y)|^2 dy = m(A) > 0$$

where m is the Lebesgue measure. However, observe

$$\int \chi_A(y + x_n)\chi_A(y) dy = \int_A \chi_A(y + x_n) dy$$

and $y + x_n \notin A$ for any $y \in A$. Indeed if $y + x_n \in A$ then $x_n = (y + x_n) - y$ so $x_n \in A - A$. Therefore,

$$\int_A \chi_A(y + x_n) dy = 0$$

which is a contradiction. Therefore, $A - A$ has non-empty interior. $\square$

Problem 5. Prove the following claim: Let$\mathcal{H}$ be a Hilbert space with he scalar product of x and y denoted by (x, y) and let $A, B : \mathcal{H} \to \mathcal{H}$ be (everywhere-defined) linear operators with

$$\forall x, y \in \mathcal{H} : (Bx, y) = (x, Ay)$$

Then A and B are both bounded (thus continuous).

Proof. Consider the family of

$$L_x(y) := (Bx, y)$$

where $||x|| \leq 1$ this is continuous since

$$|L_x(y)| \leq ||Bx||(||y||) = C(x)||y||$$

and by assumption we have

$$|L_x(y)| = |(x, Ay)| \leq (||x||)(||Ay||) \leq ||Ay||$$

i.e. the family is pointwise bounded. So by Uniform Boundness Principle, we deduce that

$$\sup_{||x|| \leq 1} ||L_x|| \leq C$$

for some C. Now observe that if $||x|| \leq 1$

$$||Bx|| = \sup_{||y|| \leq 1} (Bx, y) = \sup_{||y|| \leq 1} L_x(y) \leq C$$

Therefore, B is continuous and an identical argument shows that A is continuous.

□

Problem 6. Prove that there exists a continuous linear functional functional ϕ on $\ell^\infty(\mathbb{N})$ such that

$$\phi(x) := \lim_{n \to \infty} x_n$$

whenever the limit exists.

Also show that ϕ is not unique.

Proof. Let $S := \{x \in \ell^\infty(\mathbb{N}) : \lim_{n \to \infty} x_n \text{ exists } \}$ then S is a linear subspace of $\ell^\infty(\mathbb{N})$. So define $\phi : S \to \mathbb{R}$ via

$$\phi(x) = \lim_{n \to \infty} x_n$$

Observe also

$$|\phi(x)| \leq ||x_n||_{\ell^\infty}$$

so by Hanh Banach we can find an extension of ϕ from S to ℓ^∞ such that $||\phi|| \leq 1$. This proves that such a functional exists.

For non-uniqueness, observe that S is closed, so fix any $x \notin S$ then there is a $\delta > 0$ such that $d(x, S) = \delta > 0$ then define for any $y \in S$

$$\psi_\alpha(y + \lambda x) = \lambda\alpha||x||$$

where α is a constant we'll choose later. Observe

$$||y + \lambda x|| = \lambda||y/\lambda + x|| \geq \lambda\delta \geq \lambda|\alpha|(||x||) = |\psi_\alpha(y + \lambda x)|$$

where α is chosen so that $|\alpha|||x|| \leq \delta$. Then by Hanh Banach this function extends to a continuous functional on ℓ^∞. Notice $\psi_\alpha = 0$ on S, so if we choose $\alpha, \beta > 0$ with $\alpha \neq \beta$ such that $|\alpha|||x|| \leq \delta$ and $|\beta|||x|| \leq \delta$ then

$$\phi(x) + \psi_\alpha(x) \text{ and } \phi(x) + \psi_\beta(x)$$

are two such extensions and they are not equal since $\alpha \neq \beta$.

□

Problem 7. Let $J \subset \mathbb{R}$ be a compact interval, and let μ be a finite Borel measure whose support lies in J. For $z \in \mathbb{C} \setminus J$ define

$$F_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t)$$

Prove that the mapping $\mu \mapsto F_\mu$ is one-to-one

Proof. Assume that $J = [a, b]$ where $a \neq b$ and that $F_{\mu_1}(z) = F_{\mu_2}(z)$ then define $\mu := \mu_1 - \mu_2$ then we want to show $\mu = 0$. We have that for $z \in \mathbb{C} \setminus J$

$$F_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-t} d\mu(t) = \int_a^b \frac{1}{z-t} d\mu(t) = 0$$

Now we have for any $0 < h < 1/2$

$$\frac{F_\mu(b+1+h) - F_\mu}{h} = -\frac{1}{h} \int_a^b \frac{1}{(b+1-t)(b+1+h-t)} d\mu(t)$$

Notice that the integrand is bounded by $1/(1+h) \leq 1$ and μ is finite, so by the Dominated Convergence Theorem we have that

$$0 = \int_a^b \frac{-1}{(b+1-t)^2} d\mu(t)$$

and similarily by differentiating again with an identical argument gives for any $n \in \mathbb{N}$

$$0 = \int_a^b \frac{1}{(b+1-t)^n} d\mu(t)$$

so let $\mathcal{A}$ be the sub-algebra generated by $\{1/(b+1-t), 1/(b+1-t)^2, ...\}$ then this family of functions vanish nowhere $[a, b]$ since $1/(b+1-t) \neq 0$ on $[a, b]$. It also separates points since $1/(b+1-t)$ is injective. Therefore, as $[a, b]$ is compact, Stone Weiestrass tells us $\mathcal{A}$ is dense, from which it follows that if $f \in C([a, b])$ then

$$\int_a^b f(t) d\mu(t) = 0$$

i.e. $\mu = 0$ by Risez Representation Theorem, so the mapping $\mu \mapsto F_\mu$

□

Problem 8. A function $f : \mathbb{C} \to \mathbb{C}$ is entire and has the property that $|f(z)| = 1$ when $|z| = 1$. Prove that $f(z) = az^n$ for some integer $n \geq 0$ and some $a \in \mathbb{C}$ with $|a| = 1$.

Proof. See Spring 2016 Number 9.

□

Problem 9. Determine the number of zeros of the polynomial

$$P(z) = z^6 - 6z^2 + 10z + 2$$

in the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. Prove your claim.

Proof. Observe that if $A := \{z \in \mathbb{C} : 1 < |z| < 2\}$ then $|z^6 + 10z| > |2 - 6z^2|$ on ∂A, so by Rouche $P(z)$ has the same number of zeros on A as $z^6 + 10z$. But observe if $1 < |z| < 2$ that $z^6 + 10z = z(z^5 + 10)$ so the only roots on A have to be when $z^5 + 10 = 0$. Let ξ_i be a root of unit of order 5 i.e. $\xi_i^5 = 1$ then $-(10)^{1/5}\xi_i$ is a root of $z^5 + 10$ and there are 5 of them. And as $1 < (10)^{1/5} < 2$ it follows that these roots are in A, so $P(z)$ has 5 roots on the annulus.

□

Problem 10. Evaluate
$$\lim_{x\to\infty}\int_0^x \sin(t^2)dt$$
Justify all steps.

Proof. Recall that $\sin(t^2) = \mathrm{Im}(e^{it^2})$. Let $\gamma_{a\to b}$ be the straight line segment from a to b and $\gamma_R := Re^{i\theta}$ for $\theta \in [0, \pi/4]$ then define
$$\gamma := \gamma_{0\to R} + \gamma_R - \gamma_{0\to Re^{i\pi/4}}$$
Notice that
$$\int_{\gamma_R} e^{iz^2}dz = \int_0^{\pi/4} \exp(R^2(-\sin(2\theta) + i\cos(2\theta)))d\theta$$
Notice that
$$\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta} = 1$$
and so on $\theta \in [0, \pi/4]$ we have $\sin(2\theta)/(2\theta) \geq \delta > 0$ for some δ. Therefore,
$$\left|\int_{\gamma_R} e^{iz^2}dz\right| \leq \int_0^{\pi/4} |\exp(-R^2\delta)|d\theta = \pi/4\exp(-R^2\delta) \to 0 \text{ as } R \to \infty$$
And we also have
$$\int_{\gamma_{0\to Re^{i\pi/4}}} e^{iz^2}dz = Re^{i\pi/4}\int_{t=0}^1 \exp(i(t^2R^2e^{i\pi/2})) = Re^{i\pi/4}\int_0^1 \exp(-t^2R^2) = e^{i\pi/4}\int_0^R \exp(-t^2)dt$$
so
$$\lim_{R\to\infty}\int_{\gamma_{0\to Re^{i\pi/4}}} e^{iz^2}dz = e^{i\pi/4}\frac{\sqrt{\pi}}{2} = (1+i)\frac{\sqrt{\pi}}{2\sqrt{2}}$$
So it follows from Cauchy Theorem that
$$\lim_{R\to\infty}\int_0^R e^{it^2}dt = (1+i)\frac{\sqrt{\pi}}{2\sqrt{2}}$$
so
$$\int_0^\infty \mathrm{Im}(e^{it^2})dt = \int_0^\infty \sin(t^2)dt = \sqrt{\pi}/(2\sqrt{2})$$
□

Problem 11. Find a conformap map of the domain
$$D = \{z \in \mathbb{C} : |z-1| < \sqrt{2}, |z+1| < \sqrt{2}\}$$
onto the open unit disc centered at the origin. It suffices to write this map as a composition of explicit conformal maps.

