ELLIPTIC REGULARITY AND HOLOMORPHIC FUNCTIONS

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Abstract. We first derive from the Cauchy-Riemann equations that the real and imaginary parts of a holomorphic functions are weak solutions of $\Delta u = 0$. This is a striking difference between holomorphic functions and real differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}^2$. This leads us to explore the regularity of uniformly elliptic PDEs since $\Delta u = 0$ is the prototype of this family. We first prove continuous weak solutions of $\Delta u = 0$ are smooth then use the spectral theorem to deduce its true for elliptic constant coefficient PDE. Then we characterize the decay of integrals with C^α regularity and deduce enough estimates from elliptic constant coefficient PDE to obtain C^α regularity for a large class of uniformly elliptic PDEs via a perturbation argument known as Schauder's Estimates.

1. Introduction

A remarkable feature of holomorphic functions is that while the definition only requires that a single derivative exists, they are actually infinitely differentiable and analytic. The analogous result for real differentiable functions fails; for instance $\int_{-1}^{x} |t| \mathrm{d}t$ is C^1 but is not C^2. But one naively might expect that since $\mathbb{R}^2 \cong \mathbb{C}$ that there might be similar regularity results about real differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. However, it is not the topology of $\mathbb{C}$ that gives holomorphic functions such nice regularity results, but it is instead due to the Cauchy-Riemann equations. Indeed, recall that if $f : \Omega \to \mathbb{C}$ holomorphic with Ω being an open and bounded subset of $\mathbb{C}$ with $\partial\Omega \in C^1$ then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

where u and v are the real ad imaginary parts of f respectively. So if $\varphi \in C_c^\infty(\Omega)$ is a smooth real-valued function that is compactly supported in Ω (which will be referred to as a test function from now on) then we have

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} \right) dxdy = \int_\Omega \left(\frac{\partial v}{\partial y}\frac{\partial \varphi}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial \varphi}{\partial y} \right) = \int_\Omega v \left(-\frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y \partial x} \right) dxdy = 0$$

where for the second equality we used the Cauchy-Riemann equations and the third we integrated by parts using that $\varphi|_{\partial\Omega} = 0$. Note that the condition of smoothness on $\partial\Omega$ and Ω being bounded were used to justify the usage of the divergence theorem to integrate by parts. So in particular we have shown that from the Cauchy-Riemann equations that $\int_\Omega \nabla u \cdot \nabla \varphi dxdy = 0$ for all test functions φ. Now we define what it means to be weakly harmonic:

Definition 1.1. *We say a function $u : \Omega \to \mathbb{R}$ with $u \in H^1(\Omega)$ is weakly harmonic if for any test function $\varphi \in C_c^\infty(\Omega)$ satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi dxdy = 0$$

where $H^1(\Omega) := \{u : \Omega \to \mathbb{R} : u \in L^2(\Omega) \text{ and } \nabla u \in L^2(\Omega)\}$ and ∇u is the distributional derivative of u (and we implicitly require the distributional derivative to be an actual function).

Remark 1.2. *For a review of properties of L^p spaces see [4] and for H^1 see Chapter 5 of [1]. And note that the definition of a weakly harmonic function can be also derived formally by multiplying $\Delta u = 0$ with a test function to see $\varphi \Delta u = 0$ and integrating by parts once, which means every harmonic function is weakly harmonic. However, from just the definition, it is unclear if every weakly harmonic function is harmonic since u may not even be twice differentiable.*

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Remark 1.3. *Note that one can enlarge the class of our test functions to $H_0^1(\Omega)$ which is defined to be functions in H^1 that have trace zero on the boundary. The trace operator is an operator that assigns H^1 functions their boundary value. There is no loss of generality since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$. See [1] for more details about Sobolev Spaces.*

So we have shown that the real part of f is weakly harmonic when we only require f is continuously complex differentiable once. Hence, if we obtain regularity results for weakly harmonic functions then they also apply to the real and imaginary parts of holomorphic functions. So we are motivated to study the regularity properties of weak solutions of uniformly elliptic partial differential equations on $\mathbb{R}^d$ since weakly harmonic functions are the prototype of this family. In this exposition we focus on a sub-family of divergence form elliptic PDE to simplify the proofs. For the remainder of the report assume that $\Omega \subset \mathbb{R}^d$ is bounded, open, and connected with $\partial\Omega \in C^1$.

Definition 1.4. *Let L be the operator defined by*

$$Lu := -D_j(a_{ij}(x)D_i u) = -\nabla \cdot (A(x)\nabla u)$$

where $D_i u = \frac{\partial u}{\partial x_i}$, A is the matrix function with $A_{ij} = a_{ij}$, and we are using Einstein's summation convention. Then we say L is uniformly elliptic if there exists a positive constant $\lambda > 0$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for any } x \in \Omega \text{ and any } \xi \in \mathbb{R}^d$$

Definition 1.5. *We say that $u \in H^1(\Omega)$ is a weak solution of the elliptic operator $Lu = f$ where*

$$Lu := -D_j(a_{ij}(x)D_i u) = -\nabla \cdot (A(x)\nabla u)$$

if for any $\varphi \in H_0^1(\Omega)$ we have

$$\int_\Omega a_{ij}(x)D_i u D_j\varphi = \int_\Omega f\varphi$$

where we assume that $f \in L^{\frac{2n}{n+2}}(\Omega)$.

