Goals | MVT & Second Derivatives

MVT | If $f(x)$ is cts on $[a,b]$ and diff on (a,b) then there is a c in (a,b) s.t.
$$\underbrace{f'(c)}_{\text{instantaneous slope}} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{avg slope}}$$

Ex) What value of c satisfies the MVT for $f(x) = 6x^2-4x-5$ on $[3,10]$.

Soln | MVT says there is a c in $(3,10)$ s.t.
$$f'(c) = \frac{f(10)-f(3)}{10-3} = \frac{6(10)^2-4(10)-5-[6(3)-4(3)-5]}{7}$$
$$= \frac{6(100)-40-\cancel{5}-18+12+\cancel{5}}{7}$$
$$= 518/7 = \boxed{74}$$

$f'(x) = 12x-4$
$74 = f'(c) = 12c-4 \Rightarrow \boxed{78/12 = c}$

Ex) If $f(x)$ is diff s.t. $f'(x) \geq 1$ w/ $f(2)=8$. Find the largest $M>0$ s.t. $f(4) \geq M$

Soln | By MVT $\frac{f(4)-f(2)}{4-2} = f'(c)$ for some c in $(2,4)$. We are told $f'(x) \geq 1$ so $f'(c) \geq 1$
so
$$\frac{f(4)-f(2)}{2} \geq 1 \Rightarrow f(4) \geq 2+f(2)$$
$$f(4) \geq 10 \leftarrow$$

To show 10 is ~~the~~ the largest we find a function with $f'(x) \geq 1$, $f(2)=8$ and $f(4)=10$

$y = x+6 \leftarrow$ $y(2)=8$, $y'=1$
$y(4)=10$

So $M=10$

Work sheet #1

1) a) The avg slope is $\frac{f(b)-f(a)}{b-a}$

$f(x) = \sin(x)-x^3$ on $[0,\pi]$

$$\frac{f(\pi)-f(0)}{\pi} = \frac{\cancel{\sin(\pi)}-\pi^3-\cancel{\sin(0)+0^3}}{\pi}$$
$$= -\pi^2$$

b) Use MVT to show $\exists\, w>0$ s.t. $\cos(w) = 3w^2-\pi^2$

By MVT there is a w in $(0,\pi)$ s.t.
$$f'(w) = \frac{f(\pi)-f(0)}{\pi} = -\pi^2$$

$f'(x) = \cos(x)-3x^2$
$f'(w) = \cos(w)-3w^2$

so
$$\cos(w)-3w^2 = -\pi^2$$
$$\cos(w) = 3w^2-\pi^2$$

2) Want to show if $f(x)=e^x$ then there is a $c>0$ s.t.
$$f'(c) = \frac{e^2-1}{2} = \frac{e^2-e^0}{2-0}$$

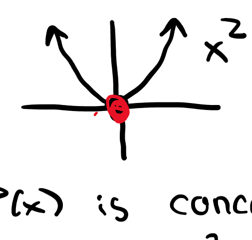
$e^0 = 1$

So apply MVT on the interval $[0,2]$

Second derivative | If $f(x)$ is diff w/ $f'(x)>0$ on (a,b) then f is increasing.

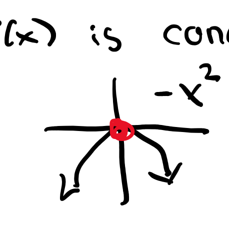
If $f(x)$ is diff w/ $f'(x)<0$ on (a,b) then f is decreasing

We say $f(x)$ is concave up if f' is increasing $[f''(x)>0$ on $(a,b)]$



$f''(x) = 2$

We say $f(x)$ is concave down if f' is decreasing $[f''(x)<0$ on $(a,b)]$



Inflection pts are where $f''(x) = 0$ or DNE $\leftarrow$ pts when concavity can change

Second deriv test | If x^* is a crit pt s.t. $f''(x^*)>0$ then it's a min
If x^* is a crit pt s.t. $f''(x^*)<0$ then it's a max

Worksheet #5 | $f(x) = 2x^6-5x^4$

$f'(x) = 12x^5-20x^3 \leftarrow$
$f''(x) = 60x^4-60x^2$
$= 60x^2(x^2-1)$

c s.t. $f''(c)=0$
$0 = 60c^2(c^2-1)$
$c=0$ or $c=\pm 1 \leftarrow$ all the inflection pts

+	−	−	+

$-1 \quad 0 \quad 1$

on $(-\infty,-1)$ and $(1,\infty)$ concave up & on $(-1,0)$ and $(0,1)$ its concave down

$f''(-2) = 60(-2)^2((-2)^2-1)$
$60(4)(4-1) > 0$

$f''(-1/2) = 60(-1/2)^2((-1/2)^2-1)$
$60/4\,(\underbrace{1/4-1}_{<0})$

$f''(1/2) = 60(1/2)^2((1/2)^2-1)$
$60/4\,(\underbrace{1/4-1}_{<0})$

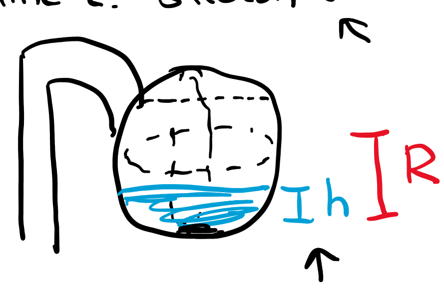
$f''(2) = 60(2)^2(2^2-1) = 60(4)(4-1) > 0$

Find the crit pts & use the second derivative test to see if the crit pts are a max or min or not enough info from the test.

