Mathematics 131BH Lecture Part II

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Wednesday 21$^{\text{st}}$ March, 2018

Exercise. Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Show that $\exists\, x_0 \in (a, b)$ such that

$$f'(x_0)(g(b) - g(a)) = g'(x_0)(f(b) - f(a)).$$

Proof. Let $h : [a, b] \to \mathbb{R}, h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. Then h is continuous on $[a, b]$ and differentiable on (a, b). We have $h(a) = f(a)g(b) - g(a)f(b)$ and $h(b) = -f(b)g(a) + g(b)f(a)$. Thus $h(a) = h(b)$. Therefore, by Rolle's Theorem, $\exists\, x_0 \in (a, b) : h'(x_0) = 0$. □

L'Hospital's rule

Theorem. *Let $-\infty \leq a < b \leq \infty$ and $f, g : (a, b) \to \mathbb{R}$ be differentiable. Assume $g'(x) \neq 0\ \forall\, x \in (a, b)$ and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{\pm\infty\}$. Assume additionally that either of the following hold.*

1. $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$
2. $\lim_{x \to a^+} |g(x)| = \infty$

Then $\lim_{x \to a^+} \frac{f(x)}{g(x)} = L$.

Remark. We can replace $\lim_{x \to a^+}$ by $\lim_{x \to x_0}$ where $x_0 \in (a, b)$.

Proof. We will only prove the theorem for $L \in \mathbb{R}$. We will prove the following two claims

Claim. $\forall\, \epsilon > 0, \exists\, \delta_1(\epsilon) > 0 : a < x < a + \delta_1(\epsilon) \implies \frac{f(x)}{g(x)} < L + \epsilon$.

Claim. $\forall\, \epsilon > 0, \exists\, \delta_2(\epsilon) > 0 : a < x < a + \delta_2(\epsilon) \implies \frac{f(x)}{g(x)} > L - \epsilon$.

Combining the claims and taking $\delta(\epsilon) = \min\{\delta_1(\epsilon), \delta_2(\epsilon)\}$, we get $\forall\, \epsilon > 0, \exists\, \delta(\epsilon) > 0 : a < x < a + \delta(\epsilon)$ then $|\frac{f(x)}{g(x)} - L| < \epsilon$.

Exercise. If $L = -\infty$, prove the following variants of the first claim. Similarly with $L = \infty$.

Claim. $\forall\, M > 0, \exists\, \delta(M) > 0 : a < x < a + \delta(M) \implies \frac{f(x)}{g(x)} < M$.

Let's prove claim 1. As g' has the intermediate value property and $g'(x) \neq 0\ \forall\, x \in (a, b)$, we must have that either $g'(x) < 0\ \forall\, x \in (a, b)$ or $g'(x) > 0\ \forall\, x \in (a, b)$. Assume WLOG that $g'(x) < 0\ \forall\, x \in (a, b)$. This means g is strictly decreasing on (a, b). In case 1, we must have $g(x) < 0\ \forall\, x \in (a, b)$. In case 2, we must have

$$\lim_{x \to a^+} g(x) = \infty \implies \exists\, c \in (a, b) : g(x) > 0\ \forall\, x \in (a, c)$$

In both cases,

$$\exists\, c \in (a, b) : g(x) \neq 0\ \forall\, x \in (a, c).$$

Fix $\epsilon > 0$. Then $\exists\, \delta > 0$ such that if $x \in (a, a + \delta)$, then $\frac{f'(x)}{g'(x)} < L + \epsilon$. Taking δ even smaller (if necessary), we have

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)}$$

for some $z \in (x, y)$. Then

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(z)}{g'(z)} < L + \epsilon.$$

- Consider case 1: $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Let $x \to a^+$ to get

$$\frac{f(y)}{g(y)} = \lim_{x\to a^+} \frac{f(x)-f(y)}{g(x)-g(y)} < L + \epsilon \;\forall\; a < y < a+\delta,$$

which is claim 1.

- Consider case 2: $\lim_{x\to a^+} g(x) = \infty$. Because g is decreasing and positive on $(a, a+\delta)$. we have

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(y)}{g(x)-g(y)}\frac{g(x)-g(y)}{g(x)} + \frac{f(y)}{g(x)} < (L+\epsilon)\left(1 - \frac{g(y)}{g(x)}\right) + \frac{f(y)}{g(x)} = L + \epsilon + \frac{f(y)-(L+\epsilon)g(y)}{g(x)}.$$

As

$$\lim_{x\to a^+} \frac{f(y)-(L+\epsilon)g(y)}{g(x)} = 0, \exists\; \tilde{\delta} > 0 : \frac{f(y)-(L+\epsilon)g(y)}{g(x)} < \epsilon \;\forall\; x \in (a, a+\tilde{\delta}).$$

For $a < x < a + \min\{\delta, \tilde{\delta}\}$, we get $\frac{f(x)}{g(x)} < L + \epsilon$, which is claim 1.

□

Exercise.

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x - \sin 0}{x - 0} = \cos 0 = 1.$$

$$\lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0.$$

$$\lim_{x\to 0} \frac{x^3}{\sin x - x} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \lim_{x\to 0} \frac{6x}{-\sin x} = -6.$$

$$\lim_{x\to\infty} x^{\sin\frac{1}{x}} = \lim_{x\to\infty} e^{\sin\frac{1}{x}\ln x} = \lim_{x\to\infty} e^{\frac{\sin\frac{1}{x}}{\frac{1}{x}}\frac{\ln x}{x}} = e^{\lim_{x\to\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}\lim_{x\to\infty}\frac{\ln x}{x}} = e^{1\cdot\lim_{x\to\infty}\frac{1}{x}} = 1.$$

$$\lim_{x\to 0} (1+2x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\ln(1+2x)} = e^{\lim_{x\to 0}\frac{\ln(1+2x)}{x}} = e^{\lim_{x\to 0}\frac{2}{1+2x}} = e^2.$$

Definition. Let $f : (a,b) \to \mathbb{R}$ and $x_0 \in (a,b)$. Assume f admits a derivatives of any order at x_0. The series

$$\sum_{n\geq 0} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

is called the **Taylor series for f at x_0**. For $n \geq 1$, we define the **remainder**

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k.$$

Taylor

Theorem. *Fix $n \geq 1$ and let $f : (a,b) \to \mathbb{R}$ be n times differentiable on (a,b). Fix $x_0 \in (a,b)$. Then for any $x \in (a,b) \setminus \{x_0\}, \exists\; y$ between x and x_0 such that*

$$R_n(x) = \frac{f^{(n)}(y)}{n!}(x-x_0)^n.$$

In particular,

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{f^{(n)}(y)}{n!}(x-x_0)^n.$$

Proof. Fix $x \in (a,b) \setminus \{x_0\}$. Let M be the unique solution to

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{M}{n!}(x-x_0)^n.$$

Look at

$$g(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(t-x_0)^k - \frac{M}{n!}(t-x_0)^n.$$

We have $g(x) = g(x_0) = 0$ For $1 \leq l \leq n-1$,

$$g^{(l)}(t) = f^{(l)}(t) - \sum_{k=l}^{n-1} \frac{f^{(k)}(x_0)}{k!} k(k-1)\dots(k-l+1)(t-x_0)^{k-l} - \frac{M}{n!} n(n-1)\dots(n-l+1)(t-x_0)^{n-l}.$$

Then $g^{(l)}(x_0) = f^{(l)}(x_0) - f^{(l)}(x_0) = 0$. By Rolle's theorem, $\exists\ x_1$ between x and x_0 such that $g'(x_1) = 0$. As $g'(x_0) = g'(x_1) = 0, \exists\ x_2$ between x_0 and x_1 such that $g''(x_2) = 0$. After iterating, we get x_n between x and x_0 such that $g^{(n)}(x_n) = 0$. But

$$g^{(n)}(x_n) = f^{(n)}(x_n) - M.$$

□

Corollary. *Let $a > 0$ and assume $f : (-a, a) \to \mathbb{R}$ is differentiable to any order on $(-a, a)$. Assume also that f and all its derivatives are uniformly bounded on $(-a, a)$. Then*

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \xrightarrow[n\to\infty]{} 0\ \forall\ x \in (-a, a).$$

Proof.

$$\exists\ M > 0 : |f^{(n)}(x)| \leq M\ \forall\ n \geq 0, x \in (-a, a).$$

By Taylor's theorem,

$$R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$$

for some y between 0 and x. Then

$$|R_n(x)| \leq M \frac{|x|^n}{n!} \leq M \frac{a^n}{n!} \xrightarrow[n\to\infty]{} 0\ \forall\ x \in (-a, a).$$

This shows that

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} x^n\ \forall\ x \in (-a, a).$$

□

Example.

- $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$. For $|x| \leq M$,

  $$|f^{(n)}(x)| = |e^x| \leq e^M.$$

  Thus

  $$e^x = \sum_{n \geq 0} \frac{x^n}{n!}.$$

  As M is aribtrary, this holds $\forall\ x \in \mathbb{R}$.

- $f : \mathbb{R} \to \mathbb{R}, f(x) = \cos x, |f^{(n)}(x)| \leq 1\ \forall\ n \geq 0, x \in \mathbb{R}$.

  $$f^{(n)}(x) = \begin{cases} -\sin x & \forall\ n = 4k+1 \\ -\cos x & \forall\ n = 4k+2 \\ \sin x & \forall\ n = 4k+3 \\ \cos x & \forall\ n = 4k \end{cases}$$

  Thus

  $$f^{(n)}(0) = \begin{cases} -1 & \forall\ n = 2(2k+1) \\ 1 & \forall\ n = 2(2k) \end{cases}$$

  so $f^{(2n)}(0) = (-1)^n$ then

  $$\cos x = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} x^{2n}$$

Exercise. Find the Taylor expansion for $\sin x$.

