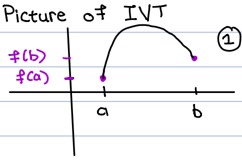
Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a,b]$ for $a<b$ then if

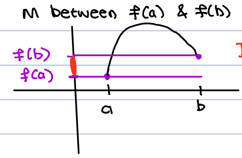
$f(a) < M < f(b)$ or $f(b) < M < f(a)$ then there is a $c \in (a,b)$ such that

$f(c) = M$.

Picture of IVT

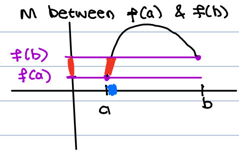


M between $f(a)$ & $f(b)$ ②



IVT guarntees all the numbers in red are obtained by $f(x)$ for x in (a,b)

M between $f(a)$ & $f(b)$ ③



And all the values are obtained in ■ x values which correspond to ■ y values

Application Show there is an x between $[0,\frac{\pi}{2}]$ such that $e^x \sin(x) = e^{\pi/4}$ ← easy IVT since endpoint given so just check function is continuous & $f(a)$ and $f(b)$ & the number they want is inbetween $f(a)$ & $f(b)$.

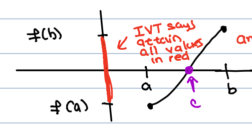
Note $e^x\sin(x)$ is continuous on $[0,\pi/2]$ since it is the product of 2 continuous functions (e^x & $\sin(x)$)

And $e^0\sin(0)=0$, $e^{\pi/2}\sin(\pi/2) = e^{\pi/2}$

And $0 < e^{\pi/4} < e^{\pi/2}$ so the Intermediate Value Thm tells us there is a c such that $0<c<\pi/2$ and $e^{\pi c}\sin(c) = e^{\pi/4}$

Theorem Let $f(x)$ be continuous on $[a,b]$ with $f(a)<0$ and $f(b)>0$ then there is a c such that $a<c<b$ with $f(c)=0$

Remark| Same result holds if $f(a)>0$ and $f(b)<0$



and 0 is in there

Harder Examples)

Example 2 Show that there is an c such that

$e^c = c^2$ ← harder since you need to find an interval

Solution 1) Note $e^c = c^2$ means $e^c - c^2 = 0$

2) Write $f(x) = e^x - x^2$ so problem wants us to find a c such that $f(c)=0$

3) Now as $f(x)$ is continuous everywhere, motivated by exercise 2 we want to find $a<b$ with $\begin{cases} f(a)>0 \text{ and } f(b)<0 \\ \text{OR } f(a)<0 \text{ and } f(b)>0 \end{cases}$

1st try $[a=0, b=1]$

$f(0) = e^0 - 0^2 = 1 > 0$

$f(1) = e - 1^2 > 0$ ← so $f(a), f(b) > 0$ so cannot use exercise 2

so try new interval (use intuition that $\lim_{x\to-\infty} e^x = 0$)

2nd try $a=-100$ $b=0$

$f(a) = e^{-100} - (-100)^2 = \underset{\text{very small}}{e^{-100}} - \underset{\text{very big}}{(100)^2} < 0$

$f(b) = 1 > 0$

So $f(b) > 0$ and $f(a) < 0$ with f continuous on $[-100, 0]$

so exercise implies there is a c such that $-100<c<0$ with $f(c)=0$

i.e. $e^c - c^2 = 0$ i.e. $e^c = c^2$ □

Idea: Rewrite $e^c = c^2$ as $e^c - c^2 = 0$

so it becomes find a zero of $f(x) = e^x - x^2$

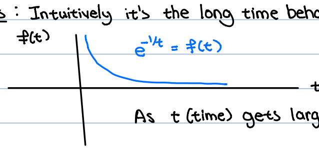
Finding an x such that $f(x)=c$ suggests IVT ($c=0$ in this case)

So our goal is to find $a<b$ such that

$\begin{cases} f(a)<0 \text{ and } f(b)>0 \text{ OR} \\ f(a)>0 \text{ and } f(b)<0 \end{cases}$

i.e. $f(a)$ & $f(b)$ have diff signs to use IVT to get there is a c w/ $a<c<b$ such that $f(c)=0$

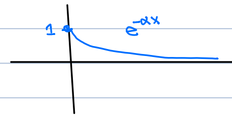
Infinite limits: Intuitively it's the long time behavior of the function



As t (time) gets larger & larger $f(t)$ goes to 0 so we say $\lim_{t\to\infty} f(t) = 0$

<u>Theorem</u> For any $\alpha > 0$ [for instance $\alpha = \frac{1}{2}, 3/4, \pi, .0001$ etc] we have

$$\lim_{x \to \infty} \frac{1}{x^\alpha} = 0 \quad \& \quad \lim_{x \to \infty} e^{-\alpha x} = 0$$



<u>In general, factor whenever you're asked to solve the limit as $x \to \infty$ of a rational function or root</u>

<u>Example</u> $\lim_{t \to \infty} \frac{8t^2 + 7t^{1/3}}{\sqrt{16t^4 + 6}}$

1st step: Factor out highest power from top & bottom

$$\frac{8t^2 + 7t^{1/3}}{\sqrt{16t^4 + 6}} = \frac{t^2(8 + 7t^{\frac{1}{3} - 2})}{\sqrt{t^4(16 + 6t^{-4})}} = \frac{t^2(8 + 7t^{-5/3})}{t^2\sqrt{16 + 6t^{-4}}}$$

$$= \frac{8 + 7t^{-5/3}}{\sqrt{16 + 6t^{-4}}}$$

Remember

$t^\alpha t^\beta = t^{\alpha + \beta}$

So $t^2(8 + 7t^{-5/3}) = 8t^2 + 7t^{2 - 5/3} = 8t^2 + 7t^{1/3}$

So $\lim_{t \to \infty} \frac{8t^2 + 7t^{1/3}}{\sqrt{16t^4 + 6}} = \lim_{t \to \infty} \frac{8 + 7t^{-5/3}}{\sqrt{16 + 6t^{-4}}} \overset{\text{Quotient Rule}}{=} \frac{\lim_{t \to \infty}(8 + 7t^{-5/3})}{\lim_{t \to \infty}\sqrt{16 + 6t^{-4}}} = \frac{8}{\sqrt{16}} = \frac{8}{4} = 2$

