Squeeze Thm

Lemma: For any number a we have

$$-|a| \leq a \leq |a|$$

Example: $-|1| \leq 1 \leq |1|$ means
$-1 \leq 1 \leq 1$

$-|-2| \leq -2 \leq |-2|$ means
$-2 \leq -2 \leq 2$

So we have $-|f(x)| \leq f(x) \leq |f(x)|$ for any function f since $f(x)$ is a number

Example) $\lim_{x \to 3} (x-3)\, e^{\cos\left(\frac{1}{x^2}\right)}$ and $\lim_{x \to 3} (x-3)^2 e^{\cos(1/x)}$

Idea: Bound "bad" function $e^{\cos\left(\frac{1}{x^2}\right)}$ and compare $(x-3)e^{\cos(1/x^2)}$ to a constant multiple of $(x-3)$ to use squeeze thm

Note: $e^{-1} \leq e^{\cos(1/x^2)} \leq e^{1}$ since $-1 \leq \cos(1/x^2) \leq 1$

Note $(x-3)^2$ is non-negative so as

$e^{-1} \leq e^{\cos(1/x^2)} \leq e^{1}$ implys

$e^{-1}(x-3)^2 \leq (x-3)^2 e^{\cos(1/x^2)} \leq e\,(x-3)^2$ ← FINE SINCE MULTIPLYING BY A NON-NEGATIVE NUMBER

squeeze implies limit is zero

BUT CANNOT SAY

$e^{-1}(x-3) \leq (x-3)e^{\cos(1/x^2)} \leq e\,(x-3)$ ← MULTIPLYING BY A NEGATIVE NUMBER IF $x<3$

SOLUTION: $-|x-3|\,|e^{\cos(1/x^2)}| \leq (x-3)e^{\cos(1/x^2)} \leq |x-3|\,|e^{\cos(1/x^2)}|$

$|e^{\cos(1/x^2)}| \leq e$

so $|x-3|\,|e^{\cos(1/x^2)}| \leq e|x-3|$

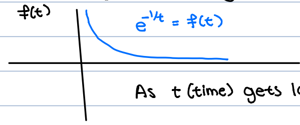
and $-|x-3|\,|e^{\cos(1/x^2)}| \geq -e|x-3|$

So $-e|x-3| \leq -|x-3|\,|e^{\cos(1/x^2)}| \leq (x-3)e^{\cos(1/x^2)} \leq |x-3|\,|e^{\cos(1/x^2)}| \leq e|x-3|$

So $-e|x-3| \leq (x-3)e^{\cos(1/x^2)} \leq e|x-3|$

So squeeze ⇒ limit is 0.

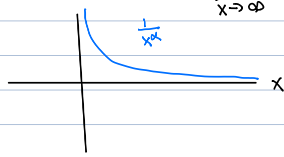
Infinite limits : Intuitively it's the long time behavior of the function

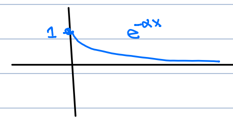


As t (time) gets larger & larger $f(t)$ goes to 0 so we say $\lim_{t\to\infty} f(t) = 0$

Theorem For any $\alpha > 0$ [for instance $\alpha = \frac{1}{2}, 3/4, \pi, .0001$ etc] we have

$$\lim_{x\to\infty} \frac{1}{x^{\alpha}} = 0 \quad \& \quad \lim_{x\to\infty} e^{-\alpha x} = 0$$





In general, factor whenever you're asked to solve the limit as $x\to\infty$ of a rational function or root

Example $\lim_{t\to\infty} \dfrac{8t^2 + 7t^{1/3}}{\sqrt{16t^4 + 6}}$

1st step: Factor out highest power from top & bottom

$$\frac{8t^2 + 7t^{1/3}}{\sqrt{16t^4+6}} = \frac{t^2(8 + 7t^{\frac{1}{3}-2})}{\sqrt{t^4(16 + 6t^{-4})}} = \frac{t^2(8+7t^{-5/3})}{t^2\sqrt{16+6t^{-4}}}$$

$$= \frac{8+7t^{-5/3}}{\sqrt{16+6t^{-4}}}$$

Remember

$t^{\alpha} t^{\beta} = t^{\alpha+\beta}$

So $t^2(8+7t^{-5/3}) = 8t^2 + 7t^{2-5/3} = 8t^2 + 7t^{1/3}$

So
$$\lim_{t\to\infty} \frac{8t^2 + 7t^{1/3}}{\sqrt{16t^4+6}} = \lim_{t\to\infty} \frac{8+7t^{-5/3}}{\sqrt{16+6t^{-4}}} \overset{\text{Quotient Rule}}{=} \frac{\lim_{t\to\infty}(8+7t^{-5/3})}{\lim_{t\to\infty}\sqrt{16+6t^{-4}}} = \frac{8}{\sqrt{16}} = \frac{8}{4} = 2$$