Remark 1.6. *Note by the Sobolev Embedding Theorem (see [1] chapter 5) we have that as $\varphi \in C_c^1(\Omega)$ so $\varphi \in L^{\frac{2n}{n-2}}(\Omega)$ so we have by Holder's Inequality that*

$$\int_\Omega |f\varphi| \leq ||f||_{L^{\frac{2n}{n+2}}} ||\varphi||_{L^{\frac{2n}{n-2}}} < \infty$$

so that the right hand side is well defined for any $\varphi \in C_c^1(\Omega)$.

Remark 1.7. *Note that this does not encompass the entire class of uniformly elliptic PDEs. For instance, our PDE is linear and is in divergence form. That is we can formally write L as the divergence of another operator and in our case $Lu = -\nabla \cdot (A(x)\nabla u)$. We choose this subclass to clearly present our ideas and proofs.*

Remark 1.8. *We note that the operator $-\Delta u$ is elliptic since $-\Delta u = -\nabla \cdot (\nabla u)$ so the matrix A in the operator is the identity operator which is strictly positive definite with constant $\lambda = 1$. And we note that our definition of weakly harmonic is exactly the same as being a weak solution of $-\nabla u = 0$.*

In this report we will derive interior regularity results of weak solutions of an elliptic operator L when we assume additional regularity results such as uniform continuity of $A(x)$ and some regularity on the inhomogeneous term $f(x)$. First we will derive regularity results of continuous weakly harmonic functions, which will imply similar results for when $A(x)$ is a constant strictly positive definite matrix with $c(x) \equiv 0$ thanks to the spectral theorem. Then we derive an integral decay condition that implies C^α regularity and use this with estimates on solutions to $Lu = 0$ with A being a constant matrix that smallest eigenvalue $\lambda > 0$. Finally we use these results to derive these regularity results by using a perturbation argument through estimating the difference between weak solutions of L compared to a constant coefficient (i.e. $A(x)$ is a constant matrix) elliptic operator to show our weak solutions are sufficiently regular inside Ω. This method of obtaining regularity of elliptic PDE is known as Schauder's Estimates.

2. Regularity of Constant Coefficient Elliptic PDEs

In this section we aim to prove that weakly harmonic functions are in fact harmonic in the usual sense, then we will prove harmonic functions are smooth. Then we will extend this result to PDEs of the form

$$a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{2.1}$$

where a_{ij} are constants with $[A]_{ij} = a_{ij}$ being a uniformly strictly positive definite matrix i.e. A smallest eigenvalue is bigger than $\lambda > 0$ for some λ. And we call solutions to such PDEs solutions to uniformly elliptic cosntant coefficient PDEs. We need these results since we are going to show that a general weak solution of the elliptic operator $Lu = 0$ can be locally approximated by a constant coefficient PDE of the form (2.1). Assume we are working on $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open, bounded subset, connected, and $\partial\Omega \in C^1$.

Now we are going to described a procedure to create smooth approximations of L^p functions known as *mollification*. For more details see the appendix of [1]. First we construct a test function η supported on the unit ball

$$\eta_: = \begin{cases} C\exp(\frac{1}{|x|^2-1} \text{ if } |x| < 1 \\ 0 \text{ if } |x| \geq 1 \end{cases}$$

where C is chosen such that $\int_{\mathbb{R}^n} \eta dx = 1$. Then for any $\varepsilon > 0$ we define $\eta_\varepsilon := \frac{1}{\varepsilon^d}\eta(x/\varepsilon)$ then $\eta_\varepsilon \in C_c^\infty(B_\varepsilon(0))$. Then by [1] we have that if $f \in L^1(\Omega)$ then for

$$f^\varepsilon := \eta_\varepsilon * f := \int_\Omega \eta_\varepsilon(x-y)f(y)dy$$

we have $f^\varepsilon \in C^\infty(\Omega_\varepsilon)$ where $\Omega_e := \{x \in \Omega |\ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$, $f^\varepsilon \to f$ a.e., and if $f \in L^p(\Omega)$ then $f^\varepsilon \to f$ in $L^p(\Omega)$. This gives us a method of approximating an arbitrary L^p function by a smooth approximation of that function as long as we are away from the boundary. We will call η_ε the standard mollifier.

Now we have from Chapter 1 of [3] that

Theorem 2.1. ***Mean Value Property Equivalence*** *A continuous function $u : \Omega \to \mathbb{R}$ is harmonic if and only if for all balls $B_r(x_0)$ such that $\overline{B_r(x_0)} \subset \Omega$ we have*

$$u(x_0) = \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)dx = \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x_0))}\int_{\partial B_r(x_0)} u(x)dS$$

where $|B_r(x_0)|$ is the volume of the ball and $\mathcal{H}^{n-1}(\partial B_r(x_0))$ is the surface area of the sphere.

