MATH 31A: WEEK 6

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0.1. **Linearization.** Given a function $f(x)$ that is differentiable at $x = x_0$ our idea is to try to approximate $f(x)$ by an easier to understand function with a controlled error. We will show that the tangent line is a good approximation to $f(x)$. Recall that as $f(x)$ is differentiable at $x = x_0$ then

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0)$$

so

$$\lim_{h \to 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \right) = 0$$

Let $\varepsilon(h)$ denote the error between the secant line of $f(x_0)$ and $f(x_0 + h)$ and $f'(x_0)$ i.e.

$$\varepsilon(h) := \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)$$

Then we have

$$\lim_{h \to 0} \varepsilon(h) = 0$$

so for h close to 0

$$\varepsilon(h) = \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \approx 0$$

So we have by multiplying both sides by h that

$$f(x_0 + h) - f(x_0) - hf'(x_0) \approx 0$$

i.e.

$$f(x_0 + h) \approx f(x_0) + hf'(x_0)$$

So we can approximate for h close to 0 $f(x_0 + h)$ by the tangent line of $f(x_0)$.

Let us see how much error we get from this approximation. Going back to

$$\varepsilon(h) = \frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0)$$

we get by multiplying both sides by h that

$$h\varepsilon(h) = f(x_0 + h) - f(x_0) - hf'(x_0)$$

so

$$h\varepsilon(h) + hf'(x_0) + f(x_0) = f(x_0 + h)$$

Recalling we are approximating $f(x_0 + h)$ by $f(x_0) + hf'(x_0)$ we get

$$f(x_0 + h) - (f(x_0) + hf'(x_0)) = h\varepsilon(h)$$

where the left hand side is the error in our approximation of $f(x_0+h)$ using the tangent line approximation. So the error in our approximation is always $h\varepsilon(h)$. It is known that

$$|h\varepsilon(h)| \leq M\frac{h^2}{2}$$

for $M := \max |f''(x)|$ where the max occurs is taken on the closed interval $[x_0, x_0 + h]$ (if $h > 0$) or $[x_0 + h, x_0]$ (if $h < 0$). This means

$$|f(x_0 + h) - (f(x_0) + hf'(x_0))| = |h\varepsilon(h)| \leq M\frac{h^2}{2}$$

i.e. at worst the error in our approximation is $M\frac{h^2}{2}$.

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0.2. **Example.** Approximate sin(.1) using linearization at $x_0 = 0$ and find the maximum between this approximation and sin(.1).

Solution: As $x_0 = 0$ we have

$$\sin(h) \approx \sin(0) + \cos(0)h$$

where we used $\frac{d}{dx}\sin(x) = \cos(x)$. So we have

$$\sin(.1) \approx .1$$

By our previous discussion the maximum error is $M\frac{h^2}{2}$ where M is the max of $|-\sin(x)|$ on $[0,h]$ since $\frac{d^2}{dx^2}\sin(x) = -\sin(x)$. We can use $|\sin(x)| \leq 1$ to get

$$M\frac{h^2}{2} \leq \frac{h^2}{2}$$

so the error is at worst $\frac{(.1)^2}{2} = \frac{1}{200}$. For a sharper bound we can use $\max_{x\in[0,\frac{1}{10}]}|-\sin(x)| = .0998.. \approx .1$ so the error is at worst approximately

$$.1\frac{h^2}{2} = .1\frac{1}{200} = \frac{1}{2000}$$

and we have

$$|\sin(.1) - .1| = .00016653 < \frac{1}{2000}$$

so we see the approximation is really good.

0.3. **Example.** Linearize $f(x) = \sqrt{8+x}$ at $x_0 = 0$ and use this to approximate $f(.4)$ and find the maximum and relative error in this approximation.
Solution The Tangent Line Approximation says

$$f(h) \approx f(0) + f'(0)h$$

Then using

$$f'(x) = \frac{1}{2\sqrt{8+x}}$$

gives

$$f'(0) = \frac{1}{2\sqrt{8}}$$

and $f(0) = \sqrt{8}$ so

$$f(h) \approx \sqrt{8} + \frac{1}{2\sqrt{8}}h$$

so

$$f(.4) \approx \sqrt{8} + \frac{4}{10}\frac{1}{2\sqrt{8}} = \sqrt{8} + \frac{1}{5\sqrt{8}}$$