Theorem. *For $n \geq 1$, let $f_n : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume $\{f_n'\}_{n\geq 1}$ converges uniformly on (a,b) and assume $\{f_n(x_0)\}_{n\geq 1}$ converges at some $x_0 \in (a,b)$. Then $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$ to some function f and*

$$f'(x) = \lim_{n\to\infty} f_n'(x).$$

Remark.

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\lim_{y\to x} \frac{f_n(y)-f_n(x)}{y-x}, \quad \lim_{y\to x}\lim_{n\to\infty} \frac{f_n(y)-f_n(x)}{y-x} = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = f'(x).$$

Proof. Let $\epsilon > 0$. As $\{f_n'\}_{n\geq 1}$ converges uniformly on (a,b),

$$\exists\, n_1(\epsilon) \in \mathbb{N} : |f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)} \ \forall\, n,m \geq n_1(\epsilon), x \in (a,b).$$

As $\{f_n(x_0)\}_{n\geq 1}$ converges,

$$\exists\, n_2(\epsilon) \in \mathbb{N} : |f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2} \ \forall\, n,m \geq n_2(\epsilon), x \in (a,b).$$

Let $n(\epsilon) = \max(n_1(\epsilon), n_2(\epsilon))$. By the Mean Value Theorem, for $x, y \in [a,b]$, we have

$$[f_n(x) - f_m(x)] - [f_n(y) - f_m(y)] = (x-y)[f_n'(z) - f_m'(z)]$$

for some z between x, y. In particular, for $n, m \geq n_1(\epsilon)$, we have

$$[f_n(x) - f_m(x)] - [f_n(y) - f_m(y)] < (x-y)\frac{\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}.$$

For $n, m \geq n(\epsilon), y = x_0, x \in [a,b]$,

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + |\,(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This shows $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$. Let $f(x) = \lim_{n\to\infty} f_n(x)$. Fix $x \in (a,b)$. For $y \in [a,b] \setminus \{x\}$, define

$$g_n(y) = \frac{f_n(y) - f_n(x)}{y-x} \quad \text{and} \quad g(y) = \frac{f(y)-f(x)}{y-x}.$$

Note

$$\lim_{y\to x} g_n(y) = f_n'(x) \quad \text{and} \quad \lim_{n\to\infty} g_n(y) = g(y).$$

Recall that for $n, m \geq n_1(\epsilon)$ we have

$$\left| \frac{f_n(x) - f_n(y)}{x-y} - \frac{f_m(x) - f_m(y)}{x-y} \right| < \frac{\epsilon}{2(b-a)} \ \forall\, x, y \in [a,b].$$

Let $m \to \infty$ to get

$$|g_n(y) - g(y)| \leq \frac{\epsilon}{2(b-a)} \ \forall\, y \in [a,b] \setminus \{x\}.$$

Let $L(x) = \lim_{n\to\infty} f_n'(x)$. Letting $m \to \infty$ in

$$|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)},$$

we get

$$|f_n'(x) - L(x)| \leq \frac{\epsilon}{2(b-a)} \ \forall\, n \geq n_1(\epsilon).$$

As

$$\lim_{y\to x} g_n(y) = f_n'(x) \implies \exists\, \delta > 0 : 0 < |x-y| < \delta,$$

then

$$|g_n(y) - f_n'(x)| < \frac{\epsilon}{2}.$$

For $0 < |y-x| < \delta$ and $n \geq n_1(\epsilon)$ we get

$$|g(y) - L(x)| \leq |g(y) - g_n(y)| + |g_n(y) - f_n'(x)| + |f_n'(x) - L(x)| \leq \frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2} + \frac{\epsilon}{2(b-a)} = \frac{\epsilon}{2} + \frac{\epsilon}{b-a}.$$

This proves f is differentiable at x and

$$f'(x) = L(x) = \lim_{n\to\infty} f_n'(x).$$

□

Integrability

Definition. 1. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For $S \subseteq [a,b]$ we write

$$M(f,S) = \sup\{f(x) : x \in S\} \quad \text{and} \quad m(f,S) = \inf\{f(x) : x \in S\}.$$

2. A **partition** P of $[a,b]$ is a finite ordered subset of $[a,b]$. We write

$$P = \{a = t_0 < \cdots < t_n = b\}.$$

3. Given a partition P of $[a,b]$ and $f : [a,b] \to \mathbb{R}$ bounded, we define the **upper Darboux sum** of f associated to P via

$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1})$$

and the **lower Darboux sum** of f associated to P via

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}).$$

4. The **upper Darboux integral** of f is given by

$$U(f) = \inf\{U(f,P) : P \text{ partition on } [a,b]\}.$$

The **lower Darboux integral** of f is given by

$$L(f) = \sup\{L(f,P) : P \text{ partition on } [a,b]\}.$$

We will show $L(f) \leq U(f)$.

Remark. Given $f : [a,b] \to \mathbb{R}$ bounded and $P = \{a = t_0 < \cdots < t_n = b\}$ we have

$$m(f,[a,b])(b-a) = \sum_{k=1}^{n} m(f,[a,b])(t_k - t_{k-1}) \leq L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1})$$
$$\leq \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) = U(f,P) \leq \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1}) = M(f,[a,b])(b-a).$$

This shows $L(f) \in \mathbb{R}, U(f) \in \mathbb{R}$.

Definition. Let $f : [a,b] \to \mathbb{R}$ be bounded. If $L(f) = U(f)$, we say that f is **(Darboux) integrable** and we write

$$U(f) = L(f) = \int_a^b f(x)dx.$$

Example. • Let $f : [a, b] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$$

Let $P = \{a = t_0 < \cdots < t_n = b\}$. Then

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} (t_k - t_{k-1}) = b - a \implies U(f) = b - a$$

but

$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = 0 \implies L(f) = 0.$$

As $0 \neq b - a$ we see that f is not integrable.

- Let $f : [0, b] \to \mathbb{R}, f(x) = x^3$ and $P = \{a = t_0 < \cdots < t_n = b\}$. Then

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} t_k^3 (t_k - t_{k-1}),$$
$$L(f, P) = \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) = \sum_{k=1}^{n} t_{k-1}^3 (t_k - t_{k-1}).$$

Let $t_k = \frac{k}{n} b$ and $0 \leq k \leq n$. Then

$$U(f, P) = \sum_{k=1}^{n} \frac{k^3}{n^3} b^3 \frac{b}{n} = \frac{b^4}{n^4} \sum_{k=1}^{n} k^3 = \frac{b^4}{n^4} \left(\frac{n(n+1)}{2} \right)^2 \xrightarrow[n \to \infty]{} \frac{b^4}{4} \implies U(f) \leq \frac{b^4}{4},$$
$$L(f, P) = \sum_{k=1}^{n} \frac{(k-1)^3}{n^3} b^3 \frac{b}{n} = \frac{b^4}{n^4} \sum_{l=1}^{n-1} l^3 = \frac{b^4}{n^4} \left(\frac{n(n-1)}{2} \right)^2 \xrightarrow[n \to \infty]{} \frac{b^4}{4} \implies L(f) \geq \frac{b^4}{4}.$$

Proposition. *Let $f : [a, b] \to \mathbb{R}$ be bounded and let P, Q be partitions of $[a, b] : P \subseteq Q$. Then*

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P).$$

Proof. By induction, it suffices to prove the claim when Q contains exactly one more point than P. Say $P = \{a = t_0 < \cdots < t_n = b\}$ and $Q = \{a = t_0 < \cdots < t_{k-1} < s < t_k < \ldots t_n = b\}$ for some $1 \leq k \leq n$. Then

$$\begin{aligned} L(f, P) - L(f, Q) &= m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) - \left[m(f, [t_{k-1}, s])(s - t_{k-1}) + m(f, [s, t_k])(t_k - s) \right] \\ &\leq m(f, [t_{k-1}, t_k]) \left((t_k - t_{k-1}) - (s - t_{k-1}) - (t_k - s) \right) = 0. \end{aligned}$$

□

Corollary. *Let $f : [a, b] \to \mathbb{R}$ be bounded and let P, Q be partitions of $[a, b]$. Then $L(f, P) \leq U(f, Q)$. In particular, $L(f) = U(f)$.*

Proof. Let $R = P \cup Q$. Then

$$\begin{aligned} & L(f, P) \leq L(f, R) \leq U(f, R) \leq U(f, Q) \\ & \implies L(f) = \sup \{ L(f, P) : P \text{ partition of } [a, b] \} \leq U(f, Q) \\ & \implies L(f) \leq \inf \{ U(f, Q) : Q \text{ partition of } [a, b] \} = U(f). \end{aligned}$$

□

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable iff $\forall\ \epsilon > 0, \exists\ P$ partition of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$.*

Proof.

- " $\Longleftarrow$ " Let $\epsilon > 0$ and P be a partition of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$. Then
$$U(f) \leq U(f,P) < L(f,P) + \epsilon \leq L(f) + \epsilon.$$
Let $\epsilon \to 0$ to get $U(f) \leq L(f)$. As $L(f) \leq U(f)$ we get $L(f) \leq U(f) \implies f$ is integrable.