Proof. Note that the two circle $B_{\sqrt{2}}(1)$ and $B_{\sqrt{2}}(-1)$ intersect at $z = \pm i$. So consider the conformal map
$$\varphi(z) := \frac{1-\sqrt{2}-i}{1-\sqrt{2}+i}\left(\frac{z+i}{z-i}\right)$$
this is a Mobius Transformation that sends $i \mapsto \infty, -i \mapsto 0, 1-\sqrt{2} \mapsto 1$. Using that Mobius Transformations maps circles to circles and lines, we deduce that φ maps D to
$$H := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) < 0\}$$
since $\varphi(1+\sqrt{2}) = -i$. Now we apply $\varphi_2 := iz$ to map this into the region $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } \mathrm{Im}(z) > 0\}$ and use the map z^2 to map it into $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and finally the Cayley Transform
$$\phi = \frac{z-i}{z+i}$$
to map this into the unit disc. Composing all of our maps give the desired conformal map.
□

Problem 12. Show that
$$F(z) := \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}dt$$
is well defined (by the integral) and analytic in $\{z \in \mathbb{C} : \mathrm{Re}(z) < 1/2\}$, and admits a meromorphic continuation to the region $\{z \in \mathbb{C} : \mathrm{Re}(z) < 3/2\}$

Proof. We define $t^z := \exp(z\log(t))$ where $\log(t)$ is the standard branch of the logarithim, so that t^z is holomorphic since $t \in [1,\infty)$. Write $z = x + iy$ and then
$$|t^{x+iy}| = |\exp(x\log(t) + iy\log(t)| = |\exp(x\log(t)| = t^x$$
Therefore,
$$|F(x+iy)| \leq \int_1^\infty \frac{t^x}{\sqrt{1+t^3}}dt \lesssim \int_1^\infty \frac{t^x}{t^{3/2}}dt$$
so if $\mathrm{Re}(z) = x < 1/2$ then $x - 3/2 < -1$ so that the integral defining $F(z)$ is absolutely integrable i.e. it is well defined. Also if $R \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 1/2\}$ is a rectangle then as the integral defining F is absolutely integrable we may apply Fubini to see
$$\int_{\partial R} F(z) = \int_{\partial R}\int_1^\infty \frac{t^z}{\sqrt{1+t^3}}dt = \int_1^\infty \int_{\partial R} \frac{t^z}{\sqrt{1+t^3}}dt = 0$$
where for the last equality we used $t^z/\sqrt{1+t^3}$ is holomorphic for $t \in [1,\infty)$. Therefore, by Morrera's Theorem since $F(z)$ is continuous we see that $F(z)$ (by the DCT) is holomorphic in $\{z \in \mathbb{C} : \mathrm{Re}(z) < 1/2\}$.

Intuitively from our earlier bound the pole should be at $z = 1/2$. So we rewrite
$$F(z) = \int_1^\infty \frac{t^z}{\sqrt{1+t^3}}dt = \int_1^\infty t^{z-3/2}\frac{t^{3/2}}{\sqrt{1+t^3}}dt$$
since we will want to integrate by parts to pick up a $1/(z-1/2)$ factor. This leads to by integration by parts
$$F(z) = \frac{1}{z-1/2} - \frac{3}{2(z-1/2)}\int_1^\infty \frac{t^z}{(1+t^3)^{3/2}}dt$$
so this integral converges by an identical computation as above if $\mathrm{Re}(z) - 9/2 < -1 \Rightarrow \mathrm{Re}(z) < 7/2$. Therefore, $F(z)$ extends meromorphically onto $\{z \in \mathbb{C} : \mathrm{Re}(z) < 3/2\}$ with a pole at $1/2$. $\square$

21. Spring 2020

Problem 1. Assume $f \in C_c^\infty(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} e^{-tx^2} f(x)dx = 0 \text{ for any } t \geq 0$$

show that $f(x) = -f(-x)$ for any $x \in \mathbb{R}$

Proof. Observe that we have for any $h > 0$

$$\int_{\mathbb{R}} \frac{(e^{-(t+h)x^2} - e^{-tx^2})}{h} f(x)dx = 0$$

so by the Mean Value Theorem and Dominated Convergence Theorem we can tkae $h \to 0$ inside the integral to conclude

$$\int_{\mathbb{R}} x^2 e^{-tx^2} f(x)dx = 0 \text{ for any } t \geq 0$$

then we can keep iterating this process since $p(x)e^{-x^2}$ for any polynomial is in L^1 to conclude that for any finite sum that

$$\sum_{n \text{ even}} \int_{\mathbb{R}} a_n x^n e^{-tx^2} f(x)dx = 0 \text{ for any } t \geq 0 \Rightarrow \sum_{n \text{ even}} \int_{\mathbb{R}} a_n x^n f(x)dx = 0$$

Now let f be supported on $[-M, M]$ then we have

$$\sum_{n \text{ even}} \int_{-M}^{M} a_n x^n f(x)dx = 0$$

By Stone Weiestrass, polynomials are dense in $C([-M, M])$ under the sup-norm. Then there is a polynomial $P(x) = \sum_{n=1}^{N} b_n x^n$ such that $||f(x) - P(x)||_{L^\infty([-M,M]} < \varepsilon/(2M)$ where $\varepsilon > 0$ is given. Now decompose $f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2} = f_{even}(x) + f_{odd}(x)$ i.e. the even and odd decomposition of f and observe that since the integral is symmetirc we have

$$\sum_{n \text{ even}} \int_{-M}^{M} a_n x^n f(x)dx = 0 \Rightarrow \sum_{n \text{ even}} \int_{-M}^{M} a_n x^n f_{even}(x)dx = 0$$

Therefore, since f_{even} is even and Stone-Weiestrass we conclude that f can be uniformly approxiamted in $[-M, M]$ by even polynomials, but then choosing a_n to be these polynomials coefficients lets us conclude that

$$\int_{-M}^{M} |f_{even}(x)|^2 dx = 0$$

so $f_{even} = 0$ i.e. f is an odd function as desired.

□

Problem 2. Assume $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of differentiable functions satisfying

$$\int_{\mathbb{R}} |f_n(x)|dx \leq 1 \text{ and } \int_{\mathbb{R}} |f_n'(x)|dx \leq 1$$

Assume also that for any $\varepsilon > 0$ there is an $R(\varepsilon) > 0$ such that

$$\sup_n \int_{|x| \geq R(\varepsilon)} |f_n(x)|dx \leq \varepsilon$$

Show that there exists a subsequence of $\{f_n\}$ that converges in $L^1(\mathbb{R})$.

Proof. **Proof One: Frechet Kolmogorov Theorem** We will prove the family is first equicontinuous in the L^p norm. Indeed, observe for any $h > 0$ that by the Fundamental Theorem of Calculus that

$$|f_n(x+h) - f_n(x)| \leq \int_x^{x+h} |f_n'(y)|dy$$

so we have

$$\int_{\mathbb{R}} |f_n(x+h)-f_n(x)|dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[x,x+h]}(y)|f_n'(y)|dydx = \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[y-h,y]}(x)|f_n'(y)|dxdy = h\int_{\mathbb{R}} |f_n'(x)|dx \leq h$$

where the interchange of integration is justified by Tonelli since the integrand is non-negative.

Now we claim this implies combined with the second given condition implies that $\{f_n\}$ is relatively compact in $L^1(\mathbb{R})$. By completeness of $L^1(\mathbb{R})$ it suffices to show that $\{f_n\}$ is totally bounded. We will do this with an approximation argument using Arzela-Ascoli. Indeed, fix any $\rho \in C_c^\infty(B_1(0))$ with $||\rho||_{L^1(\mathbb{R})} = 1$ and define $\rho_n := n\rho(x/n)$ also has mass 1. Then define for any $n, m \in \mathbb{N}$

$$g_{m,n}(x) := (\rho_m \star f_n)(x) = \int_{\mathbb{R}} \rho_m(x-y)f_n(y)dy$$

Then it is well known $g_{m,n}(x) \in C^\infty(\mathbb{R})$ and we have the bounds

$$||g_{m,n}||_{L^\infty} \leq ||f_n||_{L^1} \leq ||\rho_m||_{L^\infty} := C(m)$$

so the family $\{g_{m,n}\}_{n\in\mathbb{N}}$ is uniformly bounded in the sup norm. Also

$$|g_{m,n}(x) - g_{m,n}(x+h)| \leq \int_{\mathbb{R}} |\rho_m(y)||f_n(x+h-y) - f_n(x-y)|dy$$

$$\leq ||\rho_m||_{L^\infty}||f(x+h) - f(x)||_{L^1(\mathbb{R})} \leq h||\rho_m||_{L^\infty}$$

so the family $\{g_{m,n}\}_{n\in\mathbb{N}}$ is totally bounded in $C(|x| \leq R(\varepsilon))$. Therefore, if $\varepsilon > 0$ we can find finitely many N such that