$f'(x) = 12x^5-20x^3$
$0 = x^3(12x^2-20)$

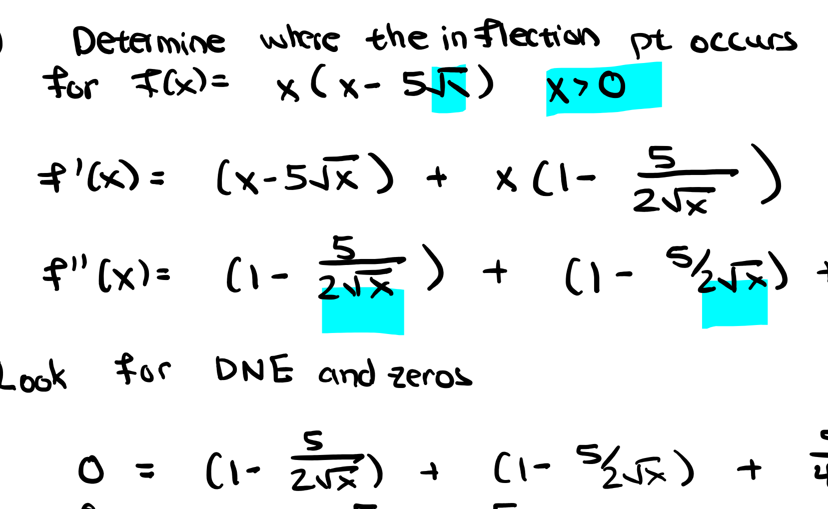
$x=0$, $\quad 12x^2-20=0$
not enough info $\quad 12x^2 = 20$
$x^2 = 20/12$
$x = \pm\sqrt{20/12} \approx \pm 1.291 \leftarrow$ min

59) Water is pumped into a sphere of radius R at a variable rate s.t. the water level rises at a constant rate.
Let $V(t)$ denote the volume of the water at the tank at time t. Sketch V



Think of V' | slowest near bottom and top
fastest at the middle

V' is increasing from bottom to middle
V' is decreasing from middle to top



Sapling #9) Determine where the inflection pt occurs for $f(x) = x(x-5\sqrt{x})$ $x>0$

Soln) $f'(x) = (x-5\sqrt{x}) + x\left(1-\frac{5}{2\sqrt{x}}\right)$

$f''(x) = \left(1-\frac{5}{2\sqrt{x}}\right) + \left(1-\frac{5}{2\sqrt{x}}\right) + x\left(\frac{5}{4x^{3/2}}\right)$

Look for DNE and zeros

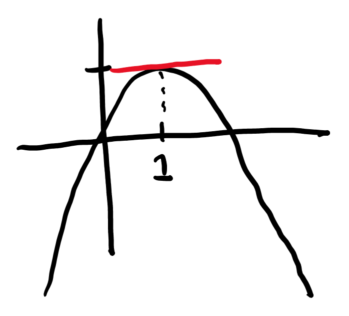
$0 = \left(1-\frac{5}{2\sqrt{x}}\right) + \left(1-\frac{5}{2\sqrt{x}}\right) + \frac{5}{4\sqrt{x}}$
$0 = 2-\frac{5}{\sqrt{x}}+\frac{5}{4\sqrt{x}}$
$0 = 2-\frac{20}{4\sqrt{x}}+\frac{5}{4\sqrt{x}}$
$0 = 2-\frac{15}{4\sqrt{x}}$
$\frac{15}{4\sqrt{x}} = 2$
$15 = 8\sqrt{x}$
$\left(\frac{15}{8}\right)^2 = x$

Concavity $\quad + \quad -$

$0 \quad (15/8)^2$

Concave up on $(0,(15/8)^2)$
down on $((15/8)^2,\infty)$

7)



Find inequalities between $f(1)$, $f'(1)$, $f''(1)$
$f(1)>0$
$f'(1)=0$
$f''(1)<0$

$f''(1) < f'(1) < f(1)$

13 $f(x) = 21x^{17/3}-51\sqrt{x}$ $\quad x\geq 0$
Do not want to include $x=0$ as a crit pt b/c f is not defined for $x<0$