Now we have enough tools to prove that every continuous weakly harmonic function is actually harmonic.

Theorem 2.2. ***Weyl's Lemma*** *Let u be a weakly harmonic function on Ω then after redefinition on a set of measure zero, we have that $\Delta u = 0$ in Ω.*

Proof. Fix an $\varepsilon > 0$ and a test function ψ and the standard mollifier η_ε then $\eta_e * \psi$ is also a test function when ε is small enough. Then we have

$$0 = \int_\Omega u\Delta_x(\eta_\varepsilon * \psi)dx = \int_\Omega \Delta_x(u * \eta_\varepsilon)\psi dy$$

where the first equality is since Δu is the distributional derivative of u so we can integrate by parts. The second equality is due to Fubinis theorem, the standard mollifier being an even function, and integration by parts. Therefore, $\Delta_x(u * \eta_\varepsilon) = 0$ a.e. due to the density of test functions in $L^1(\Omega)$ (see [4]), but continuity gives us $\Delta_x(u * \eta_\varepsilon) = 0$ in Ω_ε. So $u * \eta_\varepsilon$ is a harmonic function which converges a.e. to u, this means if we define for any $x_0 \in \Omega$

$$\overline{u}(x_0) := \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)dx$$

where r is chosen so that the closure of the ball is in Ω then from L^2 convergence of $u * \eta_\varepsilon$ to u combined with the mean value property of $u * \eta_\varepsilon$ to see $u * \eta_\varepsilon \to \overline{u}$ and as $\overline{u}$ is a continuous function with the mean value property. Then we apply Theorem 0.1 to conclude $\overline{u}$ is harmonic and notice $\overline{u} = u$ a.e. $\square$

Remark 2.1. *This proof highlights how rigid weakly harmonic functions are. Indeed, the proof highlights how from an arbitrary weakly harmonic function we can find an explicit sequence of harmonic functions that converge to u in L^2 and a.e.. This combined with the stability property of harmonic functions under L^2 convergence showed that u is actually harmonic when we redefine it on a null set.*

Now we have the following result

Theorem 2.3. *A continuous function $u \in C(\Omega)$ that satisfies the mean value property is smooth i.e. $u \in C^\infty(\Omega)$*

Proof. For brevity we omit the proof which can be found on [1] chapter 2. The idea of the proof is to fix an $\varepsilon > 0$ and show by integrating in polar coordinates that if η_ε is the standard mollifier then $(u * \eta_\varepsilon) = u$ on Ω_ε which is possible since η is a radial function. Then since $(u * \eta_\varepsilon)$ is smooth in Ω_ε, so u is smooth in Ω since $\bigcup_{\varepsilon>0} \Omega_\varepsilon = \Omega$. □

Notice that this implies if u is weakly harmonic then after a redefinition on a set of measure zero we have that u is smooth. So in fact this implies that the real parts of a holomorphic functions are smooth. However, we can do better since by using the Mean Value Property, we are able to deduce estimates almost identical to the Cauchy's Estimate for a holomorphic function. These estimates can be used to justify that in fact harmonic functions are analytic, see chapter 1 of [2] for more details.

Now consider weak solutions of

$$\sum_{i,j=1}^{n} a_{ij} D^2_{ij} u(x) = 0 \tag{2.2}$$

where the constant matrix $A_{ij} := a_{ij}$ is uniformly positive definite i.e. its smallest eigenvalue $\lambda > 0$. Then by the Spectral Theorem, we can find a unitary change of basis so that $A = \text{diag}(\lambda_1, ..., \lambda_n)$ with $\lambda_i > 0$. This implies by integration by subsitution that u is also a weak solution of

$$\sum_{i=1}^{n} \lambda_i D^2_{ii} u(x) = 0$$

in a rotated domain. Then we can scale each coordinate by defining $y_i := x_i/\sqrt{\lambda_i}$ then we have

$$\Delta u = 0$$

in this scaled and rotated domain, which implies by Theorem 2.2 and Theorem 2.3 and undoing this linear transformation that after redefining u on a null set that it is is smooth everywhere. So we have shown

Theorem 2.4. *If u is a weak solution of*

$$\sum_{i,j=1}^{n} a_{ij} D^2_{ij} u(x) = 0$$

where the matrix $A_{ij} := a_{ij}$ with a_{ij} constant is uniformly positive definite then u is smooth after a redefinition on a set of measure zero.

From now on, when we work with solutions to elliptic constant coefficient PDEs, we assume that the solution is smooth, which thanks to the theorem above implies no loss of generality.

