Goals : Sapling #8

Linearization

Max/min

Sapling #8

On $(x^2+y^2)^2 = 10(x^2-y^2)+6$ find $(x,y)$

s.t. $dy/dx = 0$

Solution

$\frac{d}{dx}(x^2+y^2)^2 = \frac{d}{dx}(10(x^2-y^2)+6)$

$2(x^2+y^2)(2x+2y\,dy/dx) = 10(2x-2y\,dy/dx)$

$(x^2+y^2)(x+y\,dy/dx) = 5(x-y\,dy/dx)$

$(x^2+y^2)x + (x^2+y^2)y\,dy/dx = 5x - 5y\,dy/dx$

$(x^2+y^2)x + (x^2+y^2)y\,dy/dx + 5y\,dy/dx = 5x$

$(x^2+y^2)y\,dy/dx + 5y\,dy/dx = 5x - (x^2+y^2)x$

$dy/dx\,((x^2+y^2)y+5y) = 5x-(x^2+y^2)x$

$dy/dx = \frac{5x-(x^2+y^2)x}{(x^2+y^2)y+5y}$

$dy/dx = 0$ so numerator has to be 0

so $0 = 5x - (x^2+y^2)x$

$0 = x(5-(x^2+y^2))$

$x=0$ or $x^2+y^2=5$ ←

Case : $x>0$ $y>0$ (Have $x^2+y^2=5$)

$(x^2+y^2)^2 = 10(x^2-y^2)+6$ find $(x,y)$

$5^2 = 10(x^2-y^2)+6$

$5^2 + 10(x^2+y^2) = 10(x^2-\cancel{y^2})+6 + 10(x^2+\cancel{y^2})$

$25 + 10(5) = 20x^2+6$

$75 = 20x^2+6$

$\frac{69}{20} = x^2$

$x = \sqrt{69/20}$ To find y

use $x^2+y^2=5$.

Linearization Idea: Replace a function w/ an easy to understand function w/ a controlled error.

Recall $\lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) = f'(x_0)$

$\rightarrow \left(\varepsilon(h) = \frac{f(x_0+h)-f(x_0)}{h} - f'(x_0)\right)$

$\lim_{h\to 0}\varepsilon(h) = 0$

And $h\varepsilon(h) = f(x_0+h) - f(x_0) - hf'(x_0)$ [Multiply by h]

$h\varepsilon(h) + hf'(x_0) + f(x_0) = f(x_0+h)$

$f(x_0+h) \approx f(x_0) + hf'(x_0)$

In fact $|h\varepsilon(h)| \leq Mh^2/2$ $M = \max|f''|$ $f(x_0+h)\approx f(x_0)+hf'(x_0)$

w/ an error at most $Mh^2/2$

Ex) Approximate $\sin(.1)$ and list the max error in the approximation

$\sin(.1) \approx \sin(0) + \cos(0)(.1)$ $[x_0=0]$

$\approx .1$ $[h=.1, x_0=0]$

$\frac{d^2}{dx^2}(\sin(x)) = -\sin(x)$ $|h\varepsilon(h)| \leq Mh^2/2 = h^2/2$ (plug in $h=.1$) so $\frac{M(1/10)^2}{2}$

$M = \max|f''(x)| = \max|-\sin(x)| = 1$ $= \frac{M}{200}$

Ex) Linearize $\sqrt{8+x}$, $\Delta x = .4$ means find linearization at .4 using 0

$f(x) = \sqrt{8+x}$, $f'(x) = \frac{1}{2\sqrt{8+x}}$ ← Redo

$f(0) = \sqrt{8}$

$f'(0) = \frac{1}{2\sqrt{8}}$

$f'(x) = \frac{(8+x)^{-1/2}}{2}$ $\quad f''(x) = \frac{-(8+x)^{-3/2}}{4} = \boxed{-\frac{1}{4(8+x)^{3/2}}}$

$f(.4) \approx \sqrt{8} + (.4)\frac{1}{2\sqrt{8}}$

$\approx f(0) + \Delta x\, f'(0)$

So the error is at most $Mh^2/2$ $\quad \Delta x = h$

$M = \max_{x \text{ in } [x_0, x_0+h]} |f''(x)|$ $\quad x_0 = 0$, $h = .4$ in this case

$M = \left| \frac{1}{4(8)^{3/2}} \right|$

So the max error is $\frac{1}{4(8)^{3/2}} \left( \frac{(.4)^2}{2} \right)$

Ex 3) $y^4 + 2xy = 11$ find linearization at $(1,5)$ ← ($x=1$, $y=5$)

$4y^3 y' + 2(y + xy') = 0$

$y'(4y^3 + 2x) + 2y = 0$

$y'(4(5)^3 + 2(1)) + 2(5) = 0$

$y'(600+2) + 10 = 0$

$y' = -10/602$

$y(x+h) \approx y(x) + y'(x)(h)$

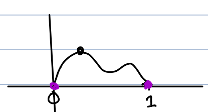
$\approx 5 - 10/602\,(h)$

Approximate $y$ at 1.1

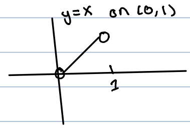
$y(1.1) \approx 5 - 10/602\,(.1)$

Maxima / Minima

EVT: If $f(x)$ is continuous on $[a,b]$ then it has a max & min



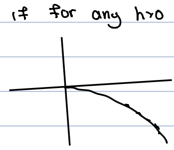
False if interval is open



$y = x$ on $(0,1)$ No max or min in $(0,1)$

$f$ is increasing if for any $h>0$ $f(x+h) > f(x)$

$f$ is decreasing if for any $h>0$ $f(x) > f(x+h)$

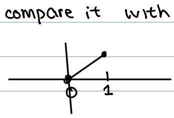


In general the max/min of $f(x)$ on $[a,b]$ occurs when

$f'(x) = 0$ $\quad f'(x)$ DNE or

at end points

Strategy Find all critical points $(x^*)$ and compute $f(x^*)$ and compare it with $f(a)$ & $f(b)$



$y(x) = x$ $\quad y' = 1$

Ex) Find max of $\sqrt{x^2+2} - 2x$ on $[0,2]$

$\frac{d}{dx}\left(\sqrt{x^2+2} - 2x\right)$

$\frac{2x}{2\sqrt{x^2+2}} - 2 = \frac{x}{\sqrt{x^2+2}} - 2 = 0$

$\frac{x}{\sqrt{x^2+2}} = 2$

$(x)^2 = (2\sqrt{x^2+2})^2$

$x^2 = 4(x^2+2)$

$x^2 = 4x^2+8$

$0 = 3x^2+8$ ← no real soln

So just compare $f(0)$ & $f(2)$

$f(0)=0$, $f(2) = \sqrt{2^2+2} - 4 = \sqrt{6} - 4 < 0$

Ex) Show $f(x) = 4x^5 + 3x^3 + 20x$ has no zeros on $(0,\infty)$

First

$f(0) = 4(0)^5 + 3(0)^3 + 20(0) = 0$

$f'(x) = 20x^4 + 9x^2 + 20 > 0$ so $f$ is inc

If $f' > 0$ then $f$ is increasing

$f' < 0$ then $f$ is dec

for any $h>0$ $f(0) < f(h)$

So for any $x>0$

$f(x) > f(0) = 0$ so

there is no zeros on $(0,\infty)$