- " $\implies$ " Assume f is integrable, then $L(f) = U(f)$. Let $\epsilon > 0$.
$$L(f) = \sup\{L(f,P) : P \text{ partition of } [a,b]\} \implies \exists\ P_1 \text{ partition of } [a,b] : L(f) - \frac{\epsilon}{2} \leq L(f,P_1),$$
$$U(f) = \inf\{U(f,P) : P \text{ partition of } [a,b]\} \implies \exists\ P_2 \text{ partition of } [a,b] : U(f) + \frac{\epsilon}{2} \geq U(f,P_2).$$
Set $P = P_1 \cup P_2$. Then
$$U(f,P) - L(f,P) \leq U(f,P_2) - L(f,P_1) \leq U(f) + \frac{\epsilon}{2} - \Big(L(f) - \frac{\epsilon}{2}\Big) = \epsilon.$$

□

Definition. Given a partition $P = \{a = t_0 < \cdots < t_n = b\}$, the **mesh of** P is
$$mesh(P) = \max_{1 \leq k \leq n} (t_k - t_{k-1}).$$

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable iff $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that*
$$P = \{a = t_0 < \cdots < t_n = b\} : mesh(P) < \delta \implies U(f,P) - L(f,P) < \epsilon.$$

Proof.

- " $\Longleftarrow$ " This follows from the previous theorem plus the observation that for any $\delta > 0, \exists\ P$ partition of $[a,b]$ with $mesh(P) < \delta$.

- " $\implies$ " Assume f is integrable, then
$$\forall\ \epsilon > 0, \exists\ P \text{ partition of } [a,b] : U(f,P) - L(f,P) < \epsilon.$$
Let $\epsilon > 0$ and let
$$P_0 = \{a = s_0 < \cdots < s_m = b\} \text{ be a partition of } [a,b] : U(f,P_0) - L(f,P_0) < \epsilon.$$
Let $\delta > 0$ to be chosen shortly and let
$$P = \{a = t_0 < \cdots < t_n = b\} : mesh(P) < \delta.$$
As f is bounded,
$$\exists\ M > 0 : |f(x)| < M\ \forall\ x \in [a,b].$$
Consider
$$U(f,P) - L(f,P) = U(f,P) - U(f,P_0) + U(f,P_0) - L(f,P_0) + L(f,P_0) - L(f,P).$$
Notice
$$L(f,P_0) - L(f,P) \leq L(f,Q) - L(f,P)$$
and
$$\begin{aligned} &|m(f,[t_{k-1},s_l])(s_l - t_{k-1}) + m(f,[s_l,t_k])(t_k - s_l) - m(f,[t_{k-1},t_k])(t_k - t_{k-1})| \\ &\leq M(s_l - t_{k-1}) + M(t_k - s_l) + M(t_k - t_{k-1}) \leq 2M mesh(P) \\ &\implies L(f,Q) - L(f,P) \leq m2M mesh(P). \end{aligned}$$
A similar argument gives
$$U(f,P) - U(f,P_0) \leq U(f,P) - U(f,Q) \leq m2M mesh(P).$$
Thus
$$U(f,P) - L(f,P) < \frac{\epsilon}{2} + 4mM mesh(P) < \epsilon \text{ provided } \delta < \frac{\epsilon}{8mM}.$$

□

Definition. Let $f : [a, b] \to \mathbb{R}$ be bounded and let $P = \{a = t_0 < \cdots < t_n = b\}$.

A **Riemann sum of f associated to** P is of the form

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

where $x \in [t_{k-1}, t_k]$ for $1 \leq k \leq n$. We say that f is **Riemann integrable** if $\exists\ r \in \mathbb{R} : \forall\ \epsilon > 0, \exists\ \delta > 0 : |S - r| < \epsilon$ for any Riemann sum S associated to a partition P with $mesh(P) < \delta$. In this case, r is called the **Riemann integral** of f on $[a, b]$ and we write

$$r = R\int_a^b f(x)dx.$$

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Darboux integrable iff it's Riemann integrable, in which case the two integrals agree.*

Proof. We prove both ways separately.

- " $\implies$ " Assume f is Darboux integrable. Let $\epsilon > 0$. Let $\delta > 0$ such that if P is a partition of $[a, b]$ with $mesh(P) < \delta$, then $U(f, P) - L(f, P) < \epsilon$. Let $P = \{a = t_0 < \cdots < t_n = b\}$ with $mesh(P) < \delta$. Let

$$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$$

for $x \in [t_{k-1}, t_k]$. Then

$$L(f, P) \leq S \leq U(f, P).$$

But

$$U(f, P) < L(f, P) + \epsilon \leq L(f) + \epsilon = \int_a^b f(x)dx + \epsilon$$

and

$$L(f, P) > U(f, P) - \epsilon \geq U(f) - \epsilon = \int_a^b f(x)dx - \epsilon.$$

Thus

$$|S - \int_a^b f(x)dx| < \epsilon \implies R\int_a^b f(x)dx = \int_a^b f(x)dx.$$

- " $\impliedby$ " Assume f is Riemann integrable and let

$$r = R\int_a^b f(x)dx.$$

Let $\epsilon > 0$. Then $\exists\ \delta > 0$ such that if P is a partition with $mesh(P) < \delta$, then $|S - r| < \delta$ for all Riemann sums S associated with P. Let $P = \{a = t_0 < \cdots < t_n = b\} : mesh(P) < \delta$. We want to show

$$U(f, P) - L(f, P) < \epsilon.$$

Let $x_k \in [t_{k-1}, t_k]$ such that

$$f(x_{k_n}) < m(f, [t_{k-1}, t_k]) + \frac{\epsilon}{2(b - a)}.$$

Then

$$\begin{aligned} r - \frac{\epsilon}{2} < \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1}) &< \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) + \sum_{k=1}^{n} \frac{\epsilon}{2(b - a)}(t_k - t_{k-1}) \\ &= L(f, P) + \frac{\epsilon}{2} \leq L(f) + \frac{\epsilon}{2} \implies L(f) > r - \epsilon. \end{aligned}$$

Let $y_k \in [t_{k-1}, t_k]$ such that

$$f(y_{k_n}) > M(f, [t_{k-1}, t_k]) - \frac{\epsilon}{2(b - a)}.$$

Then

$$r + \frac{\epsilon}{2} > \sum_{k=1}^{n} f(y_k)(t_k - t_{k-1}) > U(f,P) - \frac{\epsilon}{2} \geq U(f) - \frac{\epsilon}{2} \implies U(f) < r + \epsilon.$$

Let $\epsilon \to 0$ to get that $L(f) = U(f) = r$.

□

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be a monotonic function. Then f is integrable.*

Proof. Let $\epsilon > 0$ and let $P = \{a = t_0 < \cdots < t_n = b\} : mesh(P) < \delta$ for δ to be chosen shortly. We want to show

$$U(f,P) - L(f,P) < \epsilon.$$

Assume, WLOG, that f is increasing. Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}) = \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right)(t_k - t_{k-1}) \\ &\leq \delta \sum_{k=1}^{n} \left(f(t_k) - f(t_{k-1})\right) = \delta(f(b) - f(a)) < \epsilon, \end{aligned}$$

provided

$$\delta < \frac{\epsilon}{f(b) - f(a)}.$$

Exercise. Treat the case when f is constant.

□

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is integrable.*

Proof. Let $\epsilon > 0$. Let $P = \{a = t_0 < \cdots < t_n = b\} : mesh(P) < \delta$ for δ to be chosen shortly.

$$U(f,P) - L(f,P) = \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}).$$

As f is continuous on $[a,b]$ compact, f is uniformly continuous. So $\exists\ \delta > 0$ such that

$$|f(x) - f(y)| < \frac{\epsilon}{b-a}\ \forall\ x, y \in [a,b] : |x - y| < \delta.$$

For this δ and P as above,

$$U(f,P) - L(f,P) < \epsilon \sum_{k=1}^{n} (t_k - t_{k-1}) = \epsilon.$$

We have a strict inequality because f attains its sup and inf on $[t_{k-1}, t_k]$. □

Theorem. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable and let $\alpha \in \mathbb{R}$. Then*

1. *αf is integrable and*

$$\int_a^b (\alpha f)(x)dx = \alpha \int_a^b f(x)dx.$$

2. *$f + g$ is integrable and*

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Proof. 1. If $\alpha = 0$, this is clear. Assume $\alpha > 0$. For $S \subseteq [a,b]$, we have $M(\alpha f, S) = \alpha M(f,S)$ and $m(\alpha f, S) = \alpha m(f,S)$. For a partition P of $[a,b]$, we have $U(\alpha f, P) = \alpha U(f,P)$ and $L(\alpha f, P) = \alpha L(f,P)$. Then

$$U(\alpha f) = \inf\{U(\alpha f, P) : P \text{ partition of } [a,b]\} = \inf\{\alpha U(f,P) : P \text{ partition of } [a,b]\} = \alpha U(f).$$