3. Estimates of Constant Coefficient Elliptic PDEs

One of the most remarkable features of Sobolev Spaces is the Sobolev Embedding Theorem. Recall that $u \in W^{1,p}(\Omega)$ means u and its distributional derivative ∇u are in $L^p(\Omega)$. One consequence of the Sobolev Embedding Theorem is that if $u \in W^{1,n+q}(\Omega)$ where $q > 0$ then up to a redefinition in a set of measure zero then $u \in C^\alpha(\Omega)$ for $\alpha := 1 - \frac{n}{n+q}$; while if $q = 0$ then the Sobolev Embedding Theorem says that u is of Bounded Mean Oscillation. That is if $(u_{x,r}) := \frac{1}{|B_r(x)|} \int_{B_r(x)} u dx$ then there exists a constant independent of x and r such that

$$\int_{B_r(x)} |u - (u_{x,r})| dx \leq C r^n$$

(see [1] chapter 5 for more discussions on the Embedding Theorem). So a natural question one may pose is what happens if we improve the bound on the Bounded Mean Oscillation term to $r^{n+\beta}$ for some $\beta > 0$. And one might expect that u is actually α-holder continuous for some $\alpha > 0$ due to how close being in $W^{1,n}$ is with being α-holder continuous. We slightly modify this and have the following result:

Theorem 3.1. ***Integral Decay Characterization of α-Holder continuity*** *Let $u \in L^2(\Omega)$ satisfy*

$$\int_{B_r(x)} |u - (u_{x,r})|^2 \leq M r^{n+2\alpha} \text{ for any } B_r(x) \subset \Omega$$

then $u \in C^\alpha(\Omega)$ with the following estimate for any open $U \subset \Omega$ such that $dist(U, \partial\Omega) > 0$ we have for some constant $C > 0$

$$||u||_{C^\alpha(U)} \leq C \left(M + ||u||_{L^2(\Omega)}\right)$$

Proof. For the full proof see [2] Chapter 3. The idea of the proof is to show that the net $\{(u_{x,r})\}_{r>0}$ is Cauchy via showing that due to the integral decay we have the estimate

$$|u_{x_0,r_1} - u_{x_0,r_2}|^2 \leq CM^2 r_1^{-n}(r_1^{n+2\alpha} + r_2^{n+2\alpha})$$

for any $0 < r_1 < r_2$ with $\overline{B_{r_2}(x)} \subset \Omega$. Then by taking $r_1 = 2^{-h}R$ and $r_2 = 2^{-k}R$ and iterating the above inequality to obtain

$$|u_{x_0,2^{-h}R} - u_{x_0,2^{-k}R}| \leq C/2^{h\alpha} M R^\alpha$$

which implies

$$|\overline{u}(x_0) - u_{x_0,r}| \leq CMr^\alpha$$

from which it is easy to deduce the α-holder continuity of $\overline{u}$ along with the desired inequality. But from Lebesgue differentiation theorem (see [4]) we know that $\overline{u} = u$ a.e., so by identifying u as $\overline{u}$ we obtain the desired result

□

Remark 3.1. *Notice a major theme in the proofs of obtaining a C^α or a smooth identification of a L^p function is that we either derived that the function agrees almost everywhere with an integral representation (mean value property for weakly harmonic functions) or identified the function as the limit of an integral process. This is rather natural since we expect integration to be a procedure that increases the regularity of our functions.*

This allows us to characterize C^α functions with a local decay condition on their integrals. This is incredibly useful for weak solutions since they are defined in terms of integration against test functions. And we chose to square the integrand since our weak solutions are defined to be in $H^1(\Omega)$ instead of $W^{1,1}(\Omega)$. The proof given in [2] also seems with very slight modifications seem to show that if $\int_{B_r(x)} |u - (u_{x,r})| \leq Mr^{n+\alpha}$ for some $\alpha > 0$ and any ball in Ω then $u \in C^\alpha(\Omega)$. We can also somewhat weaken the conditions thanks to Poincare's Lemma.

Theorem 3.2. ***Poincare Lemma*** *If Ω is bounded, open, connected subset of $\mathbb{R}^d$ with $\partial\Omega \in C^1$ then there exists a constant $C(n, \Omega)$ such that*

$$||u - \frac{1}{|\Omega|}\int_\Omega u||_{L^2(\Omega)} \leq C||Du||_{L^2(\Omega)}$$

for any $u \in H^1(\Omega)$

Proof. For a proof see [1] chapter 5. Recall $|\Omega|$ is the Lebesgue measure of Ω.