Again by our discussion we have the error is at worst $M\frac{h^2}{2}$ for $M = \max_{x\in[0,.4]}|f''(x)|$. Then using

$$f''(x) = -\frac{1}{4(8+x)^{3/2}}$$

so

$$\max_{x\in[0,.4]}|f''(x)| = |f''(0)| = \frac{1}{64\sqrt{2}}$$

so the error is at most

$$\frac{1}{64\sqrt{2}}\frac{(.4)^2}{2} \approx .0009$$

and in fact we have

$$|f(.4) - \sqrt{8} - \frac{1}{5\sqrt{8}}| = .00086... < .0009$$

so our error bound is correct. The relative error is given by

$$\frac{|f(.4) - \sqrt{8} - \frac{1}{5\sqrt{8}}|}{|f(.4)|} = .0002975...$$

0.4. **Max/Min.** Extreme Value Theorem: If $f(x)$ is continuous on the closed interval $[a, b]$ then it attains its max and minimum inside the interval.

The hypothesis that the interval is closed is very important because there are continuous functions on open that do not attain a max or a min inside the open interval. For example $f(x) = x + 2$ on the open interval $(0, 1)$ has no max or min inside $(0, 1)$. In general, there may be a max or a min of a continuous function inside an open interval, but we cannot guarantee that a min or max will exist inside an open interval.

To find the max or min of a continuous function $f(x)$ on $[a, b]$ first differentiate $f(x)$ and find all the critical points i.e. x^* such that $f'(x^*) = 0$ or $f'(x^*)$ Does Not Exist. Then compare $f(a)$, $f(b)$ with $f(x^*)$ for every x^* that is a critical point. The maximum and minimum of $f(a)$, $f(b)$ with $f(x^*)$ for every x^* that is a critical point is the maximum and minimum of the function.

0.5. **Example.** Find the maximum and minimum of $f(x) = \sqrt{x^2+2} - 2x$ on $[0, 2]$.

0.6. **Solution.** First we differentiate $f(x)$ and find the critical points.

$$f'(x) = \frac{2x}{2\sqrt{x^2+2}} - 2$$

Note that this is defined everywhere so we just look for x^* such that $f'(x^*) = 0$. Then

$$\frac{2x}{2\sqrt{x^2+2}} - 2 = 0$$

implies by subtracting the -2 and multiplying by $2\sqrt{x^2+2}$ that

$$2x = 4\sqrt{x^2+2}$$

so

$$x = 2\sqrt{x^2+2}$$

so squaring both sides gives

$$x^2 = 4x^2 + 8$$

i.e.

$$0 = 3x^2 + 8$$

so there is no x^* such that $f'(x^*) = 0$ since there is no real solutions of $0 = 3x^2 + 8$. Therefore, as there is no critical points the maximum and minimum of $f(x)$ are the maximum and minimum of $f(0)$ and $f(2)$. And we have

$$f(0) = 0 \quad f(2) = \sqrt{6} - 4 < 0$$

so the maximum is $f(0) = 0$ and the minimum is $f(2) = \sqrt{6} - 4$.

0.7. **Increasing/Decreasing.** We say that a function is increasing if for any $x < y$ we have

$$f(x) < f(y)$$

And we say that a function is decreasing if for any $x < y$ we have

$$f(y) < f(x)$$

Note that if $f'(x) > 0$ on (a, b) then f is increasing on (a, b) and if $f'(x) < 0$ on (a, b) then f is decreasing on (a, b).

0.8. **Example.** Show that $f(x) = 4x^5 + 3x^3 + 20x$ has no zeros on $(0, \infty)$.

0.9. **Solution.** First notice $f(0) = 0$ and 0 is not in the open interval $(0, \infty)$. Next by differentiating we have

$$f'(x) = 20x^4 + 9x^2 + 20 > 0$$

since $20x^4 + 9x^2 \geq 0$ and $20 > 0$. Therefore, $f(x)$ is increasing on $(0, \infty)$ i.e. for any $x > 0$ we have

$$f(x) > f(0) = 0$$

so for any $x > 0$ we cannot have $f(x) = 0$.