Goals Implicit Differentiation & Related Rates

Implicit Diff: Differentiating when we have not isolated y.

Ex 1 Find y' when $x^3 + y^2 = 1$

$$\frac{d}{dx}(x^3+y^2) = \frac{d}{dx}(1)$$
$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^2) = 0$$
$$3x^2 + \frac{d}{dx}(y^2) = 0$$

Let $f(x) = x^2$ then $y^2 = f(y)$

$d/dx\, f(y(x)) = f'(y(x))\cdot y'(x)$

$f(x) = x^2 \Rightarrow f'(x) = 2x$

$f'(y(x)) = 2y$

$d/dx\,(y^2) = d/dx\,(f(y)) = 2y\,y'$

$3x^2 + 2yy' = 0$

$2y\,y' = -3x^2$ so $\boxed{y' = \frac{-3x^2}{2y}}$

Ex 2) Find dy/dx when

Reminder | $\tan(\arctan(z)) = z$

$y = \arctan(e^x)$

$\tan(y) = \tan[\arctan(e^x)] = e^x$

$\tan(y) = e^x$

Let $f(x) = \tan(x)$ then $\tan(y) = f(y)$

$d/dx\, f(y) = f'(y)\,y'$ ← chain rule

↓ quotient rule

$$\frac{d}{dx}(\tan(x)) = \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{d/dx(\sin(x))\cos(x) - d/dx(\cos(x))\sin(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$d/dx\,(\tan(y)) = \sec^2(y)y'$

As $\tan(y) = e^x$ we get $d/dx\,(\tan(y)) = d/dx\,(e^x)$

$d/dx(e^x) = e^x$

$= \sec^2(y)\,y' = e^x$

$$y' = \frac{e^x}{\sec^2(y)} = \frac{e^x}{\sec^2(\arctan(e^x))}$$

Ex 3) Find dy/dx when

$y = (1+\frac{1}{x})^x$ ←

Trick | Take logs whenever you see a power of x

$\log(a^b) = b\log(a)$

Take $(1+\frac{1}{x}) = a$, $b = x$

$\log((1+\frac{1}{x})^x) = x\log((1+\frac{1}{x}))$

$\log(y) = x\log(1+\frac{1}{x})$

$\frac{d}{dx}(\log(y)) = d/dx\,(x\log(1+\frac{1}{x}))$

↑

$f(x) = \log(x)$

$\log(y) = f(y)$ → $d/dx\,(\log(y)) = \frac{1}{y}\,y'$

$d/dx\,(\log(x)) = \frac{1}{x}$

$$y'/y = d/dx\,(x\log(1+\tfrac{1}{x}))$$
$$= \log(1+\tfrac{1}{x}) + x\frac{d}{dx}(\log(1+\tfrac{1}{x}))$$
$$\log(1+\tfrac{1}{x}) + x\left[\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)\right]$$

$y' = y\left[\log(1+\frac{1}{x}) + x\left[\frac{1}{1+1/x}\left(-\frac{1}{x^2}\right)\right]\right]$

Related Rates Word problems

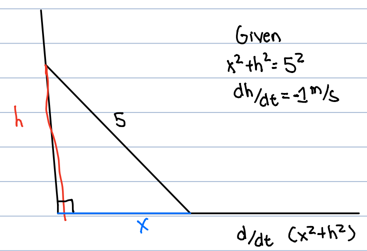
- Draw picture
- Label unknowns and known
- If triangle use $a^2+b^2=c^2$ or θ
- Physics : Usually look for conserved quantity like length of rope

Ex A 5m ladder is sliding down a wall

h = height of the ladder from top at time t

x = distance from wall to ladder bottom

If ladder sliding down at 1m/s Find dx/dt



Given

$x^2+h^2=5^2$

$dh/dt = -1\,m/s$

Want

dx/dt

$$\frac{d}{dt}(x^2+h^2) = \frac{d}{dt}(5^2)$$

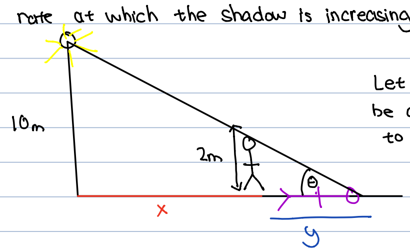
$$2x\frac{dx}{dt} + 2h\frac{dh}{dt} = 0$$

$$x\,\frac{dx}{dt} + h\,\frac{dh}{dt} = 0$$

$$x\,\frac{dx}{dt} - h = 0$$

$$\boxed{dx/dt = h/x}$$

Ex 2 A man of 2m walks away from a 10 m lamp post at a speed of 1.2m/s. Find the rate at which the shadow is increasing in length.



Let x distance be distance from man to pole

Let y length of the shadow

Given

$dx/dt = 1.2$

man is 2m

light bulb is 10m

Unknown

dy/dt

SOH CAH TOA

$\sin(\theta) = opp/hyp$

$\cos(\theta) = adj/hyp$

$\tan(\theta) = opp/adj$

$$\tan(\theta) = \frac{10}{x+y}$$

$$\tan(\theta) = \frac{2}{y}$$

$$\frac{10}{x+y} = \frac{2}{y} \Rightarrow 10y = 2(x+y) \Rightarrow 10y = 2x+2y$$

So $8y = 2x$

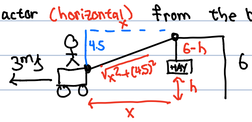
$$y = \frac{1}{4}x$$

$$dy/dt = \frac{1}{4}\frac{dx}{dt}$$

$$= \frac{1}{4}(1.2)$$

Sapling 16

Farmer John's tractor pulls a rope attached to a bale of hay through a pulley. At a certain moment his tractor speed is 3m/s and the bale is rising at 2m/s. How far is the tractor (horizontal) from the bale at this moment.



x is the horizontal distance from hay

h is the vertical height of hay

Want	Know
x	$dx/dt = 3$
	$dh/dt = 2$

Want length of rope since it's a constant

$$l = \sqrt{x^2+(4.5)^2} + 6 - h$$

$$dl/dt = 0$$

$$0 = \frac{d}{dt}\left(\sqrt{x^2+(4.5)^2} + 6 - h\right)$$

$$\rightarrow \; = \frac{1}{2\sqrt{x^2+(4.5)^2}} 2x \, dx/dt - dh/dt = 0$$

$$= \frac{x\frac{dx}{dt}}{\sqrt{x^2+20.25}} - \frac{dh}{dt} = 0$$

$$\frac{x \, dx/dt}{\sqrt{x^2+20.25}} = dh/dt$$

$$\frac{3x}{\sqrt{x^2+20.25}} = 2$$

$$3x = 2\sqrt{x^2+20.25}$$

$$9x^2 = 4(x^2+20.25)$$

$$5x^2 = 4(20.25)$$

$$x^2 = 4(20.25)/5$$

$$x = \sqrt{16.2}$$

$$\approx 4.025$$