□

Now notice this implies $||u - (u_{x,r})||_{L^2(B_r(x))} \leq Cr||Du||_{L^2(B_r(x))}$. Indeed, if $u \in B_r(x)$ then define $v(y) := u(x + ry)$ where $y \in B_1(0)$ so $v \in H^1(B_1(0))$ then we take C from $\Omega = B_1(0)$ and undo the scaling to obtain the extra factor of r from differentiating. So now we have the slightly weaker version of Theorem 3.1

Theorem 3.3. *Let $u \in H^1(\Omega)$ satisfy*

$$\int_{B_r(x)} |Du|^2 \leq Mr^{n+2\alpha-2}$$

then $u \in C^\alpha(\Omega)$

Proof. By Poincare's Lemma we have

$$\int_{B_r(x)} |u-(u_{x,r})|^2 \leq Cr^2 \int_{B_r(x)} |Du|^2 \leq Kr^{n+2\alpha}$$

now we apply theorem 3.1 to conclude. (Note we have an r^2 factor since we did not take a square root) □

Our goal is to show that under some additional regularity hypothesis on the coefficients a, b, c and f that if L is a general uniformly elliptic PDE operator then weak solutions of $Lu = f$ are actually C^α continuous, which will be done by showing the hypothesis of theorem 3.3 is satisfied for u. And the idea of our proof is to write $u = w + (u - w)$ where w is a solution to a constant coefficient elliptic PDE such that the error $u - w$ is small in a very small ball. This method leads to the famous Schauder's Estimates, which is a pertubative method. To begin these estimates we need to derive similar estimates for a solution to an elliptic constant coefficient PDE.

Theorem 3.4. ***Cacciopolli's Inequality*** *Let $\{a_{ij}\}_{i,j=1}^n$ be constants such that the matrix $A_{ij} := a_{ij}$ is postivie definite with smallest eigenvalue $\lambda > 0$ and let $u \in C^1(\Omega)$ solve*

$$\int_\Omega a_{ij} D_i u D_j \varphi = 0 \text{ for any } \varphi \in C_c^1(\Omega)$$

then if $\eta \in C_c^1(\Omega)$ we have

$$\int_\Omega \eta^2 |Du|^2 \leq C \int_\Omega |D\eta|^2 |u|^2$$

Proof. Take $\varphi = \eta^2 u$ then we have

$$\lambda \int_\Omega \eta^2 |Du|^2 dx \leq M \int_\Omega (|D_j\eta||u|)(|D_i u||\eta|)\, dx$$

for $M := 2\max|a_{ij}|$. Then the result holds by Cauchy-Schwarz. □

Notice that this implies by taking η to be a cut off function that is if $0 < r < R$ then $\eta = 1$ on $B_r(0)$ and $\eta = 0$ on $B_R(0)$ with η smooth such that $|D\eta| \leq 2(R-r)^{-1}$

$$\int_{B_r(0)} |Du|^2 \leq \frac{4}{(R-r)^2} \int_{B_R(0)} |u|^2$$

Then notice as due to theorem 2.4 that u is smooth, so that the structure of the constant coefficient PDE imply any derivatives of u are also weak solutions. Therefore, we also have for any $k \in \mathbb{N}$ and $0 < r < R$ with $\overline{B_R(x_0)} \subset \Omega$ that

$$||u||_{H^k(B_r(x))} \leq C(k) \int_{B_R(x)} |u|^2 \leq C(k) \int_\Omega |u|^2$$

where $H^k(\Omega)$ is defined to be the Sobolev Space where u and all of its distributional derivative of order k are in $L^2(\Omega)$; see [1] for more details on H^k. Then this with the Sobolev embedding theorem implies when k is sufficiently large with respect to n then we have $u \in C^\alpha(\Omega)$ and $||u||_{C^{k,\alpha}(B_r(x))} \leq C||u||_{L^2(\Omega)}$ (see [2] chapter 3 for more details) so in particular,

$$||u||_{L^\infty(B_r(x)} + ||Du||_{L^\infty(B_r(x)} \leq C||u||_{L^2(\Omega)}$$

so observe that if $r < \rho/2$ then

$$\int_{B_\rho(x)} |u|^2 \leq C\rho^n ||u||^2_{L^\infty(B_\rho(x)} \leq K\rho^n ||u||_{L^2(\Omega)} \tag{3.1}$$

and by Poincare's Inequality

$$\int_{B_\rho(x)} |u-(u_{x,\rho})|^2 \leq C\rho^2 \int_{B_\rho(0)} |Du|^2 \leq C\rho^{n+2} ||Du||_{L^\infty(B_r(x))} \leq C\rho^{n+2}||u||_\Omega$$

then observe that since u is a weak solution so is $w := u - (u_{x,r})$ then $(w_{x,\rho}) = (u_{x,\rho}) - (u_{x,r})$ so we have from plugging in w into the above inequality that

$$\int_{B_\rho(0)} |u - (u_{x,\rho})|^2 \leq C\rho^{n+2} \int_{B_1(0)} |u - (u_{x,r})|^2 \tag{3.2}$$

then notice that if $r/2 \leq \rho < r$ then the inequalities (3.1) and (3.2) are obvious when C is large. Indeed observe $C\rho^{n+2} \int_{B_1(0)} |u - (u_{1,\rho})|^2 \geq C(1/2)^{n+2} \int_{B_1(0)} |u - (u_{1,\rho})|^2$ so we choose C so large such that $C(1/2)^{n+2} \int_{B_1(0)} |u - (u_{1,\rho})|^2 \geq \sup_{r \in [1/2,1]} \int_{B_\rho(0)} |u - (u_{0,\rho})|^p$ which is a finite value since the map $\rho \mapsto \int_{B_\rho(0)} |u - (u_{0,\rho})|^p$ is continuous and also choose $C(1/2)^{n+2} \geq 1$. Therefore, by a scaling argument we have shown the following theorem