Similarly, $L(\alpha f) = \alpha L(f)$. Because f is integrable,

$$U(f) = L(f) \implies U(\alpha f) = L(\alpha f) = \alpha \int_a^b f(x)dx$$

Assume $\alpha < 0$. Then we have $U(\alpha f, P) = \alpha L(f,P)$ and $L(\alpha f, P) = \alpha U(f,P)$. Thus $U(\alpha f) = \alpha L(f)$ and $L(\alpha f) = \alpha U(f)$. We conclude as before; because f is integrable,

$$U(f) = L(f) \implies U(\alpha f) = L(\alpha f) = \alpha \int_a^b f(x)dx.$$

2. Note that for a partition P of $[a,b]$, we have

$$U(f+g,P) \leq U(f,P) + U(g,P), \quad L(f+g,P) \geq L(f,P) + L(g,P)$$

Let $\epsilon > 0$. As f is integrable,

$$\exists\, P_1 \text{ partition of } [a,b] : U(f,P_1) - L(f,P_1) < \frac{\epsilon}{2}$$
$$\exists\, P_2 \text{ partition of } [a,b] : U(g,P_2) - L(g,P_2) < \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then

$$U(f+g,P) - L(f+g,P) \leq U(f,P) - L(f,P) + U(g,P) - L(g,P) \leq U(f,P_1) - L(f,P_1) + U(g,P_2) - L(g,P_2) < \epsilon.$$

This shows $f+g$ is integrable. Moreover,

$$\begin{aligned} U(f+g) &\leq U(f+g,P) \leq U(f,P) + U(g,P) < L(f,P) + \frac{\epsilon}{2} + L(g,P) + \frac{\epsilon}{2} \\ &\leq L(f) + L(g) + \epsilon = \int_a^b f(x)dx + \int_a^b g(x)dx + \epsilon. \end{aligned}$$

Similarly,

$$\begin{aligned} L(f+g) &\geq L(f+g,P) \geq L(f,P) + L(g,P) > U(f,P) - \frac{\epsilon}{2} + U(g,P) - \frac{\epsilon}{2} \\ &\geq U(f) + U(g) - \epsilon = \int_a^b f(x)dx + \int_a^b g(x)dx - \epsilon. \end{aligned}$$

Then

$$\int_a^b f(x)dx + \int_a^b g(x)dx - \epsilon \leq L(f+g) \leq U(f+g) \leq \int_a^b f(x)dx + \int_a^b g(x)dx + \epsilon.$$

Let $\epsilon \to 0$ to get the result.

□

Lemma. *Let $f, g : [a,b] \to \mathbb{R}$ be integrable : $f(x) \leq g(x)\ \forall\, x \in [a,b]$. Then*

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

Proof. Let $h : [a,b] \to \mathbb{R}, h(x) = g(x) - f(x)$ integrable. Moreover,

$$L(h) = \sup\{L(h,P) : P \text{ partition of } [a,b]\} \geq 0 \implies \int_a^b (g-f)(x)dx \geq 0 \implies \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

□

Theorem. *Let $f : [a, b] \to \mathbb{R}$ be integrable. Then $|f|$ is integrable and*

$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx.$$

Proof. Let's show $|f|$ is integrable. For $S \subseteq [a, b]$,

$$\begin{aligned} M(|f|, S) - m(|f|, S) &= \sup_{x \in S} |f(x)| - \inf_{y \in S} |f(y)| = \sup_{x,y \in S} |f(x)| - |f(y)| \\ &\leq \sup_{x,y \in S} |f(x) - f(y)| = \sup_{x,y \in S} (f(x) - f(y)) = \sup_{x \in S} f(x) - \inf_{y \in S} f(y) \\ &= M(f, S) - m(f, S). \end{aligned}$$

If P is a partition of $[a, b]$, then

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P).$$

As f is integrable, given $\epsilon > 0$,

$$\exists\ P \text{ partition of } [a, b] : U(f, P) - L(f, P) < \epsilon.$$

Collecting both, we find that f is integrable. Moreover,

$$\begin{aligned} -|f| \leq f \leq |f| &\implies \int_a^b (-|f|)(x)dx \leq \int_a^b f(x)dx \leq \int_a^b (|f|)(x)dx \\ &\implies -\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx \implies \left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx. \end{aligned}$$

□

Theorem. *Assume $f : [a, c] \to \mathbb{R}$ is a function and $a < b < c$ are such that f is integrable on $[a, b]$ and f is integrable on $[b, c]$. Then f is integrable on $[a, c]$ and*

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Proof. Let $\epsilon > 0$. As f is integrable on $[a, b]$,

$$\exists\ P_1 \text{ partition of } [a, b] : U_a^b(f, P_1) - L_a^b(f, P_1) < \frac{\epsilon}{2}.$$

Similarly, as f is integrable on $[b, c]$,

$$\exists\ P_2 \text{ partition of } [b, c] : U_b^c(f, P_2) - L_b^c(f, P_2) < \frac{\epsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then P is a partition of $[a, c]$ and

$$U_a^c(f, P) = U_a^b(f, P_1) + U_b^c(f, P_2), \quad L_a^c(f, P) = L_a^b(f, P_1) + L_b^c(f, P_2).$$

Thus $U_a^c(f, P) - L_a^c(f, P) < \epsilon$, so f is integrable on $[a, c]$. Moreover,

$$\begin{aligned} \int_a^c f(x)dx \leq U_a^c(f, P) = U_a^b(f, P_1) + U_b^c(f, P_2) < L_a^b(f, P_1) + L_b^c(f, P_2) + \epsilon \leq \int_a^b f(x)dx + \int_b^c f(x)dx + \epsilon, \\ \int_a^c f(x)dx \geq L_a^c(f, P) = L_a^b(f, P_1) + L_b^c(f, P_2) > U_a^b(f, P_1) + U_b^c(f, P_2) - \epsilon \geq \int_a^b f(x)dx + \int_b^c f(x)dx - \epsilon. \end{aligned}$$

Let $\epsilon \to 0$ to get the result. □

Definition.

1. A function $f : [a, b] \to \mathbb{R}$ is **piecewise continuous** if there exists a partition $P = \{a = t_0 < \cdots < t_n = b\}$: f is uniformly continuous on each (t_{k-1}, t_k).

2. A function $f : [a,b] \to \mathbb{R}$ is **piecewise monotone** if there exists a partition $P = \{a = t_0 < \cdots < t_n = b\} : f$ is monotone on each (t_{k-1}, t_k).

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be either piecewise continuous or bounded piecewise monotone. Then f is integrable on $[a,b]$.*

Proof. Let $P = \{a = t_0 < \cdots < t_n = b\}$: either f is uniformly continuous on (t_{k-1}, t_k) or f is monotone on (t_{k-1}, t_k).

- If f is uniformly continuous on (t_{k-1}, t_k), then f admits a continuous extension to $[t_{k-1}, t_k]$. Let's call this extension f_k. Then f_k is integrable on $[t_{k-1}, t_k]$.
- If f is monotone on (t_{k-1}, t_k), say it's increasing, then extend if to a function $f_k : [t_{k-1}, t_k] \to \mathbb{R}$ via

$$f_k(t_{k-1}) = \inf_{t \searrow t_{k-1}} f(t), \quad f_k(t_k) = \sup_{t \nearrow t_k} f(t).$$

  As f_k is monotone on $[t_{k-1}, t_k]$, f_k is integrable on $[t_{k-1}, t_k]$.

In either case, f_k is integrable on $[t_{k-1}, t_k]$. As

$$f|_{(t_{k-1}, t_k)} = f_k|_{(t_{k-1}, t_k)},$$

f is integrable on $[t_{k-1}, t_k]$. By the previous theorem,

$$\int_a^b f(x)dx = \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} f(x)dx.$$

□

Intermediate value theorem for integrals

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then*

$$\exists\ x_0 \in [a,b] : f(x_0) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Proof. As f is continuous on $[a,b]$,

$$\begin{aligned} \exists\ \alpha, \beta \in [a,b] :& f(\alpha) = \inf_{x \in [a,b]} f(x) \leq f(x) \leq \sup_{x \in [a,b]} f(x) = f(\beta)\ \forall\ x \in [a,b] \\ &\implies f(\alpha)(b-a) = \int_a^b f(\alpha)dx \leq \int_a^b f(x)dx \leq \int_a^b f(\beta)dx = f(\beta)(b-a) \\ &\implies f(\alpha) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(\beta). \end{aligned}$$

As f is continuous, it has the intermediate value property. Thus

$$\exists\ x_0 \in [a,b] : f(x_0) = \frac{1}{b-a} \int_a^b f(x)dx.$$

□

Definition. We say that a function $f : (a,b) \to \mathbb{R}$ is integrable on $[a,b]$ if every extension of f to $[a,b]$ is integrable on $[a,b]$. In this case, the value of $\int_a^b f(x)dx$ does not depend on the values of the extensions at the points a and b.