$$\{g_{m,n}\}_{n\in\mathbb{N}} \subset \bigcup_{n=1}^N B_{\varepsilon/(2R(\varepsilon))}(g_{m,n})$$

where the ball is with respect to the sup norm. Now observe that

$$\int_{\mathbb{R}} |f_n(x)-g_{m,n}(x)|dx \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |\rho_m(y)||f_n(x)-f_n(x-y)|dxdy = \int_{B_{1/m}(0)} |\rho_m(y)|\int_{\mathbb{R}} |f_n(x)-f_n(x-y)|dxdy \leq \frac{1}{m} < \varepsilon$$

for m large enough. Notice that this bound is independent of n. Hence, now we claim that

$$\{f_n\} \subset \bigcup_{i=1}^N B_{5\varepsilon,L^1(\mathbb{R})}(f_j)$$

where these balls are taken over $L^1(\mathbb{R})$ metric. Indeed, observe if $k \in \mathbb{N}$ then we can find an $1 \leq n \leq N$ such that $||g_{m,n} - g_{m,k}||_{L^\infty} < \varepsilon$ then

$$||f_n - f_k||_{L^1(|x|\leq R(\varepsilon))} \leq ||f_n - g_{m,n}||_{L^1(|x|\leq R(\varepsilon)} + ||g_{m,k} - g_{m,n}||_{L^1(|x|\leq R(\varepsilon)} + ||f_k - g_{n,k}||_{L^1(|x|\leq R(\varepsilon)}$$

$$\leq 3\varepsilon$$

And we know that

$$||f_n - f_k||_{L^1(|x|>R(\varepsilon)} \leq ||f_n||_{L^1(|x|>R(\varepsilon)} + ||f_k||_{L^1(|x|>R(\varepsilon)} \leq 2\varepsilon$$

Hence, we conclude

$$||f_n - f_k||_{L^1(\mathbb{R})} \leq 5\varepsilon$$

so $\{f_n\}$ is totally bounded in $L^1(\mathbb{R})$; therefore, it is precompact since $L^1(\mathbb{R})$ is complete. So there exists along a sub-sequence denoted n_k and an $f \in L^1(\mathbb{R})$ such that $||f_{n_k} - f||_{L^1(\mathbb{R})} \to 0$ □

Proof Two: Helly's Selection Theorem

Proof. We use the following theorem:

Helly's Selection Theorem Given $\{f_n\}$ a sequence of monotone functions such that the family is uniformly bounded, there exists a sub-sequence that converges everywhere.

The proof is a standard application of Bolzano-Weiestrass, a diagonalization argument along $\mathbb{Q}$, and using the limiting function has to be monotone to define $f(x) := \lim_{q\to x} f(q)$ where $f(q)$ is the limit of the subsequence f_{n_k} along rationals.

Notice that this automatically extends into BV functions since they are the difference of two monotone functions. And observe that by the fundamental theorem of calculus that for all $x \in \mathbb{R}$

$$|f_n(x) - f_n(0)| \leq \int_{\mathbb{R}} |f_n'(x)|dx \leq 1$$

and from

$$\int_{\mathbb{R}} |f_n(x)|dx = 1$$

we conclude that $|f_n(x)| \leq 1$ a.e., which by continuity implies $|f_n(x)| \leq 1$ everywhere. And as the total variation of differentiable functions is just the L^1 norm of the derivative, we see that $\{f_n\}$ is a family of bounded variations that is uniformly bounded. So by Helly's Selection Theorem, we deduce that along a subsequence which we still denote by n that $f_n(x) \to f(x)$ pointwise everywhere. By Fatou's lemma we deduce

$$\int_{\mathbb{R}} |f(x)|dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|dx \leq 1$$

and for $\varepsilon > 0$

$$\int_{|x|\geq R(\varepsilon)} |f(x)|dx \leq \liminf_{n\to\infty} \int_{|x|\geq R(\varepsilon)} |f_n(x)| \leq \varepsilon$$

So now let $K := \overline{B_{R(\varepsilon)}(0)}$ which is compact so now fix $\varepsilon > 0$ and by Egorov's theorem we have a set $E \subset K$ such that $m(K \setminus E) < \varepsilon$ such that $f_n \to f$ uniformly on K. Then

$$\int_K |f_n(x) - f(x)|dx \leq \int_E |f_n(x) - f(x)|dx + \int_{K\setminus E} |f_n(x)| + |f(x)|dx$$

And as $||f_n||_{L^\infty} \leq 1$ we see that $||f||_{L^\infty} \leq 1$ so we see for n sufficiently large thanks to uniform convergence on E which is of finite measure that we have

$$\int_K |f_n(x) - f(x)|dx \leq 3\varepsilon$$

so for large enough n

$$\int_{\mathbb{R}} |f_n(x) - f(x)|dx \leq 5\varepsilon$$

Therefore, we have $||f_n - f||_{L^1} \to 0$.

□

Problem 3. Prove that $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ is a Borel subset of $L^3(\mathbb{R}^n)$

Proof. Observe that

$$L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n) = \bigcup_{n=1}^\infty \{f \in L^3(\mathbb{R}^n) : ||f||_{L^\infty(\mathbb{R}^n)} \leq n\} := \bigcup_{n=1}^\infty A_n$$

and we claim each A_n is closed in $L^3(\mathbb{R}^n)$. Indeed, if $f_n \in A$ converges to $f \in A$ in the L^3 sense we conclude that $f \in L^3(\mathbb{R}^n)$. Also along a subsequence which we denote by n_k we have $f_{n_k} \to f$ pointwise a.e., so we conclude that $||f||_{L^\infty(\mathbb{R}^n)} \leq n$ since each $||f_{n_k}||_{L^\infty} \leq n$. Therefore, $f \in A_n$ so we have written $L^\infty(\mathbb{R}^n) \cap L^3(\mathbb{R}^n)$ as a countable union of closed sets in $L^3(\mathbb{R}^n)$, so it is Borel subset of $L^3(\mathbb{R}^n)$.

□

Problem 4. Fix $f \in L^1(\mathbb{R})$. Show that

$$\lim_{n\to\infty} \int_0^2 f(x)\sin(x^n)dx = 0$$

Proof. We will first show the statement for step functions. By linearity of the integral it suffices to show that it is true for characteristic functions of intervals to show its true for all step functions. First observe that if $0 \le a < b < 1$ then

$$\int_a^b \sin(x^n)dx \to 0$$

since $\sin(x^n) \to 0$ as $0 < a, b < 1$ and by applying DCT with $f(x) = 1$ to get the above convergence.

Now we show that if $1 < a < b < 2$ then

$$\lim_{n\to\infty} \int_a^b \sin(x^n)dx = 0$$

via the method of non-stationary phase. Indeed observe that

$$\int_a^b \sin(x^n)dx = \mathrm{Im}\left(\int_a^b e^{ix^n}dx\right)$$

And we have

$$\int_a^b e^{ix^n}dx = \int_a^b \frac{1}{inx^{n-1}}\frac{d}{dx}\left(e^{ix^n}\right)dx$$

so integration by parts gives

$$= \frac{1}{in}\left(\frac{e^{ib^n}}{b^{n-1}} - \frac{e^{ia^n}}{a^{n-1}}\right) - \int_a^b e^{ix^n}\left(\frac{1-n}{inx^n}\right)$$

so it follows that

$$\left|\int_a^b e^{ix^n}dx\right| \le \frac{1}{n}\left(\frac{1}{b^{n-1}} + \frac{1}{a^{n-1}}\right) + (b-a)\frac{(1-n)}{na^n} \to 0$$

since $a, b > 1$

This implies that if $0 \le a < b < 2$ that

$$\int_a^b \sin(x^n)dx \to 0$$

Therefore, if $f(x)$ is a step function we have $\int_a^b f(x)\sin(x^n)dx \to 0$. Now if $f \in L^1([0,2])$ we can find a sequence of step functions f_n such that $f_n \to f$ in $L^1([0,2])$ so this gives

$$\left|\int_0^2 f(x)\sin(x^n)\right| \le \int_0^2 |f_m(x) - f(x)|dx + \left|\int_0^2 f_m(x)\sin(x^n)dx\right| \to 0$$

where the first term is small due to L^1 convergence and the second is true by our earlier computation. □

Problem 5. Rigorously determine the infimum of

$$\int_{-1}^1 |P(x) - |x||^2 dx$$

over all choices of polynomials $P(x) \in \mathbb{R}[x]$ of degree not exceeding three.