Theorem 3.5. *Let u satisfy the same hypothesis as that in Cacciopolli's Inequality with $\Omega = B_r(x)$, then we have there is a $C > 0$ such if $0 < \rho < r$*

$$\int_{B_\rho(x)} |u|^2 \leq C \left(\frac{\rho}{r}\right)^n \int_{B_r(x)} |u|^2$$

and

$$\int_{B_\rho(x)} |u - (u_{x,\rho})|^2 \leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x)} |u - (u_{x,1})|^2$$

Now notice that as u is smooth and its derivatives solve the PDE so we have

Theorem 3.6. *Let u satisfy the same hypothesis as that in Cacciopolli's Inequality with $\Omega = B_r(x)$, then we have there is a $C > 0$ such if $0 < \rho < r$*

$$\int_{B_\rho(x)} |Du|^2 \leq C \left(\frac{\rho}{r}\right)^n \int_{B_r(x)} |Du|^2$$

and

$$\int_{B_\rho(x)} |Du - (Du_{x,\rho})|^2 \leq C \left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x)} |Du - (Du_{x,1})|^2$$

and the motivation for deriving such an error estimate is that it looks very similar to our integral characterization of α-Holder continuous functions. Now as our goal is to approximate a solution of $Lu = f$ where L is a uniformly elliptic operator locally by w where w solves (2.1), we will now derive a general estimate for v in terms of $v - w$ and w where $v \in H^1(\Omega)$.

Theorem 3.7. ***Comparison with constant coefficient solutions*** *Let w solve (2.1) in $\Omega := B_r(x_0)$ where the constant matrix $A_{ij} := a_{ij}$ is uniformly positive definite, then for any $u \in H^1(B_r(x_0))$ and $0 < \rho \leq r$ we have*

$$\int_{B_\rho(x_0)} |Du|^2 \leq C \left[\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 dx + \int_{B_r(x_0)} |D(u-w)|^2 \right]$$

and

$$\int_{B_\rho(x_0)} |Du - (Du_{x_0,r})|^2 \leq C \left[\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Du - (Du_{x_0,r})|^2 dx + \int_{B_r(x_0)} |D(u-w)|^2 \right]$$

Proof. We will only prove the first inequality as the second case is an almost identical argument; see [2] chapter 3 for the full proof. First write $u = w + (u - w)$ then we have from $|a - b|^2 \leq 4|a| + 4|b|$ that

$$\int_{B_\rho(x_0)} |Du|^2 \leq 4 \int_{B_\rho} |Dw|^2 + 4 \int_{B_\rho(x_0)} |D(u-w)|^2 \leq 4C \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2 + 4 \int_{B_\rho(x_0)} |D(u-w)|^2$$

where we applied Theorem (0.6) for the final inequality. But as $w = u - (u - w)$ we have again that

$$\leq 16C^2 \left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 + \left(16C^2 \left(\frac{\rho}{r}\right)^n + 4\right) \int_{B_r(x_0)} |D(u-w)|^2$$

which implies the claim by taking C large enough since $(\rho/r)^n \leq 1$

∎

Now we have enough estimates to prove the regularity of solutions of $Lu = 0$ where L is an elliptic PDE operator.

4. Schauder Estimates

Now we are almost ready to prove the following result

Theorem 4.1. ***Schauder's Estimates*** *Suppose that $a_{ij}(x) \in L^\infty(\Omega) \cap C(\overline{\Omega})$ is uniformly elliptic in Ω that is there exists constants $\lambda > 0$ such that*

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \text{ for any } x \in \Omega, \xi \in \mathbb{R}^n$$

Then if u is an $H^1(\Omega)$ solves

$$\int_\Omega a_{ij}(x) D_i u D_j \varphi = \int_\Omega f\varphi \text{ for any } \varphi \in H_0^1(\Omega)$$

where $f \in L^q$ where $q \in (\frac{n}{2}, n)$ then $u \in C^\alpha(\Omega)$ for $\alpha := 2 - \frac{n}{q}$. And we have the following interior estimate: there exists an $R_0 > 0$ such that if $r < R_0(x_0, \lambda, \tau, q)$ then $B_r(x_0) \subset \Omega$

$$\int_{B_r(x_0)} |Du|^2 \leq C r^{n-2+2\alpha}\left(||f||^2_{L^q(\Omega)} + ||Du||_{L^2(\Omega)}\right)$$

where R depends on $\lambda, \Lambda,$ and τ only where

$$|a_{ij}(x) - a_{ij}(y)| \leq \tau(|x - y|)$$

i.e. τ is the modulus of continuity of a_{ij}

Remark 4.1. *Note by the uniform continuity of a_{ij} we can assume that its modulus of continuity is radial, which will be necessary to apply the lemma below, so the assumption $a_{ij} \in C(\overline{B_1})$ is crucial for this proof.*

However, first we will need a following technical lemma to help us get the desired inequality.