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If f' is integrable on $[a,b]$, then*

$$\int_a^b f'(x)dx = f(b) - f(a).$$

Proof. Let $\epsilon > 0$. As f' is integrable on $[a,b]$,

$$\exists \text{ partition } P = \{a = t_0 < \cdots < t_n = b\} : U(f', P) - L(f', P) < \epsilon.$$

On one hand,

$$L(f', P) \leq \int_a^b f'(x)dx \leq U(f', P) < L(f', P) + \epsilon. \tag{1}$$

On the other hand, we will show that $f(b) - f(a)$ is the value of the Riemann sum S associated to the partition P. Then

$$L(f', P) \leq S = f(b) - f(a) \leq U(f', P) < L(f', P) + \epsilon. \tag{2}$$

Collecting (1) and (2), we get

$$\left| \int_a^b f'(x)dx - (f(b) - f(a)) \right| < 2\epsilon.$$

Let $\epsilon \to 0$ to get the claim. Notice

$$f(b) - f(a) = \sum_{k=1}^n f(t_k) - f(t_{k-1}).$$

By the Mean Value Theorem,

$$\forall\, 1 \leq k \leq n, \exists\, x_k \in (t_{k-1}, t_k) : \frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}} = f'(x_k).$$

Then

$$f(b) - f(a) = \sum_{k=1}^n f'(x_k)(t_k - t_{k-1}),$$

which is a Riemann sum associated to P. □

Integration by parts

Theorem. *Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that f', g' are Riemann integrable on $[a,b]$. Then*

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a).$$

Proof. Let $h : [a,b] \to \mathbb{R}, h(x) = f(x)g(x)$. Then h is continuous on $[a,b]$, differentiable on (a,b). For $x \in (a,b)$,

$$h'(x) = f'(x)g(x) + f(x)g'(x)$$

is integrable on $[a,b]$ since products and sums of Riemann integrable functions are integrable. Then

$$\int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx = \int_a^b [f'(x)g(x) + f(x)g'(x)]dx = \int_a^b h'(x)dx = h(b) - h(a) = (fg)(b) - (fg)(a).$$

□

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be Riemann integrable and define $F : [a,b] \to \mathbb{R}$ via*

$$F(x) = \int_a^x f(t)dt.$$

Then F is continuous on $[a,b]$. Moreover, if f is continuous at some $x_0 \in (a,b)$, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

Proof. As f is Riemann integrable,
$$\exists\ M > 0 : |f(x)| \leq M\ \forall\ x \in [a,b].$$
Let $x, y \in [a,b]$, then
$$F(x) - F(y) = \int_a^x f(t)dt - \int_a^y f(t)dt = \int_y^x f(t)dt,$$
with the convention that if $x < y$, then
$$\int_y^x f(t)dt = -\int_x^y f(t)dt.$$
Then on $[a,b]$,
$$|F(x) - F(y)| \leq |\int_y^x f(t)dt| \leq M|x-y|.$$
Thus F is uniformly continuous. Assume f is continuous at some $x_0 \in (a,b)$. For $x \in [a,b] \setminus \{x_0\}$,
$$\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) = \frac{1}{x-x_0}\int_{x_0}^x f(t)dt - \frac{1}{x-x_0}\int_{x_0}^x f(x_0)dt = \frac{1}{x-x_0}\int_{x_0}^x f(t) - f(x_0)dt.$$
As f is continuous at x_0, given $\epsilon > 0$,
$$\exists\ \delta > 0 : |f(t) - f(x_0)| < \epsilon\ \forall\ t \in [a,b] : |t - x_0| < \delta.$$
Then for $x \in [a,b] \setminus \{x_0\} : |x - x_0| < \delta$, we have
$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| \leq \frac{1}{x-x_0}\int_{x_0}^x |f(t) - f(x_0)|dt \leq \frac{1}{x-x_0}\int_{x_0}^x \epsilon dt = \epsilon.$$
This proves F is differentiable at x_0 and $F'(x_0) = f(x_0)$. □

Change of variables

Theorem. *Let $J \subseteq \mathbb{R}$ be an open interval and let $u : J \to \mathbb{R}$ be differentiable with u' continuous. Let $I \subseteq \mathbb{R}$ be an open interval such that $I \supseteq u(J)$ and let $f : I \to \mathbb{R}$ be a continuous function. Then $f \circ u : J \to \mathbb{R}$ is a continuous function and*
$$\int_a^b (f \circ u)(x)u'(x)dx = \int_{u(a)}^{u(b)} f(x)dx\ \forall\ a, b \in J.$$
Proof. Pick $c \in I$ and define
$$F(x) = \int_c^x f(t)dt.$$
As f is continuous, F is differentiable and $F'(x) = f(x)\ \forall\ x \in I$. Let $g = F \circ u : J \to \mathbb{R}$ differentiable and
$$g'(x) = F'(u(x))u'(x) = (f \circ u)(x)u'(x)$$
continuous on J and so integrable on any $[a,b] \subseteq J$. Then
$$\int_a^b (f \circ u)(x)u'(x)dx = \int_a^b g'(x)dx = g(b) - g(a) = F(u(b)) - F(u(a)) = \int_c^{u(b)} f(t)dt - \int_c^{u(a)} f(t)dt = \int_{u(a)}^{u(b)} f(t)dt.$$
□

Theorem. *Let $f_n : [a.b] \to \mathbb{R}$ be Riemann integrable functions. Assume $\{f_n\}_{n\geq 1}$ converges uniformly on $[a,b]$ to a function f. Then f is Riemann integrable and*
$$\int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

Proof. Let $d_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$, then

$$f_n \xrightarrow[n\to\infty]{u} f \implies d_n \xrightarrow[n\to\infty]{} 0 \implies f_n(x) - d_n \le f(x) \le f_n(x) + d_n \;\forall\; x \in [a,b].$$

If $S \subseteq [a,b]$, then

$$M(f_n,S) - d_n \le M(f,S) \le M(f_n,S) + d_n, \quad m(f_n,S) - d_n \le m(f,S) \le m(f_n,S) + d_n.$$

For P a partition of $[a,b]$, we have

$$U(f_n,P) - d_n(b-a) \le U(f,P) \le U(f_n,P) + d_n(b-a), \quad L(f_n,P) - d_n(b-a) \le L(f,P) \le L(f_n,P) + d_n(b-a).$$

Then

$$U(f,P) - L(f,P) \le U(f_n,P) - L(f_n,P) + 2d_n(b-a).$$

Let $\epsilon > 0$. Let

$$n_\epsilon \in \mathbb{N} : d_n < \frac{\epsilon}{4(b-a)} \;\forall\; n \ge n_\epsilon.$$

Fix $n \ge n_\epsilon$. Let

$$P(n,\epsilon) : U(f_n,P) - L(f_n,P) < \frac{\epsilon}{2}.$$

Thus

$$U(f,P) - L(f,P) < \epsilon$$

and f is integrable. Then

$$\int_a^b f(x)dx \le U(f,P) \le U(f_n,P) + d_n(b-a) \le L(f_n,P) + \frac{\epsilon}{2} + d_n(b-a) \le \int_a^b f(x)dx + \frac{3\epsilon}{4}$$

for $n \ge n_\epsilon$ fixed and P as above. Now

$$\int_a^b f(x)dx \ge L(f,P) \ge L(f_n,P) - d_n(b-a) \ge U(f_n,P) - \frac{\epsilon}{2} - d_n(b-a) \ge \int_a^b f(x)dx - \frac{3\epsilon}{4}.$$

Thus

$$\left| \int_a^b f(x)dx - \int_a^b f_n(x)dx \right| \le \frac{3\epsilon}{4}.$$

□

Definition. We say that a set $A \subseteq \mathbb{R}$ has **zero content** if $\forall\; \epsilon > 0$, there exists a sequence of open sets $\{(a_n,b_n)\}_{n\ge1}$ such that

$$A \subseteq \bigcup_{n\ge1} (a_n,b_n), \quad \sum_{n\ge1} (b_n - a_n) < \epsilon.$$

Remark.
1. If A has zero content and $B \subseteq A$, then B has zero content.
2. If A is at most countable, then A has zero content. Indeed, write $A = \{a_1, \dots\}$ and let $\epsilon > 0$. Then

$$A \subseteq \bigcup_{n\ge1} (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}}), \quad \sum_{n\ge1} \frac{\epsilon}{2^n} = \epsilon.$$

3. If we have a sequence of sets $\{A_n\}_{n\ge1}$ such that for all $n \ge 1, A_n$ has zero content, then $\cup_{n\ge1} A_n$ has zero content. Let $\epsilon > 0$. Then $\forall\; m \ge 1$,

$$\exists\; \{(a_n^m, b_n^m)\}_{n\ge1} : A_m \subseteq \bigcup_{n\ge1} (a_n^m, b_n^m), \quad \sum_{n\ge1} (b_n^m - a_n^m) < \frac{\epsilon}{2^m}.$$

Thus

$$\bigcup_{m\ge1} A_m \subseteq \bigcup_{m,n\ge1} (a_n^m, b_n^m), \quad \sum_{m,n\ge1} (b_n^m - a_n^m) < \sum_{m\ge1} \frac{\epsilon}{2^m} = \epsilon.$$

Lebesgue criterion of Riemann integrability

Theorem. *A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable iff it's bounded and $\{x \in [a, b] : f$ is discontinuous at $x\}$ has zero content.*

Proof. We prove both ways separately.