Proof. Consider the subspace V of at most three degree polynomials in $L^2([-1,1])$. It is clear that V is a subspace of $L^2(\mathbb{R})$. Observe then that if $f(x) \in L^2([-1,1])$

$$\inf_{P(x)\in V} ||f(x) - P(x)||^2 = \inf_{P(x)\in V} \int_{-1}^{1} |P(x) - |x||^2 dx$$

so this reduces to an orthogonal projection question. Indeed, if $f_\perp$ is the orthogonal project of f onto V then for any $P(x) \in V$

$$||P - f||^2 = ||P - f_\perp + f_\perp - f||^2 = ||V - f_\perp||^2 + ||f_\perp - f||^2$$

where in the third equality we used Pythagerous Theorem since $V \in (P - f_\perp) \perp (f_\perp - f)$, so it follows that $f_\perp$ is the infimum. To find $f_\perp$ we do Grahm Schmit on $\{1, x, x^2, x^3\}$. Doing this gives us that $\{1/2, \sqrt{3/2}x, (x^3 - 1/3)\sqrt{8/45}, (x^3 - 3/5x)\sqrt{175/8}\} := \{v_1, v_2, v_3, v_4\}$ is an orthonormal basis of V. Therefore, noting that

$$f_\perp = \sum_{i=1}^{4} (f, v_i)v_i$$

gives us the minimizer if $\sum_{i=1}^{4}(|x|, v_i)v_i$ as desired.

□

Problem 6. Let us define a sequence of linear functionals on $L^\infty(\mathbb{R})$ as follows:

$$L_n(f) := \frac{1}{n!}\int_0^\infty x^n e^{-x} f(x)dx$$

(1) Prove that no subsequence of this sequences converges weak-*.
(2) Explain why this does not contradict Banach-Alaoglu Theorem.

Proof. For the first part observe that $L_n(1)$ is just $\Gamma(n+1)$ so $L_n(1) = n!$. Now we need the following lemma

Lemma: For a fixed $\varepsilon > 0$ there exists a sequence of intervals $\{[a_n, b_n]\}$ for n large where $a_n, b_n \to \infty$ as $n \to \infty$ such that

$$\left| \frac{1}{n!}\int_{a_n}^{b_n} x^n e^{-x} dx - \frac{1}{n!}\int_0^\infty x^n e^{-x} dx \right| \le \varepsilon$$

i.e.most of the mass of $L_n(1)$ is concentrated on the intervals $[a_n, b_n]$.

Proof of Lemma Recall stirling's approximation i.e. $n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$ so we see that

$$\frac{1}{n!}\int_0^{n/3} x^n e^{-x} dx \sim \frac{e^n}{n^n\sqrt{2\pi n}}\int_0^{n/3} x^n e^{-x} dx \le \frac{n^n e^n}{\sqrt{2\pi}3^n n^{n+1/2}} = (\frac{e}{3})^n \frac{1}{\sqrt{2\pi}\sqrt{n}} \to 0$$

and as exponential growth beats polynomials decay we can find such a b_n. Hence, we have proved for n large that most of the mass lives in $[n/3, b_n]$ for some b_n that tends to ∞ as $n \to \infty$.

Now fix a subsequence n_k and by choosing a further subsequence if necessary we can assume that $[n_k/3, 3n_k]$ are disjoint. Then define

$$f := \sum_{k \text{ even}} \chi_{[n_k/3, 3n_k]}(x) - \sum_{k \text{ odd}} \chi_{[n_k/3, 3n_k]}(x)$$

which is well defined since all the intervals are disjoint. Then for n large enough, thanks to our lemma we know that $L_{n_k}(f) < 1/2$ if k is odd and $L_{n_k}(f) > 1/2$ if k is even. Therefore, this subsequence does not converge weak* and as this subsequence was arbitrary we conclude that L_n does not converge weak* on any subsequence.

For the second part, this does not contradict Banach Alagou since the weak* topology is not metrizable, so compactness is not equivalent to subsequential compactness.

□

Problem 7. Let $\mathcal{F}_M$ denote the set of functions holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ that satisfy

$$\int_0^{2\pi} |f(e^{i\theta})| d\theta \leq M < \infty$$

Show that every $\{f_n\}$ contains a subsequence that converges uniformly locally on $\mathbb{D}$.

Proof. Fix a compact set $K \subset \mathbb{D}$ then there is a $\delta > 0$ such that $d(K, \partial\mathbb{D}) = \delta$. Then we have by Cauchy's Integral Formula that for $w \in K$ and any $f \in \mathcal{F}_M$

$$f(w) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z-w} dz = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - w} e^{i\theta}$$

so

$$|f(w)| \leq \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \frac{|f(e^{i\theta})|}{\delta} d\theta \leq \frac{M}{2\pi\delta}$$

hence for any subsequence $\{f_n\} \subset \mathcal{F}_M$ we have that it is uniformly bounded on every compact subset, so it follows by Montel's theorem that along a subsequence f_n converges uniformly locally on $\mathbb{D}$.

□

Problem 8. For each $z \in \mathbb{C}$ define

$$F(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{(n!)^2}$$

(1) Show that the resulting function is entire and $|F(z)| \leq e^{|z|}$
(2) Show that there is an infinite sequence $a_n \in \mathbb{C}$ so that

$$F(z) = \prod_{n=1}^{\infty} (1 - \frac{z^2}{a_n^2})$$

and the product converges locally uniformly on $\mathbb{C}$.

Proof. For the first part it suffices to show the sum defining $F(z)$ converges locally uniformly since each term in the sum is holomorphic. Observe that since

$$e^{|z|} = \sum_{n=0}^{\infty} \frac{|z|^n}{n!}$$

converges for all $z \in \mathbb{C}$ that if $K \subset \mathbb{C}$ is compact then there is an $R > 0$ such that $K \subset B_R(0)$ so

$$|F(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{2^{2n}(n!)^2} \leq \sum_{n=0}^{\infty} (\frac{|R|^n}{(n!)}) \frac{|R|^n}{n!} \leq C(R) \sum_{n=0}^{\infty} \frac{|R^n|}{n!} = C(R)e^R < \infty$$

Therefore, by the Weiestrass M-test the series defining $F(z)$ converges locally uniformly on $\mathbb{C}$ so $F(z)$ is an entire function since each term in the series is entire.

For the bound, observe that

$$|F(z)| \leq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{2^{2n}(n!)^2}$$

and

$$e^{|z|} \geq \sum_{n=0}^{\infty} \frac{|z|^n}{n!} \geq \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!}$$

since we only omitted the odd terms in the series expansion. So it suffices to show that $(2n)! \leq 2^{2n}(n!)^2$. The case of $n = 1$ is obvious , so assume it is true for n and then we will show its true for $n+1$. Indeed, observe if

$$(2n)! \leq 2^{2n}(n!)^2 \Rightarrow (2n+1)! \leq (2n+1)2^{2n}(n!)^2$$

and it's clear that $(2n+1) \leq 4(n+1)^2$ so we conclude that $(2n+1)! \leq 2^{2n+2}(n+1)!^2$ so we obtain that

$$|F(z)| \leq e^{|z|}$$

as desired.

For the second part, recall from part a) that $F(z)$ is an entire function of order one, so by Hadamard's factorization theorem we can write

$$F(z) = e^{az+b}z^m \prod_{n=1}^{\infty}(1 - \frac{z}{a_n})e^{z/a_n}$$

for some $a, b \in \mathbb{C}$, m is the multiplicity of the zero of $F(z)$ at $z = 0$ and a_n are the zeros of $F(z)$ and the product converges locally uniformly on $\mathbb{C}$. Notice that $F(0) = 1$ so we conclude $m = 0$. Also as $F(z) = F(-z)$ we conclude for every $n \in \mathbb{N}$ we can find an $m \in \mathbb{N}$ such that $a_n = -a_m$. Now observe

$$(1 - \frac{z}{a_n})e^{z/a_n}(1 - \frac{z}{a_m})e^{z/a_m} = (1 - \frac{z^2}{a_n^2})$$

since $a_m = -a_n$, so by rearranging terms in the product which is allowed since the product converges locally uniformly we have that

$$F(z) = e^{az+b}\prod_{n=1}^{\infty}(1 - \frac{z^2}{a_n^2})$$

Now using $F(z) = F(-z)$ we conclude that $a = 0$. Using $F(0) = 1$ gives us $1 = e^b$ so

$$F(z) = \prod_{n=1}^{\infty}(1 - \frac{z^2}{a_n^2})$$

so it suffices to show that there are infinitely many zeros of $F(z)$.

Now assume for the sake of contradiction that there are only finitely many roots of $F(z)$ say $\{z_1, .., z_n\}$. Then $F(z)/\prod_{j=1}^{n}(z - z_j)$ is a entire function with no zeros so there is an entire function $g(z)$ such that

$$F(z) = \prod_{j=1}^{n}(z - z_j)e^{g(z)}$$

but as $|F(z)| \leq e^{|z|}$ we see that $|g(z)| \leq C|z|$, which by Cauchy's inequalities imply $g(z) = az+b$ for some $a, b \in \mathbb{C}$. Then observe as F is even that $\prod_{j=1}^{n}(z-z_j)$ is an even function since if $F(z_j) = 0 \Rightarrow F(-z_j) = 0$. Therefore, we see using $F(z) = F(-z)$ that $a = 0$ so $F(z) = \prod_{j=1}^{n}(z - z_j)e^b$ which means F is a polynomial. But clearly F is not a polynomial, which is our contradiction. Therefore, $F(z)$ has infinitely many zeros i.e. there are infinitely many a_n which lets us conclude.