Theorem 4.2. *Let $\phi(t)$ be a non-negative and non-decreasing function on $[0, R]$. Suppose that*

$$\phi(\rho) \leq A\left[\left(\frac{\rho}{r}\right)^\alpha + \varepsilon\right]\phi(r) + Br^\beta$$

for any $0 < \rho \leq r \leq R$ where $A, B, \alpha, \beta > 0$ are constants with $\beta < \alpha$. Then for any $\gamma \in [\beta, \alpha)$ there exists an $\varepsilon_0 > 0$ that depends only on A, α, β, γ such that if $\varepsilon < \varepsilon_0$ then we have for all $0 < \rho \leq r \leq R$

$$\phi(\rho) \leq C\left[\phi(r)\left(\frac{\rho}{r}\right)^\gamma + B\rho^\beta\right]$$

Proof. See [2] chapter 3 for the proof of this lemma. This theorem will be important since we will derive an estimate that is of the assumption form with $\phi(r) = \int_{B_r(x_0)} |Du|^2 dx$ and $\varepsilon = \tau^2(r)$. Intuitively this theorem is saying we can remove the ε in our bound by compensating with making the (ρ/r) factor smaller by raising it to the power of γ instead of α. And this compensation in the power of (ρ/r) also allows us to replace r^β with ρ^β.

□

Now we are ready to prove our main theorem

Proof of Theorem 4.1 Let $\overline{B_r(x_0)} \subset \Omega$ then from the Lax Milgram Lemma (see chapter 6 of [1]) there exists a unique weak solution of

$$\begin{cases} -\nabla \cdot (a_{ij}(x_0)\nabla w) = 0 \text{ in } B_r(x_0) \\ w = u \text{ on } \partial B_r(x_0) \end{cases} \tag{4.1}$$

where $w = u$ is understood in the trace sense (see Chapter 5 of [1]) since u is only defined a.e. (the trace operator gives a way to assign for H^1 functions boundary values). Then as for any $\varphi \in H_0^1(B_r(x_0))$ we have from definition that

$$\int_{B_r(x_0)} a_{ij}(x_0)D_iuD_j\varphi = \int_{B_r(x_0)} f\varphi - (a_{ij}(x_0) - a_{ij}(x))D_iuD_j\varphi$$

so writing $u = (u - w) + w$ and using that w solves (4.1) we obtain for $v := u - w \in H_0^1(\Omega)$ that

$$\int_{B_r(x_0)} a_{ij}(x_0)D_ivD_j\varphi = \int_{B_r(x_0)} f\varphi - (a_{ij}(x_0) - a_{ij}(x))D_iuD_j\varphi$$

Therefore, by the usual trick of taking $\varphi = v$ and the uniform ellipticity condition we obtain that

$$\lambda \int_{B_r(x_0)} |Dv|^2 \leq \int_{B_r(x_0)} |fv| + |(a_{ij}(x_0) - a_{ij}(x))D_iuD_jv|$$

Now again that from Sobolev Embedding Theorem that $v \in L^{\frac{2n}{n-2}}(B_r(x_0))$ and the Gagliardo–Nirenberg–Sobolev inequality (which is used to prove the Embedding Theorem) implies $||v||_{L^{\frac{2n}{n-2}}(B_r(x_0))} \leq C||Dv||_{L^2(B_r(x_0))}$ [see chapter 5 of [1]]. Therefore, by Holder's Inequality we have

$$\int_{B_r(x_0)} |fv| \leq C||f||_{L^{\frac{2n}{n+2}}(B_r(x_0))}||Dv||_{L^2(B_r(x_0))}$$

and Holder also implies combined with τ being a non-increasing and non-negative function that

$$\int_{B_r(x_0)} |(a_{ij}(x_0) - a_{ij}(x))D_iuD_jv| \leq \tau(r)||Du||_{L^2(B_r(x_0)}||Dv||_{L^2(B_r(x_0)}$$

Therefore, combining these inequalities give

$$\left(\int_{B_r(x_0)} |Dv|^2\right)^{1/2} \leq K\left(\tau(r)||Du||_{L^2(B_r(x_0)} + ||f||_{L^{\frac{2n}{n+2}}(B_r(x_0))}\right)$$

this implies

$$\int_{B_r(x_0)} |Dv|^2 \leq 4K\left(\tau^2(r)\int_{B_r(x_0)} |Du|^2dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}\right) \tag{4.2}$$

Therefore, applying Theorem 3.7 we see for any $0 < \rho \leq r$

$$\int_{B_\rho(x_0)} |Du|^2 \leq c\left[\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Du|^2 + \int_{B_r(x_0)} |Dv|^2\right] \leq C\left[\left\{\tau^2(r) + \left(\frac{\rho}{r}\right)^n\right\}\int_{B_r(x_0)} |Du|^2dx + \left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}}\right]$$

