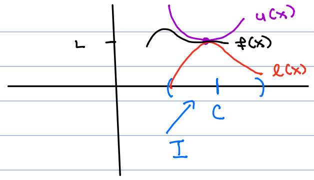
Squeeze Theorem

If $\lim_{x\to c} \ell(x) = L$ and $\lim_{x\to c} u(x) = L$ and

there is an open interval I with $c \in I$ such that

$\ell(x) \le f(x) \le u(x)$ for all x in I (potentially except c) [f may be indeterminate at $x=c$]

Then $\lim_{x\to c} f(x) = L$



↙ f is "sandwiched" between $u(x)$ & $\ell(x)$

Application 1

Find $\lim_{x\to 0} x^3 \cos(\frac{5}{x})$ (*)

Solution

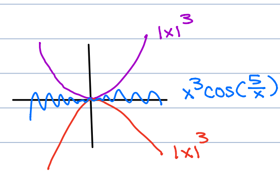
Trick: $-1 \le \cos(\frac{5}{x}) \le 1$

and $-|x^3| \le x^3 \le |x^3|$

so $-|x^3| \le x^3\cos(\frac{5}{x}) \le |x^3|$

So as $\lim_{x\to 0} -|x^3| = \lim_{x\to 0} |x^3| = 0$

$\Rightarrow$ By squeeze $\lim_{x\to 0} x^3\cos(\frac{5}{x}) = 0$



$\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})}$

As $-1 \le \sin(\frac{1}{x}) \le 1$

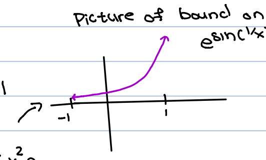
$\Rightarrow e^{-1} \le e^{\sin(\frac{1}{x})} \le e$

So $x^2 e^{-1} \le x^2 e^{\sin(\frac{1}{x})} \le x^2 e$

As $\lim_{x\to 0} x^2 e^{-1} = \lim_{x\to 0} x^2 e = 0$

Squeeze thm implies $\lim_{x\to 0} x^2 e^{\sin(\frac{1}{x})} = 0$

Picture of bound on $e^{\sin(1/x)}$



Exercise If $\lim_{x\to a} f(x) = 0$ & there is a C_1 & C_2 constants such that

$C_1 \le g(x) \le C_2$ then $\lim_{x\to 0} f(x)g(x) = 0$

Note both examples above are examples of this.

Hint: Use squeeze thm + Idea of example (*)

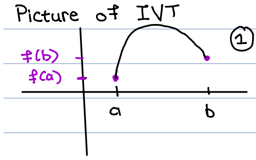
Intermediate Value Theorem

Let $f(x)$ be a continuous function on $[a,b]$ for $a<b$ then if

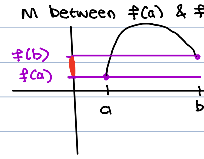
$f(a) < M < f(b)$ or $f(b) < M < f(a)$ then there is a $c \in (a,b)$ such that

$f(c) = M$.

Picture of IVT ①

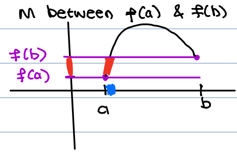


M between $f(a)$ & $f(b)$ ②



IVT guarantees all the numbers in red are obtained by $f(x)$ for x in (a,b)

M between $f(a)$ & $f(b)$ ③



And all the values are obtained in x values which correspond to y values

Application

Show there is an x between $[0,\frac{\pi}{2}]$ such that $e^x\sin(x) = e^{\pi/4}$ ← easy IVT since endpoint given so just check function is continuous & $f(a)$ and $f(b)$ & the number they want is inbetween $f(a)$ & $f(b)$.

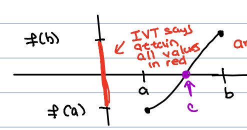
Note $e^x\sin(x)$ is continuous on $[0,\pi/2]$ since it is the product of 2 continuous functions (e^x & $\sin(x)$)

And $e^0\sin(0) = 0$, $e^{\pi/2}\sin(\pi/2) = e^{\pi/2}$

And $0 < e^{\pi/4} < e^{\pi/2}$ so the Intermediate Value Thm tells us there is a c such that $0<c<\pi/2$ and $e^{\pi c}\sin(c) = e^{\pi/4}$

Theorem Let $f(x)$ be continuous on $[a,b]$ with $f(a)<0$ and $f(b)>0$ then there is a c such that $a<c<b$ with $f(c)=0$

Remark Same result holds if $f(a)>0$ and $f(b)<0$



and 0 is in there

Harder Examples)

Idea: Rewrite
$e^c = c^2$ as $e^c - c^2 = 0$
so it becomes find a zero
of $f(x) = e^x - x^2$
Finding an x such that $f(x)=c$
suggests IVT ($c=0$ in this case)
So our goal is to find $a<b$
such that
$\begin{cases} f(a)<0 \text{ and } f(b)>0 \text{ OR} \\ f(a)>0 \text{ and } f(b)<0 \end{cases}$
i.e. $f(a)$ & $f(b)$ have diff signs to use IVT to get there is a c w/ $a<c<b$ such that $f(c)=0$

Example 2 Show that there is an c such that
$$e^c = c^2$$ ← harder since you need to find an interval

Solution 1) Note $e^c = c^2$ means $e^c - c^2 = 0$

2) Write $f(x) = e^x - x^2$ so problem wants us to find a c such that $f(c)=0$

3) Now as $f(x)$ is continuous everywhere, so we want to find
$a<b$ with $\begin{cases} f(a)>0 \text{ and } f(b)<0 \\ \text{OR } f(a)<0 \text{ and } f(b)>0 \end{cases}$ [Want $f(a)$ & $f(b)$ to have different signs to get 0 is between $f(a)$ & $f(b)$ to use IVT to show there is a c s.t. $f(c)=0$]

1st try [a=0 , b=1]
$f(0) = e^0 - 0^2 = 1 > 0$
$f(1) = e - 1^2 > 0$ ← so $f(a), f(b) > 0$ so cannot use exercise 2

so try new interval (use intuition that $\lim_{x \to -\infty} e^x = 0$)

2nd try $a = -100$ $b = 0$
$f(a) = e^{-100} - (-100)^2 = \underset{\text{very small}}{e^{-100}} - \underset{\text{very big}}{(100)^2} < 0$
$f(b) = 1 > 0$

So $f(b) > 0$ and $f(a) < 0$ with f continuous on $[-100, 0]$
so 0 is between $f(a)$ & $f(b)$ which implies there is a c such that $-100<c<0$ with $f(c)=0$
i.e. $e^c - c^2 = 0$ i.e. $e^c = c^2$ □