- " $\implies$ " It suffices to show

$$\{x \in [a,b] : f \text{ is discontinuous at } x\} = \{x \in [a,b] : w(f,x) > 0\} = \bigcup_{n\geq 1}\{x \in [a,b] : w(f,x) \geq \frac{1}{n}\}$$

  has zero content. Then it suffices to show that $\forall\ n \geq 1$,

$$F_n = \{x \in [a,b] : w(f,x) \geq \frac{1}{n}\}$$

  has zero content. Fix $N \geq 1$. As f is Riemann integrable,

$$\exists\ P = \{a = t_0 < \cdots < t_n = b\} : U(f,P) - L(f,P) < \frac{\epsilon}{2N}.$$

  Let

$$I = \{k \in (1,\ldots,n) : (t_{k-1}, t_k) \cap F_N \neq \emptyset\}.$$

  Then

$$F_N \subseteq \bigcup_{k\in I}(t_{k-1}, t_k) \cup P.$$

  As P is finite, it has zero content. Thus it suffices to control $\sum_{k\in I}(t_{k-1}, t_k)$. Note that for $k \in I$,

$$w(f, [t_{k-1}, t_k]) \geq \frac{1}{N}.$$

  Then

$$\frac{\epsilon}{2N} > U(f,P) - L(f,P) = \sum_{k=1}^{n} w(f, [t_{k-1}, t_k])(t_{k-1}, t_k) \geq \sum_{k\in I} w(f, [t_{k-1}, t_k])(t_{k-1}, t_k) \geq \frac{1}{N}\sum_{k\in I}(t_{k-1}, t_k).$$

  Thus

$$\sum_{k\in I}(t_{k-1}, t_k) < \frac{\epsilon}{2}.$$

- " $\impliedby$ "

$$f \text{ bounded } \implies \exists\ M > 0 : |f(x)| \leq M\ \forall\ x \in [a,b].$$

  Let $\epsilon > 0$. Let $\alpha > 0, \delta > 0$ to be chosen shortly. We know

$$\{x \in [a,b] : w(f,x) > 0\}$$

  has zero content. Thus

$$F_\alpha = \{x \in [a,b] : w(f,x) \geq \alpha\}$$

  has zero content and $F_\alpha \cup \{a, b\}$ has zero content. Thus $\exists\ \{(a_n, b_n)\}_{n\geq 1}$ such that

$$F_\alpha \cup \{a,b\} \subseteq \bigcup_{n\geq 1}(a_n, b_n), \quad \sum_{n\geq 1}(b_n - a_n) < \delta.$$

  Then

$$w(f,x) < \alpha\ \forall\ x \in (a,b) \setminus F_\alpha \implies \exists\ c_x, d_x : w(f, [c_x, d_x]) < \alpha.$$

  and

$$[a,b] = (F_\alpha \cup \{a,b\}) \cup ((a,b) \setminus F_\alpha) \subseteq \bigcup_{n\geq 1}(a_n, b_n) \cup \bigcup_{x\in(a,b)\setminus F_\alpha}(c_x, d_x) \text{ open cover of compact } [a,b].$$

Thus $\exists\, n_0 \geq 1$ and $J \subseteq (a,b) \setminus F_\alpha$ finite such that

$$[a,b] \subseteq \bigcup_{n=1}^{n_0} (a_n, b_n) \cup \bigcup_{x \in J} (c_x, d_x)$$

Let P be a partition of $[a,b]$ consisting of the points in

$$\bigcup_{n=1}^{n_0} \{a_n, b_n\} \cup \bigcup_{x \in J} \{c_x, d_x\}$$

that belong to $[a,b]$. Write $P = \{a = t_0 < \cdots < t_n = b\}$. Note that $\forall\, 1 \leq k \leq n$ we have either

$$[t_{k-1}, t_k] \subseteq [a_m, b_m] \text{ for some } 1 \leq m \leq n_0$$

or

$$[t_{k-1}, t_k] \subseteq [c_x, d_x] \text{ for some } x \in J.$$

Let

$$I_1 = \{1 \leq k \leq n : [t_{k-1}, t_k] \subseteq [a_m, b_m] \text{ for some } 1 \leq m \leq n_0\}, \quad I_2 = \{1, \ldots, n\} \setminus I_1.$$

Then

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k \in I_1} [M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])](t_k - t_{k-1}) \\ &\quad + \sum_{k \in I_2} [M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])](t_k - t_{k-1}) \\ &\leq 2M \sum_{k \in I_1} (t_k - t_{k-1}) + \sum_{k \in I_2} w(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &\leq 2M \sum_{m \geq 1} (b_m - a_m) + \alpha \sum_{k=1}^{n} (t_k - t_{k-1}) \leq 2M\delta + \alpha(b-a) < \epsilon, \end{aligned}$$

provided

$$\delta < \frac{\epsilon}{4M+1}, \quad \alpha < \frac{\epsilon}{2(b-a)}.$$

□

Multivariable functions

Definition. Let $G \in \mathbb{R}^n$ be open and let $f : G \to \mathbb{R}^n$ be a function. Let $a \in G$. Then for a unit vector $u \in \mathbb{R}^n$ ($||u|| = 1$), the function

$$\mathbb{R} \ni t \mapsto a + tu \in \mathbb{R}^n$$

is continuous. Then

$$A_u = \{t \in \mathbb{R} : a + tu \in G\}$$

is open. Note $0 \in A_n$, thus it contains an open interval centered at $t = 0$.

- We say the function f is **differentiable at a in the direction u** if

$$\lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

exists. In this case, we denote the derivative of f at a in the direction of u by

$$(D_u f)(a).$$

If f is differentiable in the direction of u at every point in G, we denote the derivative of f in the direction of u by

$$D_u f : G \to \mathbb{R}^n.$$

- Let G open in $\mathbb{R}^n, a \in G, f : G \to \mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ denote the canonical vectors in $\mathbb{R}^n$. If f is differentiable at a in the direction of e_i, then we denote the derivative of f at a in the direction of e_i, $(D_{e_i} f)(a)$, by

$$\frac{\partial f}{\partial x_i}(a)$$

and we call it the **partial derivative of f at a with respect to x_i.**

Remark.

- The notation $\frac{\partial f}{\partial x_i}(a)$ comes from the following observation.

$$(D_{e_i} f)(a) = \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t} = \lim_{x_i \to a_i} \frac{f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) - f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n)}{x_i - a_i}$$

So $(D_{e_i} f)(a)$ exists iff the function

$$\mathbb{R} \ni x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) \in \mathbb{R}^n$$

is differentiable at a_i.

- Let G open in $\mathbb{R}^n, a \in G, f : G \to \mathbb{R}^m$. Write $f = (f_1, \ldots, f_m)$ where each $f_k : G \to \mathbb{R}$. Note that $(D_{e_i} f)(a)$ exists ($||u||_2 = 1$) iff $(D_{e_i} f_k)(a)$ exist $\forall\ 1 \leq k \leq n$. In this case,

$$(D_u f)(a) = ((D_u f_1)(a), \ldots, (D_u f_m)(a)).$$

Similarly, f admits partial derivatives at a iff each f_k admits partial derivatives at a. We write the matrix of partial derivatives

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \ldots & & \\ \frac{\partial f_m}{\partial x_1}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a) \end{bmatrix}$$

Definition. A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation if

$$T(ax + by) = aT(x) + bT(y)\ \forall\ a, b \in \mathbb{R}, x, y \in \mathbb{R}^n.$$

Recall T is represented by multiplication by an $m \times n$ matrix $[T]$. Indeed, the j^{th} column of $[T]$ is $T_{e_j} \in \mathbb{R}^m$. For $\overrightarrow{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$,

$$T(\overrightarrow{u}) = \sum_{i=1}^{n} u_i T_{e_i} = [T] \cdot \overrightarrow{u}.$$

Let $G \in \mathbb{R}^n$ be open and let $a \in G$. A function $f : G \to \mathbb{R}^n$ is differentiable at a if there exists a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{x \to a} \frac{f(x) - f(a) - T(x - a)}{||x - a||} = 0.$$

Remark. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ satisfying the previous equation is unique and is denoted by $f'(a)$. Let's assume $T, \tilde{T} : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations satisfying the previous equation. Then

$$\lim_{x \to a} \frac{T(x - a) - \tilde{T}(x - a)}{||x - a||} = 0.$$

We want to show $T = \tilde{T}$. Clearly, $T(0) = \tilde{T}(0) = 0$. Let $y \in \mathbb{R}^n \setminus \{0\}$. Since $a \in G$ open, $\exists\ r > 0 : B_r(a) \subseteq G$. Choose $t > 0 : t||y|| < r$. Then $x = a + ty \in B_r(a)$. Thus

$$0 \xleftarrow[x \to a]{} \frac{T(x - a) - \tilde{T}(x - a)}{||x - a||} = \frac{T(ty) - \tilde{T}(ty)}{t||y||} = \frac{1}{||y||}[T(y) - \tilde{T}(y)] \implies T(y) = \tilde{T}(y).$$

Example. Let $A = \{(x, y) \in \mathbb{R}^2, x \geq 0, 0 \leq y \leq x^2\}$. Let $f : A \to \mathbb{R}^2, f \equiv 0$. For $\lambda \in \mathbb{R}$, let $T_y : \mathbb{R}^2 \to \mathbb{R}, T_\lambda(x, y) = \lambda y$, a linear transformation. For $(x, y) \in A \setminus \{(0, 0)\}$,

$$\left| \frac{f(x, y) - f(0, 0) - T_\lambda((x, y) - (0, 0))}{||(x, y) - (0, 0)||} \right| = \frac{|-\lambda y|}{\sqrt{x^2 + y^2}} \leq \frac{|\lambda||y|}{\sqrt{x^2 + y^2}} \leq \frac{|\lambda| x^2}{\sqrt{x^2 + y^2}} \leq |\lambda||x| \xrightarrow[(x,y) \to (0,0)]{} 0.$$

This example shows that the transformation T is the previous equation need not be unique if the point a doesn't belong to the interior of G.

Definition. Let $T : \mathbb{R}^n : \mathbb{R}^m$ be a linear transformation. Then the norm of T is given by $||T|| = \sup_{||x||=1} ||T_x||$. Note for $x \in \mathbb{R}^n \setminus \{0\}$,

$$||T_x|| = ||x||\ ||T(\frac{x}{||x||})|| \leq ||x||\ ||T||.$$

Remark.