Then by Holder's Inequality we have

$$\left(\int_{B_r(x_0)} |f|^{\frac{2n}{n+2}}\right)^{\frac{n+2}{n}} \leq \left(\int_{B_r(x_0)} |f|^q\right)^{\frac{2}{q}} r^{n-2+2\alpha}$$

where $\alpha := 2 - \frac{n}{p}$. Therefore, we have

$$\int_{B_\rho(x_0)} |Du|^2 \leq C\left[\left\{\tau^2(r) + \left(\frac{\rho}{r}\right)^n\right\}\int_{B_r(x_0)} |Du|^2dx + ||f||^2_{L^q(B_r(x_0))}r^{n-2+2\alpha}\right]$$

Now we apply Theorem 4.2 with $\varepsilon(r) := \tau^2(r)$ and $\phi(r) = \int_{B_r(x_0)} |Du|^2$ we see that there exists a $R > 0$ such that if $\rho < R$ then $\overline{B_\rho(x_0)} \subset \Omega$ and we can apply the theorem to see

$$\int_{B_\rho(x_0)} |Du|^2 \leq c\left[\left\{\left(\frac{\rho}{R}\right)^{n-2+2\alpha}\right\}\int_{B_R(x_0)} |Du|^2dx + ||f||^2_{L^q(B_R(x_0))}\rho^{n-2+2\alpha}\right]$$

and since R is fixed

$$\leq K\left[\rho^{n-2+2\alpha}\int_{B_R(x_0)} |Du|^2dx + ||f||^2_{L^q(B_R(x_0))}\rho^{n-2+2\alpha}\right] := M\rho^{n-2+2\alpha}$$

and now we apply Theorem 3.3 to conclude that $u \in C^\alpha(B_R(x_0))$ and now by repeating this argument for any $x_0 \in \Omega$ we conclude $u \in C^\alpha(\Omega)$. □

Remark 4.2. *If we strength the assumptions to $a_{ij} \in C^\alpha(\overline{\Omega})$ then we can modify this argument with using the second inequality on Theorem 3.7 to conclude $Du \in C^\beta(\Omega)$ for some $\beta > 0$. And this argument can also be modified to show that with the same assumptions as above with $c \in L^n(\Omega)$ then weak solutions of*

$$-\nabla \cdot (a_{ij}(x)\nabla u) + c(x)u = f$$

are also $C^\alpha(\Omega)$ for the same α. See Chapter 3 of [2] for more details.

5. Conclusion

In summary, we first showed the real and imaginary parts of holomorphic functions are weakly harmonic functions. This is a major difference between holomorphic functions and differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then we explored how their real parts being continuous weakly harmonic actually implies they are smooth, which explains why holomorphic functions are so regular compared to their real counterpart. And we were able to deduce from the spectral theorem combined with harmonic functions being smooth that solutions to constant coefficient uniformly elliptic PDEs are also smooth. Then from the intuition that weak solutions of $\nabla \cdot (a_{ij}(x)\nabla u(x)) = f$ near x_0 should behave like solutions with a_{ij} fixed at x_0 in a very small neighborhood [in fact in our estimates we instead consider the approximation $\nabla \cdot (a_{ij}(x)\nabla u(x)) = 0$]. This combined with integral decay conditions for C^α regularity of H^1 functions let us to obtain estimates for solutions of constant coefficient uniformly elliptic PDEs. These estimates were combined with a technical lemma to prove that under some regularity on a_{ij} and f that u is in fact C^α in the interior.

However, this methods is strongly dependent on the uniform continuity of $a_{ij}(x)$, so it does not extend to the case where a_{ij} is say only L^p for some p. To overcome this, one typically uses De Giorgi-Nash-Moser iteration arguments to obtain regularity results with rough coefficients; see for instance Chapter 4 of [2]. And both of these methods are limited in that they require the operator to be in divergence for; for otherwise, a notion of a weak solution is hard to define.

Another interesting question is whether or not our α is optimal. Intuitively we might expect if we instead look at solutions of $\nabla \cdot (a_{ij}(x_0)\nabla u(x)) = f(x_0)$ as our perturbation we would have a better approximation, but since in our theorem we did not assume continuity of f then this intuitively should not lead to a better estimate. However, I conjecture that if f is continuous then we can have even better regularity of u since intuitively solutions of $\nabla \cdot (a_{ij}(x_0)\nabla u(x)) = f(x_0)$ should better approximate $\nabla \cdot (a_{ij}(x)\nabla u(x)) = f(x)$ near x_0 then our original approximation scheme of $\nabla \cdot (a_{ij}(x_0)\nabla u(x)) = 0$.

References

1. Lawrence Evans, *Partial differential equations*, American Mathematical Society, 1998.
2. Qing Han and Qing Lin, *Elliptic partial differential equations*, Courant Lecture Notes, 2011.
3. Jürgen Jost, *Partial differential equations*, Springer, 2010.
4. Elias M Stein and Rami Shakarchi, *Real analysis: measure theory, integration, and hilbert spaces*, Princeton University Press, 2009.