□

Problem 9. Let $f(z) \in L^1(\mathbb{C}) \cap C^1(\mathbb{C})$. Show that the integral

$$u(z) = -\frac{1}{2\pi}\int\int_{\mathbb{C}} \frac{f(\xi)}{\xi - z} d\lambda(\xi)$$

defines a C^1 function on the entire plane such that

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) u(x+iy) = f(x+iy)$$

In this problem $d\lambda$ represents the Lebesgue measure on $\mathbb{C}$ and C^1 is meant in the real variable sense.

Proof. Note that the integral is well defined since $1/z \in L^1(B_1(0), d\lambda(z))$ and $f \in L^1(\mathbb{C})$. In particular, DCT implies that $u(z)$ is continuous. In addition, by a change of variables we have that

$$u(z) = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{f(\xi + z)}{\xi} d\lambda(\xi)$$

so we have that for h real that

$$\frac{u(z+h) - u(z)}{h} = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{f(\xi + z + h) - f(\xi + z)}{h\xi}$$

By Taylor Expansion of f we know that since h is real

$$f(\xi + z + h) = f(\xi + z) + \frac{\partial f}{\partial x}(\xi + z)h + o(h)$$

so

$$\lim_{h\to 0} \frac{u(z+h) - u(z)}{h} = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial f}{\partial x}(\xi + z)\frac{1}{\xi} d\lambda(\xi)$$

And similarily

$$\frac{\partial u}{\partial y}(z) = -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial_y f(\xi + z)}{\xi} d\lambda(\xi)$$

so we have

$$(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y})(z) = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{1}{\xi}\left(\frac{\partial f}{\partial x}(\xi + z) + \frac{\partial f}{\partial y}(\xi + z)\right)$$

$$= \lim_{\varepsilon\to 0, R\to\infty} -\frac{1}{2\pi}\int_{\varepsilon \le |z| \le R} \frac{1}{\xi}\left(\frac{\partial f}{\partial x}(\xi + z) + \frac{\partial f}{\partial y}(\xi + z)\right) d\lambda(z) = -\frac{1}{2\pi}\int_{\varepsilon \le |z| \le R} \frac{1}{\xi}\frac{\partial}{\partial \overline{z}} f(\xi + z) d\lambda(z)$$

so by the Generalized Stokes Theorem we have that

$$= \frac{1}{2\pi i}\int_{|z|=R} \frac{1}{\xi} f(\xi + z)dz - \frac{1}{2\pi i}\int_{|z|=\varepsilon} \frac{1}{\xi} f(\xi + z)dz)$$

Note the first integral goes to zero as $R \to \infty$ since $f(\xi)/\xi \in L^1(\mathbb{C})$. Now using that $\int_{|\xi|=\varepsilon} \frac{1}{\xi} d\xi = 2\pi i$ we have that

$$f(z) - \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{1}{\xi} f(\xi + z)dz = \frac{1}{2\pi i}\int_{|\xi|=\varepsilon} \frac{f(z) - f(\xi + z)}{\xi}$$

$$= \frac{1}{2\pi}\int_{\theta=0}^{2\pi} \frac{f(z) - f(\varepsilon e^{i\theta} + z)}{\varepsilon e^{i\theta}} \varepsilon e^{i\theta} d\theta \to 0$$

where the convergence is due to uniform continuity of f on $B_1(z)$. Therefore,

$$\partial_x + i\partial_y u = f$$

so it solves the PDE. □

Problem 10. Evaluate the improper Reimann Integral

$$\int_0^\infty \frac{x^2 - 1}{x^2 + 1}\frac{\sin(x)}{x} dx$$

Justify all manipulations

Proof. □

Problem 11. Let $K \subset \mathbb{T}$ be a compact proper subset.

(1) Show there is a sequence of polynomials $P_n(z) \to \overline{z}$ uniformly on K.
(2) Show there is no sequence of polynomials $P_n(z)$ that uniformly converges to $\overline{z}$ on $\mathbb{T}$

Proof. For the first part this is just simply a consequence of Runge's theorem since $\overline{z} = 1/z$ on $\mathbb{T}$. Recall that Runge's Theorem implies that if K is compact such that $\mathbb{C} \setminus K$ is connected, then any holomorphic function on a neighborhood of K can be uniformly approximated by polynomials in K. Indeed, $1/z$ is holomorphic on $\mathbb{C} \setminus \{0\}$ and since K is a proper subset of $\mathbb{T}$ we have $\mathbb{C} \setminus K$ is connected.

Indeed, recall that Runge's Theorem shows by using the Cauchy Integral Formula and a Riemann Sum Approximation that if f is holomorphic on a neighborhood of a compact set K, then there is a sequence of rational functions $\{R_n\}$ with poles outside of K such that R_n uniformly converges to f. It suffices to show that the rational function $1/(z - z_0)$ where $z_0 \notin K$ can be uniformly approximated in K by polynomials when $\mathbb{C} \setminus K$ is connected since every R_n can be written as a polynomial combination of such functions.

Indeed, fix $z_0 \notin K$, then choose a $z_1 \notin K$ far away from z_0. Then as $\mathbb{C} \setminus K$ is open and connected, we can find a curve $\gamma : [0,1] \to K^c$ such that $\gamma(0) = z_1$ and $\gamma(1) = z_0$. Then we have

$$\frac{1}{z - z_1} = -\frac{1}{z_1}\frac{1}{1 - z/z_1} = -\frac{1}{z_1}\sum_{n=0}^{\infty}(\frac{z}{z_1})^n$$

where the sum converges when we choose z_1 such that $|z/z_1| < 1$ for all $z \in K$. Choosing such a z_1 we see that $1/(z - z_1)$ can be uniformly approximated in K by its partial sums due to Weiestrass M-Test, so $1/(z - z_1)$ can be uniformly approximated by polynomials in K. Now if we let $\rho := 1/2d(K, \gamma)$ then we can choose points $\{w_i\}_{i=0}^N$ such that $w_i \in \gamma$ with $|w_i - w_{i+1}| < \rho$ and $w_0 = z_1$ and $w_N = z_0$. Then we claim we can uniformly approximate $1/(z - w_{i+1})$ by polynomials in $1/(z - w_i)$. Indeed, observe

$$\frac{1}{z - w_{i+1}} = \frac{1}{(z - w_i)(1 - \frac{w_{i+1} - w_i}{z - w_i})} = \frac{1}{z - w_i}\sum_{n=0}^{\infty}\left(\frac{w_{i+1} - w_i}{z - w_i}\right)^n$$

which converges since $|\frac{w_{i+1} - w_i}{z - w_i}| < 1/2$. Then this means we can uniformly approximate $1/(z - w_0)$ in K by polynomials since we can do it for $1/(z - w_1)$ combined with our earlier observation that we can uniformly approximate $1/(z - w_0)$ by polynomials of $1/(z - w_1)$. Then by iterating this process N we deduce that $1/(z - z_0)$ can be uniformly approximated by polynomials in K and hence so can every rational function with poles outside of K. This proves Runges Theorem.

Assume for the sake of contradiction that $1/z$ can be uniformly approximated by polynomials P_n on $\mathbb{T}$. Then we have from uniform convergence

$$2\pi i = \int_{|z|=1} \frac{1}{z} dz = \int_{|z|=1} \lim_{n\to\infty} P_n(z) dz = \lim_{n\to\infty} \int_{|z|=1} P_n(z) = 0$$

which is our contradiction.

Alternative Proof: Assume that $P_n(z)$ a sequence of polynomials uniformly converge to $\overline{z} = 1/z$ on $\mathbb{T}$. Then the uniform limit of these polynomials denoted by $f(z)$ extends to be a holomorphic function on $\mathbb{D}$ such that $f(z) = 1/z$ on $\mathbb{T}$. Then we have that for large enough n that

$$\sup_{|z|=1} |P_n(z) - 1/z| < 1 \Rightarrow \sup_{|z|=1} |zP_n(z) - 1| < 1$$

so by the maximum modulus principle $|zP_n(z) - 1| < 1$ on $\mathbb{D}$ but taking $z = 0$ gives a contradiction.

□

Problem 12. Let u be a continuous subharmonic function on $\mathbb{C}$ that satisfies

$$\limsup_{|z|\to\infty} \frac{u(z)}{\log|z|} \leq 0$$

Show that u is constant.

Proof. This given condition implies if $\varepsilon > 0$ that $u(z) - \varepsilon \log|z| \to -\infty$ as $|z| \to \infty$. Now consider the annular domain $\Omega_{R,1} := \{1 \leq |z| \leq R\}$ for $R > 1$. As $\log|z|$ is harmonic on $\Omega_{R,1}$ we have that $u(z) - \varepsilon \log|z|$ is subharmonic on $\Omega_{R,1}$, so by the maximum principle

$$\sup_{z\in\Omega_{R,1}} u(z) - \varepsilon\log(z) = \max_{z\in\partial\Omega_{R,1}} u(z) - \varepsilon\log(z)$$

But from the given conditions of $u(z) - \varepsilon \log|z| \to -\infty$ we see by taking $R \to \infty$ that

$$\sup_{z\in|z|>1} u(z) - \varepsilon\log(z) = \max_{|z|=1} u(z)$$

where we used log(1)=0 and continuity of u to deduce theres a max over $\{|z| = 1\}$. Now by letting $\varepsilon \to 0$ we see that

$$\sup_{z\in|z|>1} u(z) = \max_{|z|=1} u(z)$$

but then this implies that $u(z)$ has an interior maximum on $\overline{\mathbb{D}} \subset \mathbb{C}$. Therefore, $u(z)$ is constant (subharmonic functions with an interior maximum are constant thanks to the submean value inequality).