- Let G open $\subseteq \mathbb{R}^n, a \in G, f : G \to \mathbb{R}^n$ be differentiable at a. Let

$$\epsilon_f(x) = \begin{cases} \frac{f(x)-f(a)-f'(a)(x-a)}{||x-a||} & x \neq a \\ 0 & x = a. \end{cases}$$

  Note f is differentiable at a iff ϵ_f is continuous at a. Write

$$f(x) = f(a) + f'(a)(x-a) + \epsilon_f(x)||x-a||.$$

  Then

$$||f(x) - f(a)|| \leq ||f'(a)(x-a)|| + ||\epsilon_f(x)||\ ||x-a|| \leq ||f'(a)||\ ||x-a|| + ||\epsilon_f(x)||\ ||x-a|| \xrightarrow[x\to a]{} 0.$$

  This shows f is continuous at a.

- Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then $\forall\ a \in \mathbb{R}^n, T$ is differentiable at a and $T'(a) = T$. Indeed,

$$\frac{T(x) - T(a) - T(x-a)}{||x-a||} \equiv 0\ \forall\ x \in \mathbb{R}^n.$$

Theorem. *Let G open $\supseteq \mathbb{R}, a \in G, f : G \to \mathbb{R}^m$ be differentiable at a. Then for any unit vector $u \in \mathbb{R}^n, f$ is differentiable at a in the direction of u and $(D_u f)(a) = f'(a)u$. In particular, letting $u \in \{e_1, \ldots, e_n\}$, we deduce that the partial derivative of f exists and*

$$\frac{\partial f}{\partial x_i}(a) = f'(a)e_i.$$

This is the i^{th} column in the matrix representing $f'(a)$. This shows that the matrix representing $f'(a)$ is the matrix of partial derivatives wr wrote before. Moreover, if $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$, then

$$f'(a)u = f'(a)\sum_{i=1}^{n} u_i e_i = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}(a).$$

Proof. Let $u \in \mathbb{R}^n$ be a unit vector. Then

$$(D_u f)(a) = \lim_{t\to 0} \frac{f(a+tu) - f(a)}{t}.$$

With $x = a + tu$, use

$$\lim_{x\to a} \frac{f(x) - f(a) - f'(a)(x-a)}{||x-a||} = 0.$$

We get

$$0 = \lim_{t\to 0} \frac{f(a+tu) - f(a) - f'(a)(tu)}{|t|\ ||u||} = \lim_{t\to 0} \frac{f(a+tu) - f(a) - tf'(a)u}{|t|} \implies (D_u f)(a) = f'(a)u.$$

$\square$

Exercise. Assume $G \subseteq \mathbb{R}^n$ is open and the functions $f, g : G \to \mathbb{R}^m, h : G \to \mathbb{R}$ are differentiable at some $a \in G$. Then

1. $f + g$ is differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a).$$

2. $f, g : G \to \mathbb{R}^m$ is differentiable at a and

$$(fh)'(a)(u) = h(a)f'(a)u + f(a)h'(a)u.$$

Proposition. *Let G open $\subseteq \mathbb{R}^n, D$ open $\subseteq \mathbb{R}^m$. Assume $f : G \to D$ is differentiable at some $a \in G$ and that $g : D \to \mathbb{R}^k$ is differentiable at $f(a) = b$. Then $g \circ f : G \to \mathbb{R}^k$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a)$$

is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^k$.

Proof. Recall that f is differentiable at a iff

$$\epsilon_f(x) = \begin{cases} \frac{f(x)-f(a)-f'(a)(x-a)}{||x-a||} & x \neq a \\ 0 & x = a. \end{cases}$$

is continuous at a and g is differentiable at $b = f(a)$ iff

$$\epsilon_g(y) = \begin{cases} \frac{g(y)-g(b)-g'(b)(y-b)}{||y-b||}, & y \neq b \\ 0 & y = b. \end{cases}$$

is continuous at b. Write

$$f(x) = f(a) + f'(a)(x-a) + \epsilon_f(x)||x-a||$$

on $B_r(a) \subseteq G$, and

$$g(y) = g(b) + g'(b)(y-b) + \epsilon_g(y)||y-b||$$

on $B_\rho(b) \subseteq D$. As f is continuous at a, choosing r sufficiently small, we have

$$f(B_r(a)) \subseteq B_\rho(b).$$

Let $y = f(x)$ for $x \in B_r(a)$ to get

$$g(f(x)) = g(f(a)) + g'(f(a))[f(x) - f(a)] + \epsilon_g(f(x)) \cdot \big|\big|f'(a)(x-a) + \epsilon_f(x) \cdot ||x-a||\big|\big|.$$

For $x \neq a$,

$$\frac{g(f(x)) - g(f(a)) - (g'(f(a)) \circ f'(a))\,(x-a)}{||x-a||} = g'(f(a))\epsilon_f(x) + \frac{\epsilon_g(f(x)) \cdot \big|\big|f'(a)(x-a) + \epsilon_f(x) \cdot ||x-a||\big|\big|}{||x-a||}.$$

We want to show $RHS \xrightarrow[x \to a]{} 0$. For the first term,

$$||g'(f(a))\epsilon_f(x)|| \leq ||g'(f(a))||\ ||\epsilon_f(x)|| \xrightarrow[x \to a]{} 0$$

by continuity of ϵ_f at $x = a$. For the second term,

$$\begin{aligned} \left|\left|\epsilon_g(f(x))\frac{\big|\big|f'(a)(x-a) + \epsilon_f(x) \cdot ||x-a||\big|\big|}{||x-a||}\right|\right| &\leq ||\epsilon_g(f(x))||\frac{||f'(a)||\ ||x-a|| + ||\epsilon_f(x)||\ ||x-a||}{||x-a||} \\ &\leq ||\epsilon_g(f(x))||\left(||f'(a)|| + ||\epsilon_f(x)||\right). \end{aligned}$$

Notice

$$||\epsilon_g(f(x))|| \xrightarrow[x \to a]{} ||\epsilon_g(f(a))|| = 0,$$

and the other terms are finite. ☐

Remark. There is no mean value theorem in higher dimensions.

Example. Let $f : \mathbb{R} \to \mathbb{R}^2 \sim \mathbb{C}, f(x) = e^{ix} = \cos x + i \sin x$. Clearly,

$$f(0) = f(2\pi) = 1.$$

But we have

$$f'(x) = -\sin x + i \cos x \implies ||f'(x)|| = 1 \neq 0\ \forall\ x \in \mathbb{R}.$$

Lagrange inequality

Proposition. *Let G open $\subseteq \mathbb{R}^n$ and let $a, b \in G$ such that*

$$[a,b] = \{(1-t)a + tb : t \in [0,1]\} \subseteq G.$$

Assume $f : G \to \mathbb{R}^m$ is continuous at every point on $[a,b]$ and differentiable at every point on $(a,b) = [a,b] \setminus \{a,b\}$. Then $\exists\ x_0 \in (a,b)$ such that

$$||f(b) - f(a)|| \leq ||f'(x_0)||\ ||b-a||$$

Proof. Let $\phi : [0,1] \to \mathbb{R}$ be defined as

$$\phi(t) = \langle f((1-t)a+tb), f(b)-f(a)\rangle = f(a+t(b-a)) \cdot [f(b)-f(a)] = \sum_{i=1}^{m} f_i(a+t(b-a)) \cdot [f_i(b) - f_i(a)],$$

where $f = (f_1, \dots, f_m)$. So ϕ is continuous on $[0,1]$ and differentiable on $(0,1)$. By the mean-value theorem, $\exists\ t_0 \in (0,1) : \phi(1) - \phi(0) = \phi'(t_0)$. Then

$$\phi(1) - \phi(0) = \langle f(b), f(b)-f(a)\rangle - \langle f(a), f(b)-f(a)\rangle = \langle f(b)-f(a), f(b)-f(a)\rangle = ||f(b)-f(a)||^2.$$

For $t \in (0,1)$,

$$\begin{aligned} |\phi'(t)| = |\langle f'(a+t(b-a))(b-a), f(b)-f(a)\rangle| &\leq ||f'(a+t(b-a))(b-a)||\ ||f(b)-f(a)|| \\ &\leq ||f'(a+t(b-a))||\ ||(b-a)||\ ||f(b)-f(a)||. \end{aligned}$$

We have

$$||\phi(1) - \phi(0)|| = ||\phi'(t_0)||, \quad ||f(b)-f(a)||^2 \leq ||f'(a+t_0(b-a))||\ ||b-a||\ ||f(b)-f(a)||.$$

Let $x_0 = a + t_0(b-a) \in (a,b)$. ☐

Corollary. *Assume G is open, connected subset of $\mathbb{R}^n$. Assume $f : G \to \mathbb{R}^n$ is differentiable and $f' \equiv 0$ on G. Then f is constant on G.*

Proof. Let $a, x \in G$ and let

$$L_{a,x} = \bigcup_{k=1}^{N} [x_k, x_{k+1}]$$

be a polygonal path on G connecting a and x. Applying the Lagrange inequality on each $[x_k, x_{k+1}]$, we find $\xi_k \in (x_k, x_{k+1})$ such that

$$||f(x_{k+1} - f(x_k))|| \leq ||f'(\xi_k)||\ ||x_{k-1} - x_k|| = 0.$$

This shows $f(x) = f(a)$. ☐

Theorem. *Let $f : [0,1] \to \mathbb{R}^n$ be continuous and such that there exists a set $A \subseteq [0,1]$ which is at most countable such that f is differentiable on $(0,1) \setminus A$. Then*

$$||f(1) - f(0)|| < \sup_{x \in (0,1)\setminus A} ||f'(x)||.$$

Proof. Let $\epsilon > 0$ and

$$M = \sup_{x \in (0,1)\setminus A} ||f'(x)||.$$

We want to show

$$||f(1) - f(0)|| \leq M + 2\epsilon.$$

Assume $A = \{a_1, \dots\}$ is countable. Let $\{\epsilon_n\}_{n\geq 1} \subseteq (0, \infty)$ such that

$$\sum_{n\geq 1} \epsilon_n < \epsilon.$$

Let

$$B = \{t \in [0,1] : ||f(s) - f(0)|| \leq \epsilon s + Ms + \sum_{a_k \leq s} \epsilon_k \ \forall\ 0 \leq s \leq t\}.$$

Clearly,

$$0 \in B, t \in B \implies [0,t] \subseteq B.$$

Thus B is an interval and $\exists\ b \in [0,1]$ such that

$$[0,b) \subseteq B \subseteq [0,b].$$

Claim. $b \in B$, i.e. B is closed in $[0,1]$.

