Common Denominator

If we have an expression of the form

$$\lim_{x\to c} \frac{f(x)}{g(x)} - \frac{h(x)}{u(x)}$$

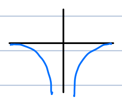
& $\frac{f(c)}{g(c)} - \frac{h(c)}{u(c)} =$ " $\frac{\text{non-zero number}}{0} - \frac{\text{non-zero number}}{0}$ "

Then make them have the same denominator

Example $\lim_{x\to 0} \left[\frac{3}{x} - \frac{3}{x^2+x} \right] = \lim_{x\to 0} \left[\frac{3}{x} - \frac{3}{x(x+1)} \right] = \lim_{x\to 0} \left[\frac{3(x+1)}{x(x+1)} - \frac{3}{x(x+1)} \right] = \lim_{x\to 0} \left[\frac{3x}{x(x+1)} \right] = \lim_{x\to 0} \left[\frac{3}{x+1} \right] = 3$

Warning some limits do not exist! Algebra tricks help us see if the indeterminate has a limit or not

$\lim_{x\to 0} \left[\frac{1}{x} - \frac{1}{x^2} \right] = \lim_{x\to 0} \left[\frac{x-1}{x^2} \right]$ " $\frac{-1}{0}$ " infinite discontinuity



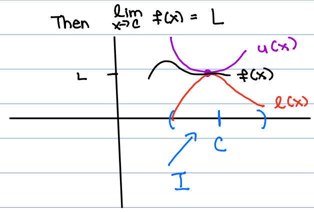
Squeeze Theorem

If $\lim_{x\to c} \ell(x) = L$ and $\lim_{x\to c} u(x) = L$ and

there is an open interval I with c in I such that

$\ell(x) \le f(x) \le u(x)$ for all x in I (potentially except c) [f may be indeterminate at $x=c$]

Then $\lim_{x\to c} f(x) = L$



↙ f is "sandwiched" between $u(x)$ & $\ell(x)$

Application 1

Find $\lim_{x\to 0} x^3 \cos\left(\frac{5}{x}\right)$ (*)

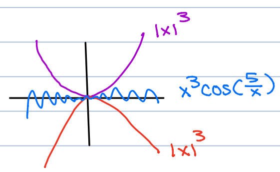
Trick: $-1 \le \cos\left(\frac{5}{x}\right) \le 1$

and $-|x|^3 \le x^3 \le |x|^3$

so $-|x|^3 \le x^3\cos\left(\frac{5}{x}\right) \le |x|^3$

So as $\lim_{x\to 0} -|x|^3 = \lim_{x\to 0} |x|^3 = 0$

$\Rightarrow$ By squeeze $\lim_{x\to 0} x^3\cos\left(\frac{5}{x}\right) = 0$



$\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$

As $-1 \le \sin\left(\frac{1}{x}\right) \le 1$

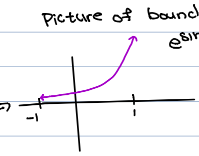
$\Rightarrow e^{-1} \le e^{\sin(\frac{1}{x})} \le e$

So $x^2e^{-1} \le x^2 e^{\sin(\frac{1}{x})} \le x^2 e$

As $\lim_{x\to 0} x^2 e^{-1} = \lim_{x\to 0} x^2 e = 0$

Squeeze thm implies $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$

Picture of bound on $e^{\sin(1/x)}$



Exercise If $\lim_{x\to a} f(x) = 0$ & there is a C_1 & C_2 constants such that $C_1 \le g(x) \le C_2$ then $\lim_{x\to 0} f(x)g(x) = 0$

Note both examples above are examples of this.

Hint: Use squeeze thm + Idea of example (*)

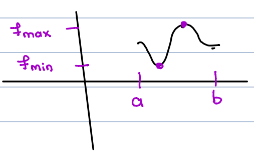
Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a,b]$ for $a<b$ then if

$f(a) < M < f(b)$ or $f(b) < M < f(a)$ then there is a $c \in (a,b)$ such that

$f(c) = M$.

Geometrically it says continuous functions map intervals to intervals between f_{min} and f_{max}
"min of $f(x)$ on $[a,b]$" "max of $f(x)$ on $[a,b]$"



Since f is continuous on $[a,b]$ it just says
f attains every value of $[f_{min}, f_{max}]$
"Topologically continuous maps preserve intervals"

Application

Show there is an x between $[0, \frac{\pi}{2}]$ such that

$e^x \sin(x) = e^{\pi/4}$ on $[0, \pi/2]$

Note $e^x\sin(x)$ is continuous since it is the product of 2 continuous functions (e^x & $\sin(x)$)

And $e^0 \sin(0) = 0$, $e^{\pi/2}\sin(\pi/2) = e^{\pi/2}$

And $0 < e^{\pi/4} < e^{\pi/2}$ so the Intermediate Value Thm tells us there is a c such that $0 < c < \pi/2$ and $e^{\pi c}\sin(c) = e^{\pi/4}$