Wednesday, November 11, 2020 11:04 PM

Goals: Linearization + Extrema
Sapling 8,23,2 & Written Hwk 2b)

SMC Hours 11 AM - 12 PM Today
Office Hours 1 PM - 2 PM Today

Linearization | Approximate a function $f(x)$ by a line

Derivatation | We have
$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a)$$
So when Δx is small
$$\frac{f(a+\Delta x) - f(a)}{\Delta x} \approx f'(a)$$
$$(*) \quad f(a+\Delta x) \approx \underbrace{\Delta x f'(a) + f(a)}_{\text{this is just the tangent line}}$$

Example | Sapling #2

Estimate Δf using a linear approximation and use a calculator to approximate

$f(x) = \sqrt{3+x}$ $a=2$, $\Delta x = .1$

Solution | $\Delta f = f(a+\Delta x) - f(a)$

$f(a+\Delta x) \approx \Delta x f'(a) + f(a)$

So $f(a+\Delta x) - f(a) \approx \Delta x f'(a) = .1 f'(a)$

$f'(x) = \dfrac{1}{2\sqrt{3+x}}$ so $f'(a) = f'(2) = \dfrac{1}{2\sqrt{3+2}} = \dfrac{1}{2\sqrt{5}}$

So $\Delta f \approx .1 \left[\frac{1}{2\sqrt{5}}\right]$ ← approximate error

Actual error is $\left|\Delta f - .1\left[\frac{1}{2\sqrt{5}}\right]\right|$ ← plug in calc to get Δf

Percentage error is
$$\frac{|\text{error}|}{|\Delta f|}$$

Sapling #8 | A player located at 19.7 ft from a basket launches a successful jumpshot at a height of 10 ft at an angle $\theta = 30°$ and initial velocity $v = 27$ ft/sec. Find an approximation for Δs - if the toss angle is changed by $\Delta\theta$. [in degrees]

Hint | $s = \frac{1}{32} v^2 \sin(2\theta)$

$\Delta s = s(\theta + \Delta\theta) - s(\theta) \approx s'(\theta)\Delta\theta$

$s' = \frac{1}{32} v^2 \cos(2\theta) \cdot 2 = \frac{1}{16} v^2 \cos(2\theta) = \frac{(27)^2}{16} \cos(2\theta)$

At $\theta = 30° = \pi/6$ radian

$s'(\theta) = \frac{(27)^2}{16} \cos(\pi/6)$

$\Delta s = \frac{(27)^2}{16} \cos(\pi/6) \, \Delta\theta°$ ← in degrees

$= \frac{(27)^2}{16} \cos(\pi/6) \left(\frac{\Delta\theta}{180}\pi\right)$ ← in radian

Extrema |

Sapling 23) Let $\phi(\omega) = \dfrac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4D^2\omega^2}}$ $\omega > 0$

ω_0 & D are positive constants

a) Find the values of D for which ϕ obtains a maximum in $(0,\infty)$

Note that $\lim_{\omega\to\infty} \phi(\omega) = 0$ [because $\phi(0) > 0$]

can restrict $[0, M]$ for large M to find max b/c eventually $\phi(\omega) < \phi(0)$ for large enough ω

If there was a max on $(0,\infty)$ then $\phi'(\omega) = 0$ or DNE

$\phi'(\omega) = \dfrac{-1}{2\left((\omega_0^2-\omega^2)^2 + 4D^2\omega^2\right)^{3/2}} \cdot 2(\omega_0^2 - \omega^2)(-2\omega) + 8D^2\omega$

$= \dfrac{-4\omega(\omega_0^2 - \omega^2) + 8D^2\omega}{2\left((\omega_0^2-\omega^2)^2 + 4D^2\omega^2\right)^{3/2}}$ (bottom not zero on $(0,\infty)$)

$\phi'(\omega) = 0$ means

$8D^2\omega - 4\omega(\omega_0^2 - \omega^2) = 0$

$\cancel{\omega}\left(8D^2 - 4(\omega_0^2 - \omega^2)\right) = 0$

$8D^2 - 4(\omega_0^2 - \omega^2) = 0$

$2D^2 - (\omega_0^2 - \omega^2) = 0$

$2D^2 - \omega_0^2 = -\omega^2$

$\omega^2 = \omega_0^2 - 2D^2$

$\omega = \sqrt{\omega_0^2 - 2D^2}$

has to be real

need $\omega_0^2 - 2D^2 > 0$

$\omega_0^2 > 2D^2$

$\frac{\omega_0}{\sqrt{2}} > D$ ← (might be reversed)

Compare $\phi(0)$ with $\phi(\omega^*)$ where ω^* in $(0,\infty)$ with $\frac{d\phi}{d\omega}(\omega^*) = 0$

See that the critical point is a max

b) Find ω_r s.t. $\phi(\omega_r)$ is max if a max exists in $(0,\infty)$

ω_r has to be the critical pt that is not zero

$\omega_r = \sqrt{\omega_0^2 - 2D^2}$

Written Hwk 2b) Modified

Prove that $x^7 + 20x^3 + 6x$ has no zeros on $(0,\infty)$

Soln | Let $f(x) = x^7 + 20x^3 + 6x$

Note that $f(0) = 0^7 + 20 \cdot 0^3 + 6 \cdot 0 = 0$

$f'(x) = \boxed{7x^6 + 60x^2} + 6 > 0$ on $(0,\infty)$

This means that $f(x)$ is increasing on $(0,\infty)$ i.e. for any $x > 0$

$f(x) > f(0) = 0$ so

$f(x)$ cannot be zero on $(0,\infty)$. □