□

Problem 1. Suppose $f : [0,1] \times [0,\infty) \to [0,1]$ is continuous. Prove that
$$F(x) := \limsup_{y\to\infty} f(x,y)$$
is Borel Measurable.

Also show that for any borel set $E \subset [0,1]$ there is a choice of continuous function $f : [0,1] \times \mathbb{R} \to [0,1]$ so that F agrees with the indicator function almost everywhere.

Proof. Note that intervals of the form $[a,b]$ for $0 < a < b < 1$ generate the borel σ-algebra on $[0,1]$, so it suffices to show $F^{-1}(a,b)$ is borel. Observe
$$F^{-1}([a,b]) = \bigcup_{M=0}^{\infty} \bigcap_{q\in\mathbb{Q}:q\geq M} \{x : b \geq f(x,q) \geq a\}$$
i.e. there is an $M \in \mathbb{N}$ such that for any rational $q \geq M$ we have $b \geq f(x,q) \geq a$. Note that we are using continuity to conclude that if for all $\mathbb{Q} \ni q \geq M$ we have $b \geq f(x,q) \geq a$ then for any $y \geq M$ that $b \geq f(x,y) \geq a$ i.e. $b \geq F(x) \geq a$.

Define $f_q := f(x,q)$ then as f is continuous so is f_q so we know that $\{x : b > f(x,q) > a\} = f_q^{-1}((a,b))$ so this set is open. Therefore, $F^{-1}([a,b])$ is Borel Measurable. It is also easy to verify that $\{E \text{ Borel} : F^{-1}(E) \text{ is Borel }\}$ is a σ-algebra and we showed it contains the closed set so it contains the borel σ-algebra, so F is Borel Measurable.

For the second part, let A denote the set of Borel Subsets of $[0,1]$ such that there is a continuous function f so that F agrees a.e. with the indicator function of that set. We claim A is a σ-algebra. Indeed, observe that $[0,1] \in A$ and $\emptyset \in A$ by defining $f(x,y) = 1$ and $f(x,y) = 0$ respectively.

Now if $E \in A$ then there is a continuous f such that the corresponding F agrees a.e. with the indicator function of E. Then notice $1 - F$ then agrees almost everywhere with the indicator function of $E^c \cap [0,1]$, which corresponds to the function $1 - f$. So this set is closed under compliments.

Now we claim that A is closed under finite intersections. Indeed, if $E_1, E_2 \in A$ then there is a corresponding f_1 and f_2 continuous such that F_i agrees a.e. with the indicator function of E_i. Then observe $\chi_{A\cap B} = \chi_A \chi_B$ so we can take $g := f_1 f_2$ then $\limsup_{y\to\infty} g(x,y)$ will agree with $\chi_{E_1\cap E_2}$ a.e.

Now let $\{E_i\}_{i=1}^{\infty} \subset A$. As A is closed under finite intersections and compliments, we may assume WLOG that E_i are disjoint. Then if $\{f_i\}$ are the corresponding continuous functions we can define $f := \sum_{i=1}^{\infty} f_i(x,y)$ which is well defined when y is large since E_i are disjoint. Therefore, this set is closed under countable union.

So now it suffices to show that it A contains intervals of the form (a,b). So define for $y \geq 1$
$$f(x,y) := \begin{cases} 0 \text{ for } x \leq a - 1/y \\ y(x - (a - 1/y)) \text{ for } a - 1/y \leq x \leq a \\ 1 \text{ for } a \leq x \leq b \\ -y(x - (b + 1/y)) + 1 \text{ for } b \leq x \leq b + 1/y \\ 0 \text{ else} \end{cases}$$
i.e. we are adjoining a small line to the end of $\chi_{[a,b]}$ that dissapears as $y \to \infty$. This f has all the desired properties for $y \geq 1$ and for $y < 1$ just make the function equal to $f(x,1)$. Therefore, A contains the borel σ-algebra, which implies the problem.

□

Problem 2. Show that there is a constant $c \in \mathbb{R}$ so that

$$\lim_{n\to\infty}\int_0^1 f(x)\cos(\sin(2\pi nx))dx = c\int_0^1 f(x)dx$$

for every $f \in L^1([0,1])$ where n is taken over $\mathbb{N}$.

Proof. Observe that

$$\int_0^1 \cos(\sin(\pi nx))dx = \frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1}\cos(\sin(\pi x))dx = \int_0^1 \cos(\sin(\pi x))dx =: c$$

Now fix $f \in C([0,1])$ then observe that

$$\int_0^1 f(x)\cos(\sin(\pi nx))dx = \frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} f(\frac{x}{n})\cos(\sin(\pi x))dx$$

So by uniform continuity of f if $\varepsilon > 0$ we can find a $\delta > 0$ such that if $x, y \in [0,1]$ and $|x-y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Choose n so large such that $1/n < \delta$ then

$$\left|\frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} f(\frac{x}{n})\cos(\sin(\pi x))dx - \frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} f(\frac{j}{n})\cos(\sin(\pi x))\right| \leq \frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1}|f(x/n) - f(j/n)| \leq \varepsilon$$

And observe that

$$\frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} f(\frac{j}{n})\cos(\sin(\pi x)) = c\frac{1}{n}\sum_{j=0}^{n-1} f(\frac{j}{n}) \to c\int_0^1 f(x)dx$$

where the last convergence is due to f is continuous so the Riemann Sums converge to the integral of f. Therefore, we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int_j^{j+1} f(\frac{x}{n})\cos(\sin(\pi x))dx = \lim_{n\to\infty}\int_0^1 f(x)\cos(\sin(\pi nx))dx = c\int_0^1 f(x)dx$$

so this shows the result for $f \in C([0,1])$, so now by density if $f \in L^1([0,1])$ then there is a sequence $f_n \in C([0,1])$ such that $||f_n - f||_{L^1([0,1])} \to 0$ so

$$\left|\int_0^1 f(x)\cos(\sin(2\pi nx))dx - c\int_0^1 f(x)dx\right| \leq \int_0^1 |f(x) - f_m(x)|dx + \left|\int_0^1 f_m(x)\cos(\sin(2\pi nx))dx - c\int_0^1 f_m(x)dx\right|$$

$$+c\int_0^1 |f(x) - f_m(x)|dx$$

so all three terms converge to 0 in the limit. Therefore,

$$\lim_{n\to\infty}\int_0^1 f(x)\cos(\sin(2\pi nx))dx = c\int_0^1 f(x)dx$$

for all $f \in L^1([0,1])$ for $c := \int_0^1 \cos(\sin(\pi x))dx$.

$\square$

Problem 3. Let $d\mu_n$ be a sequence of probability measures on $[0,1]$ so that
$$\int_0^1 f(x)d\mu_n(x)$$
converges for every $f \in C([0,1])$.

(1) Show that
$$\int\int_{[0,1]^2} g(x,y)d\mu_n(x)d\mu_n(y)$$
converges for every $g \in C([0,1]^2)$.

(2) Show by example that under hypothesis, it is possible that
$$\int\int_{0\le x\le y\le 1} d\mu_n(x)d\mu_n(y)$$
does not converge.

Proof. As $[0,1]$ is compact and μ_n is a sequence of Borel Probability Measures, we know that they are Radon. Therefore, by Risez Representation Theorem and Banach Alagou there is a subsequence such that $\mu_{n_k} \rightharpoonup \mu$ i.e. for any $f \in C([0,1])$ we have
$$\int_0^1 f(x)d\mu_{n_k}(x) \to \int_0^1 f(x)d\mu$$
And by the given assumption we know that $\int_0^1 f(x)d\mu_n$ converges so it must converge to $\int_0^1 f(x)d\mu$. Now if we fix two sub-sequence, then each by Banach Alaogu has a further weakly convergent sub-sequence and if we test any $f \in C([0,1])$ along the limit measures we obtain the same value. Therefore, by Risez Representation Theorem since measures are uniquely determined by their action on $C([0,1])$ we conclude that the two weak limits are the same, so as every subsequence has a further subsbequence that converges to the same limit we know that $\mu_n \rightharpoonup \mu$.

Now we claim that $\mu_n\otimes\mu_n \rightharpoonup \mu\otimes\mu$ which will prove the claim. Indeed let $\mathcal{A} \subset C([0,1]^2)$ be the algebra generated by functions of the form $f(x)g(y)$ where $f,g \in C([0,1])$. Then constants are in this space, so $\mathcal{A}$ vanishes nowhere and if $(s_1,t_1) \neq (s_2,t_2)$ then WLOG $x \neq s$, then byd efining $\mathcal{A} \ni g(x,y) = x$ then $g(s_1,t_1) \neq g(s_2,t_2)$, so this family vanishes nowhere. Therefore, as $[0,1]^2$ is compact, we know by Stone Weiestrass that $\mathcal{A}$ is dense in $C([0,1]^2)$.