<u>Ex</u> 2) $\lim_{x \to \infty} (\sqrt{4x^4 + 9x} - 2x^2)$

<u>Multiply by the conjugate to make into rational function</u>

$$\sqrt{4x^4 + 9x} - 2x^2 = (\sqrt{4x^4 + 9x} - 2x^2)\left(\frac{\sqrt{4x^4 + 9x} + 2x^2}{\sqrt{4x^4 + 9x} + 2x^2}\right)$$

$$= \frac{4x^4 + 9x - 4x^4}{\sqrt{4x^4 + 9x} + 2x^2} = \frac{9x}{\sqrt{4x^4 + 9x} + 2x^2}$$

<u>Factor highest power as it is a rational/root function</u>

$$\frac{9x}{\sqrt{4x^4 + 9x} + 2x^2} = \frac{x(9)}{\sqrt{x^4(4 + 9x^{-3})} + x^2(2)} = \frac{x(9)}{x^2\sqrt{4 + 9x^{-3}} + 2x^2} = \frac{x(9)}{x^2(\sqrt{4 + 9x^{-3}} + 2)} = \frac{9}{x(\sqrt{4 + 9x^{-3}} + 2)}$$

So $\lim_{x \to \infty} (\sqrt{4x^4 + 9x} - 2x^2) = \lim_{x \to \infty}\left(\frac{9}{x(\sqrt{4 + 9x^{-3}} + 2)}\right) = 0$ ◻

<u>Derivatives + Tangent line</u>

We say $f(x)$ is differentiable at x_0 if

$\lim_{h \to 0}\left[\frac{f(x_0 + h) - f(x_0)}{h}\right]$ exists. The limit is written as $f'(x_0)$

Differentiable functions are continuous !

<u>Example</u> Compute derivative of $f(x) = \frac{1}{x^2}$ at $x = 2$

<u>First</u> Write the "difference quotient"

$$\longrightarrow \frac{f(2+h) - f(2)}{h} = \frac{\frac{1}{(2+h)^2} - \frac{1}{(2)^2}}{h} \overset{\text{Common Denominator}}{=} \frac{\frac{2^2}{2^2(2+h)^2} - \frac{(2+h)^2}{2^2(2+h)^2}}{h}$$

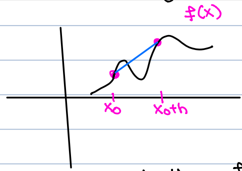
In general write difference quotient & use algebra tricks

$$= \frac{4 - (4 + 4h + h^2)}{h(4(2+h)^2)} = \frac{-4h + h^2}{h(4(2+h)^2)} = \frac{h(-4 + h)}{h(4(2+h)^2)}$$

$$= \frac{-4 + h}{4(2+h)^2}$$

So $\lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0}\left(\frac{-4 + h}{4(2+h)^2}\right) = \frac{-4}{4(2+0)^2} = \frac{-4}{16}$ $\boxed{= -1/4}$

Secant line & tangent line interpretation of derivative



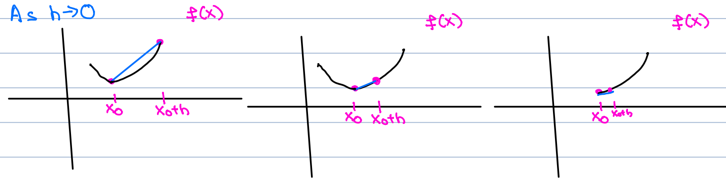
- secant line from $f(x_0+h)$ to $f(x_0)$

The line joining $f(x_0+h)$ to $f(x_0)$ is the secant line

Recall $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$ ← slope of secant line

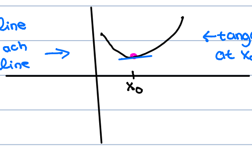
[Compare to $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$]

As $h \to 0$



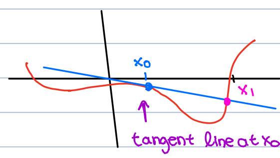
The tangent line at x_0 of f is a line that touches the point x_0 and looks "parallel" to $f(x_0)$

blue line approach → this line



← tangent line at x_0

The derivative is just the slope of the tangent line!



↙ not tangent here

tangent line at x_0

So if $f(x)$ is differentiable at x_0 then the tangent line at x_0 of $f(x)$ is

$$\ell(x) = \underbrace{f'(x_0)}_{\text{slope}}(x - x_0) + \underbrace{f(x_0)}_{\text{touches } f(x_0) \text{ at } x = x_0}$$

[- Tangent line at x_0 of $f(x)$ slope is $f'(x_0)$
- Has to touch $f(x_0)$ at $x = x_0$]

<u>Example</u> Compute tangent line at $x=2$ of $f(x) = \frac{1}{x^2}$

By previous example $f'(2) = -1/4$ and $f(2) = 1/4$

So $\ell(x) = -\frac{1}{4}(x-2) + \frac{1}{4} = \frac{3}{4} - \frac{x}{4}$ ◻