Ex 2) $\lim_{x\to\infty} \left(\sqrt{4x^4+9x} - 2x^2\right)$

Multiply by the conjugate to make into rational function

$$\sqrt{4x^4+9x} - 2x^2 = \left(\sqrt{4x^4+9x} - 2x^2\right)\left(\frac{\sqrt{4x^4+9x} + 2x^2}{\sqrt{4x^4+9x} + 2x^2}\right)$$

$$= \frac{4x^4+9x - 4x^4}{\sqrt{4x^4+9x} + 2x^2} = \frac{9x}{\sqrt{4x^4+9x} + 2x^2}$$

Factor highest power as it is a rational/root function

$$\frac{9x}{\sqrt{4x^4+9x}+2x^2} = \frac{x(9)}{\sqrt{x^4(4+9x^{-3})} + x^2(2)} = \frac{x(9)}{x^2\sqrt{4+9x^{-3}} + 2x^2} = \frac{x(9)}{x^2(\sqrt{4+9x^{-3}}+2)} = \frac{9}{x(\sqrt{4+9x^{-3}}+2)}$$

So $\lim_{x\to\infty}\left(\sqrt{4x^4+9x} - 2x^2\right) = \lim_{x\to\infty}\left(\dfrac{9}{x(\sqrt{4+9x^{-3}}+2)}\right) = 0$ ◻

Derivatives + Tangent line

We say $f(x)$ is differentiable at x_0 if

$$\lim_{h\to 0}\left[\frac{f(x_0+h)-f(x_0)}{h}\right]$$ exists. The limit is written as $f'(x_0)$

Differentiable functions are continuous!

Example Compute derivative of $f(x)=\frac{1}{x^2}$ at $x=2$

First Write the "difference quotient"

Common Denominator

$$\rightarrow \frac{f(2+h)-f(2)}{h} = \frac{\frac{1}{(2+h)^2}-\frac{1}{(2)^2}}{h} = \frac{\frac{2^2}{2^2(2+h)^2}-\frac{(2+h)^2}{2^2(2+h)^2}}{h}$$

In general write difference quotient & use algebra tricks

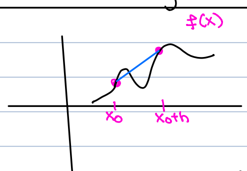
$$= \frac{4-(4+4h+h^2)}{h(4(2+h)^2)} = \frac{-4h+h^2}{h(4(2+h)^2)} = \frac{h(-4+h)}{h(4(2+h)^2)}$$

$$= \frac{-4+h}{4(2+h)^2}$$

So $$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\left(\frac{-4+h}{4(2+h)^2}\right) = \frac{-4}{4(2+0)^2} = \frac{-4}{16}$$

$$\boxed{= -1/4}$$

Secant line & tangent line interpetation of derivative



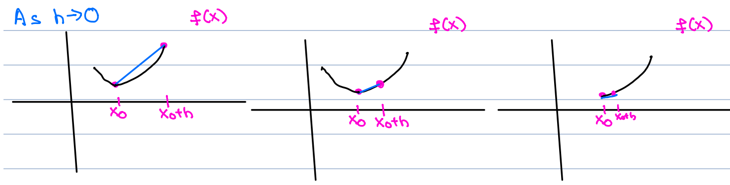
- secant line from $f(x_0+h)$ to $f(x_0)$

The line joining $f(x_0+h)$ to $f(x_0)$ is the secant line

Recall $m=\frac{y_2-y_1}{x_2-x_1} = \frac{f(x_0+h)-f(x_0)}{(x_0+h)-x_0} = \frac{f(x_0+h)-f(x_0)}{h} \leftarrow$ slope of secant line

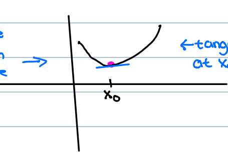
[Compare to $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$]

As $h\to 0$



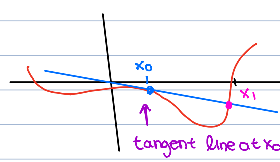
The tangent line at x_0 of f is a line that touches the point x_0 and looks "parallel" to $f(x_0)$

blue line approach this line →



← tangent line at x_0

The derivative is just the slope of the tangent line!



↙ not tangent here

↑ tangent line at x_0

So if $f(x)$ is differentiable at x_0 then the tangent line at x_0 of $f(x)$ is

$$\ell(x) = \underbrace{f'(x_0)}_{\text{slope}}(x-x_0) + \underbrace{f(x_0)}_{\text{touches } f(x_0) \text{ at } x=x_0}$$

[- Tangent line at x_0 of $f(x)$ slope is $f'(x_0)$
- Has to touch $f(x_0)$ at $x=x_0$]

Example Compute tangent line at $x=2$ of $f(x)=\frac{1}{x^2}$

By previous example $f'(2)=-1/4$ and $f(2)=1/4$

So $\ell(x) = -1/4(x-2)+\frac{1}{4} = 3/4 - x/4$ ▫