So there is a sequence $\{f_n(x)h_n(y)\}$ that converges to $g(x,y)$ in the sup norm. Then
$$\int\int_{[0,1]^2} f_n(x)h_n(y)d\mu_m(x)d\mu_m(y) = \int_0^1 f_n(x)d\mu_m(x)\int_0^1 h_n(y)d\mu_m(y) \to \int_0^1 f_n(x)d\mu(x)\int_0^1 h_n(y)d\mu(y)$$
$$= \int_{[0,1]^2} f_n(x)h_n(y)d\mu(x)d\mu(y)$$
where the interchange in derivatives is justified since $f_nh_n \in L^1([0,1]^2, \mu_m\otimes\mu_m)$ for any m because f_nh_n is bounded and $\mu_m\otimes\mu_m$ is a probability measure.

So uniform convergence implies
$$\lim_{m\to\infty}\int\int_{[0,1]^2} g(x,y)d\mu_m(x)d\mu_m(y) = \int\int_{[0,1]^2} g(x,y)d\mu(x)d\mu(y)$$
so the limit exists.

□

Problem 4. Let X be a separable Banach space over $\mathbb{R}$ and let $F : X \to \mathbb{R}$ be norm-continuous and convex. Suppose $x_n \rightharpoonup x$ show that

$$F(x) \leq \sup_n F(x_n)$$

Proof. We claim that $F(x)$ is weakly lower semi continuous which implies the claim. Indeed, it suffices to show if $\alpha \in \mathbb{R}$ then $A := \{x : F(x) \leq \alpha\}$ is weakly closed. Indeed, notice that A is convex since F is convex and is norm-closed. So now let $y \in A^c$, then $\{y\}$ is compact and convex since it is a singleton, so by Hanh Banach there exists a linear functional ℓ and $\beta \in \mathbb{R}$ such that

$$A \subset \{x : \ell(x) < \beta\} \text{ and } y \in \{x : \ell(x) > \beta\}$$

Notice as linear functionals generate the weak topology that $\{x : \ell(x) > \beta\}$ is open in the weakly open. Therefore, A is weakly closed, so F is weakly lower semi-continuous. So fix $0 < \varepsilon \ll 1$ then as $\{y : F(y) > F(x) - \varepsilon\}$ is weakly open and x is i this set, we see that

$$\liminf_{n\to\infty} F(x_n) > F(x) - \varepsilon$$

since $x_n \rightharpoonup x$ so letting $\varepsilon \to 0$ gives

$$F(x) \leq \liminf_{n\to\infty} F(x_n) \leq \sup_n F(x_n)$$

as desired.

□

Problem 5. Suppose $f \in L^1([0,1])$ has the property that

$$\int_E |f(x)|dx \leq m(E)^{1/2}$$

for any Borel Set $E \subset [0,1]$. Here m represents the Lebesgue measure on $[0,1]$.

(1) Show that if $1 \leq p < 2$ then $f \in L^p([0,1])$.

(2) Show that there is a function f satisfying the above bounds and is in $L^1([0,1])$ but is not in $L^2([0,1])$.

Proof. Note that the original question has a typo and it was supposed to be $1 \leq p < 2$.

Observe by the Layer Cake Decomposition that

$$\int_0^1 |f(x)|^p dx = p\int_{t=0}^{\infty} t^{p-1} m(\{x : |f(x)| \geq t\})dt \leq p\int_{t=0}^{\infty} t^{p-2}\left(\int_{\{|f|\geq t\}} |f(x)|dx\right) dt$$

where the second inequality is due to Chebyshev's Inequality. Therefore, combining this with $m(\{x : |f(x)| \geq t\}) \leq 1$ gives

$$\begin{aligned}\int_0^1 |f(x)|^p &\leq p\int_{t=0}^{\infty} t^{p-2} \min(1, \left(\int_{\{|f|\geq t\}} |f(x)|dx\right))\\ &\leq p\int_{t=0}^{1} t^{p-2}dt + \int_{t=1}^{\infty} t^{p-2}\left(\int_{\{|f|\geq t\}} |f(x)|dx\right)\end{aligned}$$

and as $1 < p < 2$ notice the first term is bounded. So we focus on bounding the second term. Using the given inequality gives the second term is bounded by

$$\leq p\int_{t=1}^{\infty} t^{p-2}(m(\{x : |f(x)| \geq t\}))^{1/2}dt \leq p\int_{t=1}^{\infty} t^{p-2-1/2}\left(\int_{\{|f|\geq t\}} |f(x)|dx\right)^{1/2} dt$$

and by reiterating the above argument n times we deduce that

$$\int_{t=1}^{\infty} t^{p-2}\left(\int_{\{|f|\geq t\}} |f(x)|dx\right) \leq p\int_{t=1}^{\infty} t^{p-2-\sum_{j=1}^{n} 1/2^j}\left(\int_{|f|\geq t} |f(x)|dx\right)^{1/2n} dt$$

And for any fixed n we can bound $\left(\int_{|f|\geq t} |f(x)|dx\right)^{1/2n}$ since $f \in L^1([0,1])$ and this integral converges as long as

$$p - 2 - \sum_{j=1}^{n} 1/2^j < -1$$

i.e.

$$p < 1 + \sum_{j=1}^{n} 1/2^k$$

and as $n \to \infty$ the sum converges to 1, so we conclude that $f \in L^p([0,1])$ for $1 \leq p < 2$ from our inequalities.

For the counter example take $f(x) := 1/2x^{-1/2}$ which is non-negative. Then notice for any $0 < a < b < 1$ that

$$\int_a^b f(x)dx = \sqrt{b} - \sqrt{a} \leq \sqrt{b-a}$$

where the inequality can be seen by squaring both sides. Then observe that

$$\{E \text{ Borel } : \int_E f(x)dx \leq m(E)^{1/2}\}$$

forms a σ-algebra, which contains the open intervals, so this is true for any Borel set and $f \notin L^2$ since $1/|x| \notin L^1([0,1])$.

□

Problem 6. Suppose $f : [-1,1] \to \mathbb{R}$ is C^1 and odd. Show that

$$\int_{-1}^{1} |f(x)|dx \leq \int_{-1}^{1} |f'(x)|dx$$

Proof. First observe that as f is odd that f' is even, so it suffices to show the inequality on $[0,1]$. Now as $f(x)$ is odd we know that $f(0) = 0$ so by the fundamental theorem of calculus if $0 < x < 1$ we have

$$f(x) = \int_0^x f'(y)dy \Rightarrow \int_0^1 |f(x)|dx \leq \int_0^1 \int_0^x |f'(y)|dydx$$

$$= \int_0^1 \int_0^1 \chi_{[0,x]}(y)|f'(y)|dydx = \int_0^1 |f'(y)| \int_0^1 \chi_{[0,x]}(y)dx = \int_0^1 y|f'(y)|dy \leq \int_0^1 |f'(y)|dy$$

where the interchange in integrals is justified by Fubini since the integrand is non-negative. In particular, we have shown

$$\int_0^1 |f(x)|dx \leq \int_0^1 |f'(y)|dy$$

which implies from the odd condition on f that

$$\int_{-1}^{1} |f(x)|dx \leq \int_{-1}^{1} |f'(y)|dy$$

□

Problem 7. Let $\Delta_j = \{z : |z - a_j| \leq r_j\}$ where $1 \leq j \leq n$ be a collection of closed disks with radii $r_j \geq 0$, all contained in $\mathbb{D}$. Let $\Omega := \mathbb{D} \setminus (\bigcup_j \Delta_j)$ and let $u : \Omega \to \mathbb{R}$ be harmonic. Prove that there exists real numbers $c_1, .., c_n$ such that

$$u(z) - \sum_{i=1}^{n} c_i \log |z - a_i|$$

is the real part of a holomorphic function on Ω. Show also that $c_1, ..., c_n$ is unique.

Proof. By considering the Cauchy Riemann equations we define $g(z) := \partial_x u(z) - i\partial_y u(z)$ which is holomorphic in Ω since it is C^1 in the real sense and satisfies the Cauchy Riemann Equation since u is harmonic. Now as $u(z)$ is holomorphic in a small ball that contains Δ_j that is disjoint from the other Δ_i, we conclude that $u(z)$ admits a Laurent Series Expansion near each Δ_j. So let c_j be the residue of g at a_j then consider

$$h(z) := g(z) - \sum_{i=1}^{n} \frac{c_j}{z - a_j}$$

then by the Residue Theorem we know that $h(z)$ integrates to zero along any closed curve. Therefore, $h(z)$ admits a primitive which we call $v(z)$. We claim that up to a constant that

$$w(z) := u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$

is the real part of $v(z)$. Indeed, observe that by computation one has

$$\partial_x w - i\partial_y w = g(z) = \partial_x \text{Re}(v(z)) - i\partial_y \text{Re}(v(z))$$

so one has $w = \text{Re}(v(z)) + C$ for some constant. Therefore, we have that $w(z)$ is the real part of a holomorphic functions. So now it remains to verify uniqueness of c_i.

So if

$$w(z) := u(z) - \sum_{j=1}^{n} c_j \log |z - a_j|$$

is the real part of a holomorphic function on Ω of say f then by the Cauchy Riemann equations ($f' = w_x - iw_y$)

$$f'(z) = g(z) - \sum_{i=1}^{n} \frac{c_j}{z - a_j}$$

and as this function must integrate to zero along any closed curve, we see that c_j must be the residue of $g(z)$ since we can take a small circular curve around each Δ_j and apply the Residue Theorem.

□

Problem 8. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and satisfy $f(1/2) = f(-1/2) = 0$. Show that

$$|f(0)| \leq 1/4$$

Proof. Consider the automorphism of the disk

$$\psi(z) := \frac{z - (1/2)}{1 - (1/2)z}$$

then $g := f/\psi$ is a holomorphic function since there is a removable singularity at $z = 1/2$ and $g(-1/2) = 0$. It is still a map to the disk because of the maximum principle since $|\psi_1| = 1$ on $\partial\mathbb{D}$. Then as $g \circ \psi_1(0) = 0$ we have from the Schwarz lemma that

$$|g \circ \psi| \leq |z| \Rightarrow |g(0)| \leq |\psi^{-1}(0)| = 1/2$$

But observe that $g(0) = -2f(0)$ so we conclude that

$$|f(0)| \leq 1/4$$

as desired.

□

Problem 9. Consider the following region
$$\Omega := \{x+iy : 0<x<\infty \text{ and } 0<y<1/x\}$$
Exhibit an explicit conformal map from Ω to $\mathbb{D}$.

Proof. Observe $\partial\Omega = \{(x,0) : x \geq 0\} \cup \{(x,1/x) : x > 0\} \cup (0,y) : y \geq 0$ then
$$z^2(\partial\Omega) = \{(x,0) : x \in \mathbb{R}\} \cup \{(x,2i) : x \in \mathbb{R}\}$$
so we know that $z^2(\Omega)$ is either contained in $\{(x,y) : x \in \mathbb{R} : 0 < y < 2\}$ or its complement since z^2 is continuous and Ω is connected. But by plugging in a point we see that $z^2(\Omega) = \{(x,y) : x \in \mathbb{R} : 0 < y < 2\} := \Omega_1$ since z^2 is conformal on Ω. Now consider $\frac{\pi}{2}z(\Omega_1) = \{(x,y) : x \in \mathbb{R}, 0 < y < \pi\} := \Omega_2$. Then $\exp(\Omega_2) = \{(x,y) : x \in \mathbb{R}, y > 0\}$ so then we take the Cayley Transform
$$\phi(z) := \frac{z-i}{z+i}$$
then this is a conformal map from the upper half plane to the disk. Then $\varphi(z) := \phi \circ \exp \circ (\pi/2z) \circ z^2$ is our desired map.

□

Problem 10. Let $K \subset \mathbb{C}$ be a compact set of positive area but empty interior and define a function $F : \mathbb{C} \to \mathbb{C}$ via
$$F(z) := \int_K \frac{1}{w-z} d\mu(w)$$
where $d\mu$ denotes the planar measure on $\mathbb{C}$.

(1) Prove that $F(z)$ is bounded and continuous on $\mathbb{C}$ and analytic on $\mathbb{C} \setminus K$.
(2) Prove that $F(K) = F(\mathbb{C})$.

Proof. As K is compact there is an $R > 0$ such that $K \subset B_R(0)$ then
$$|F(z)| \leq \int_{B_R(0)} \frac{1}{|w-z|} d\mu(w) \leq \int_{B_R(0)} \frac{1}{|w|} d\mu(w) := C < \infty$$
where we are using $1/|x| \in L^1(B_R(0), d\mu(w))$ since $\mathbb{C} \cong \mathbb{R}^2$. This implies $F(z)$ is bounded on $\mathbb{C}$. Note that $F(z)$ is continuous since $1/w \in L^1(B_R(0), d\mu(w))$ due to the continuity of the Lebesgue integral with respect to translation.

Now let $R \subset \mathbb{C} \setminus K$ be a rectangle, then as F is bounded we have that it is locally integrable, so
$$\int_R F(z)dz = \int_R \int_K \frac{1}{w-z} d\mu(w)dz = \int_K \int_R \frac{1}{w-z} dz d\mu(w) = 0$$
where the interchange in derivative is justified by Fubini and the last expression is 0 since $w \in K$ and $R \subset \mathbb{C} \setminus K$ so $1/(w-z)$ is holomorphic on R. Therefore, as $F(z)$ is continuous and integrates to zero over any rectangle $\mathbb{C} \setminus K$, Morrera's theorem tells us that $F(z)$ is analytic on $\mathbb{C} \setminus K$

Part 2: Missing

□

Problem 11. Let f_n be a sequence of analytic functions on a connected domain Ω such that $|f_n| \leq 1$ for all $n \in \mathbb{N}$ and $z \in \Omega$. Suppose the sequence $\{f_n(z)\}$ converges for infinitely many z in a compact set K in Ω. Prove that $\{f_n(z)\}$ converges for all $z \in \Omega$.

Proof. Let $K \subset \Omega$ be compact then there is a $\delta > 0$ such that $d(K, \partial\Omega) = 2\delta$ then for any $z \in K$ we have $B_\delta(z) \subset \Omega$. Therefore, by Cauchy's estimate we have for $z \in \Omega$

$$|f_n'(z)| \leq 1/\delta$$

so the family $\{f_n(z)\}$ is uniformly bounded and equicontinuous so by Arzela-Ascoli there is a sub-sequence such that f_n converges uniformly on K. Then by a standard diagonlization argument, we can find a sub-sequence that converges locally uniformly on Ω and by Cauchy's Integral Theorem we see local uniform convergence implies the limit is holomorphic.

Let $\{z_j\}$ be infinitely many points in K such that $\{f_n(z_j)\}$ converges. As K is compact there is a limit point $z \in K$ and by looking at a subsequence if necessary assume that $z_n \to z$. Then given two susbequence of f_n, by our previous argument we can find further subsequences where both converge locally uniformly to a holomorphic function $f(z)$ and $g(z)$ respectively. But we know that $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$ and $f(z) = g(z)$ since $\{f_n(z_j)\}$ converges. Therefore, have $f = g$ since holomorphic functions are determined by their values on any infinite set with an accumulation point. So every sub-sequence has a further sub-sequence that converges where the limit is the same, so the whole sequence converges.

□

Problem 12. Let $\Omega := \{z \in \mathbb{C} : -2 < z < 2\}$. Show that there is a finite $C > 0$ such that

$$|f(0)|^2 \leq C \int_{-\infty}^{\infty} [|f(x-i)|^2 + |f(x+i)|^2] dx$$

for every holomorphic function $f : \Omega \to \mathbb{D}$ for which the right hand side is finite.

Proof. We will prove this by using Cauchy's Integral Formula over a large rectangle. Indeed, let $S_R := \{(x, y) : -R \leq x \leq R, -1 \leq y \leq 1\}$ with counter clockwise orientation

$$f(0) = \int_{\partial S_R} \frac{f(w)}{w} dw = i \int_{t=-1}^{1} \frac{f(R+it)}{R+it} dt - \int_{-R}^{R} \frac{f(t+i)}{t+i} dt - i \int_{t=-1}^{1} \frac{f(-R+it)}{-R+it} dt + \int_{-R}^{R} \frac{f(t-i)}{t-i} dt$$

so by the triangle inequality combined with $|f| \leq 1$ gives

$$|f(0)| \leq \int_{t=-1}^{2} \frac{1}{|R+it|} dt + \int_{-R}^{R} \frac{|f(t+i)|}{|t+i|} dt + \int_{-R}^{R} \frac{|f(t-i)|}{|t-i|} dt$$

and taking $R \to \infty$

$$|f(0)| \leq \int_{-\infty}^{\infty} \frac{|f(t+i)|}{|t+i|} + \frac{|f(t-i)|}{|t-i|} dt$$

so using Cauchy-Schwarz gives

$$\begin{aligned} |f(0)| &\leq ||f(t+i)||_{L^2(\mathbb{R},dt)} ||1/(t+i)||_{L^2(\mathbb{R},dt)} + ||f(t-i)||_{L^2(\mathbb{R},dt)} ||1/(t-i)||_{L^2(\mathbb{R},dt)} \\ &\leq C(||f(t+i)||_{L^2(\mathbb{R},dt)} + ||f(t-i)||_{L^2(\mathbb{R},dt)}) \end{aligned}$$

where we used $1/(t+i) \in L^2(\mathbb{R}, dt)$ since $1/|t+i|^2$ decays fast enough at ∞ and thanks to the i factor it is integrable near the origin and similarily for $1/|t-i|$. Now using $(|a| + |b|)^2 \leq 4(|a|^2 + |b|^2)$ gives

$$|f(0)|^2 \leq 4C \int_{\mathbb{R}} |f(x+i)|^2 + |f(x-i)|^2 dx$$

as desired.

□