As f is continuous at b,

$$||f(b) - f(0)|| = \lim_{t \nearrow b} ||f(t) - f(0)|| \leq \lim_{t \nearrow b} (\epsilon t + Mt + \sum_{a_k \leq t} \epsilon_k) \leq \epsilon b + Mb + \sum_{a_k \leq b} \epsilon_k \implies b \in B.$$

Claim. $b = 1$.

Assume, towards a contradiction, that $b < 1$.

1. Assume $b \in A$, then

$$\exists\ n_0 \geq 1 : b = a_{n_0}.$$

As f is continuous at b,

$$\exists\ b < c < 1 : ||f(t) - f(b)|| < \epsilon_{n_0}\ \forall\ t \in [b,c].$$

By the triangle inequality, for $t \in [b,c]$ we have

$$\begin{aligned} ||f(t) - f(0)|| &\leq ||f(t) - f(b)|| + ||f(b) - f(0)|| \\ &< \epsilon_{n_0} + \epsilon b + \sum_{a_k < b} \epsilon_k = \epsilon b + Mb + \sum_{a_k \leq b} \epsilon_k \leq \epsilon t + Mt + \sum_{a_k \leq t} \epsilon_k. \end{aligned}$$

So $[b,c] \subseteq B$, contradiction.

2. Assume $b \notin A$. As f is differentiable at b,

$$\lim_{t \to b} \frac{f(t) - f(b) - f'(b)(t-b)}{|t-b|} = 0.$$

Then

$$\exists\ b < c < 1 : ||f(t) - f(b) - f'(b)(t-b)|| < \epsilon(t,b)\ \forall\ t \in [b,c]$$

and

$$||f(t) - f(b)|| < \epsilon(t-b) + ||f'(b)||(t-b) \leq \epsilon(t-b) + M(t-b).$$

By the triangle inequality, for $t \in [b,c]$,

$$\begin{aligned} ||f(t) - f(0)|| &\leq ||f(t) - f(b)|| + ||f(b) - f(0)|| \\ &\leq \epsilon(t-b) + M(t-b) + \epsilon b + Mb + \sum_{a_k < b} \epsilon_k \leq \epsilon t + Mt + \sum_{a_k \leq t} \epsilon_k. \end{aligned}$$

So $[b,c] \subseteq B$, contradiction.

□

Definition. Let G open $\supseteq \mathbb{R}^n$. Then $f : G \to \mathbb{R}^m$ is said to be **continuously differentiable on** G if f is differentiable on G and the derivative $f' : G \to L(\mathbb{R}^n, \mathbb{R}^m)$ (the space of linear transformations) is continuous on G. In this case, we say f **is of class** C' **on** G and write $f \in C'(G)$.

Theorem. *Let G open $\supseteq \mathbb{R}^n$ and let $f : G \to \mathbb{R}^m$ be a function. Then $f \in C'(G)$ iff the partial derivatives $\frac{\partial f}{\partial x_j}$ exist on G for all $1 \leq j \leq n$ and are continuous on G.*

Proof. We prove both ways separately.

- " $\implies$ " Fix $a \in G$. Let $\epsilon > 0$. As $f \in C'(G)$,

  $$\exists\, r > 0 : ||f'(x) - f'(a)|| < \epsilon \;\forall\, x \in B_r(a) \subseteq G.$$

  Recall

  $$\frac{\partial f}{\partial x_j}(a) = f'(a)e_j \;\forall\, 1 \le j \le n.$$

  Then

  $$\begin{aligned}\left|\left|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)\right|\right| &= ||f'(x)e_j - f'(a)e_j|| = ||(f'(x) - f'(a))e_j|| \\ &\le ||f'(x) - f'(a)||\; ||e_j|| < \epsilon \;\forall\, x \in B_r(a), 1 \le j \le n.\end{aligned}$$

- " $\impliedby$ " Fix $a \in G$. Take $\epsilon > 0$. For $1 \le j \le n$ fixed, as $\frac{\partial f}{\partial x_j}$ is continuous at a, we know

  $$\exists\, r_j > 0 : \left|\left|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)\right|\right| < \frac{\epsilon}{n} \;\forall\, x \in B_{r_j}(a) \subseteq G.$$

  Let $r = \min_{1 \le j \le n} r_j$. Then

  $$\left|\left|\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)\right|\right| < \frac{\epsilon}{n} \;\forall\, x \in B_r(a), 1 \le j \le n.$$

  Let

  $$T = \begin{bmatrix} \frac{\partial f}{\partial x_1}(a) & \dots & \frac{\partial f}{\partial x_n}(a). \end{bmatrix}$$

  This is an $m \times n$ matrix. We want to show $f'(a) = T$. Notice

  $$\begin{aligned} f(x) - f(a) - T(x-a) &= f(x) - f(a) - \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a)(x_j - a_j) \\ &= f(x_1, \dots, x_n) - f(a_1, \dots, x_n) - \frac{\partial f}{\partial x_1}(a)(x_1 - a_1) \\ &+ f(a_1, \dots, x_n) - f(a_1, a_2, \dots, x_n) - \frac{\partial f}{\partial x_2}(a)(x_2 - a_2) \\ &+ \dots \\ &+ f(a_1, \dots, a_{n-1}, x_n) - f(a_1, \dots, a_n) - \frac{\partial f}{\partial x_n}(a)(x_n - a_n). \end{aligned}$$

  For every $1 \le i \le n$, consider the map $\phi_i(t)$ given by

  $$t \mapsto f(a_1, \dots, a_{i-1}, t, x_{i+1}, \dots, x_n) - \frac{\partial f}{\partial x_j} t.$$

  ϕ_i is continuous on $[a, x]$ and differentiable on (a, x), with derivative

  $$\phi_i'(t) = \frac{\partial f}{\partial x_i}(a_1, \dots, a_{i-1}, t, x_{i+1}, \dots, x_n) - \frac{\partial f}{\partial x_i}(a).$$

  By the Lagrange inequality,

  $$\exists\, \xi_i \in (a_i, x_i) : ||\phi_i(x_i) - \phi_i(a_i)|| \le ||\phi_i'(\xi_i)||\; |x_i - a_i|.$$

  Then

  $$||f(a_1, \dots, a_{i-1}, x_i, \dots, x_n) - f(a_1, \dots, a_i, x_{i+1}, \dots, x_n) - \frac{\partial f}{\partial x_i}(a)(x_i - a_i)|| < \frac{\epsilon}{n}|x_i - a_i|| \le \frac{\epsilon}{n}||x - a||.$$

  This gives

  $$||f(x) - f(a) - T(x-a)|| \le \epsilon ||x - a|| \;\forall\, x \in B_r(a).$$

By definition, f is differentiable at a and $f'(a) = T$. For $x \in B_r(a)$ and any $n \in \mathbb{R}^n \setminus 0$,

$$[f'(x) - f'(a)]u = || \sum_{j=1}^{n} [\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)]u_j||$$

$$\leq \sum_{j=1}^{n} ||\frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(a)||\ |u_j| \leq \frac{\epsilon}{n} \left(\sum_{j=1}^{n} |u_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} 1^2 \right)^{\frac{1}{2}} \leq \frac{\epsilon}{\sqrt{n}}||u||.$$

Then

$$||f'(x) - f'(a)|| = \sup_{||u||=1} ||(f'(x) - f'(a))u|| \leq \frac{\epsilon}{\sqrt{n}}\ \forall\ x \in B_r(a).$$

□

Lemma. *Assume A is an $n \times n$ invertible matrix and assume that B is another $n \times n$ matrix such that*

$$\alpha = ||A - B||\ ||A^{-1}|| < 1.$$

Then B is invertible.

Proof. B is invertible iff $\ker B = 0$. By the triangle inequality,

$$||x|| = ||A^{-1}Ax|| = ||A^{-1}[(A - B)x + Bx]||$$
$$\leq ||A^{-1}||\ (||(A - B)x|| + ||Bx||) \leq ||A^{-1}||\ (||A - B||\ ||x|| + ||Bx||) = \alpha||x|| + ||Bx||\ ||A^{-1}||.$$

Then

$$\frac{(1-\alpha)||x||}{||A^{-1}||} \leq ||Bx||.$$

If $x \in \ker B$, then $Bx = 0$. Then $x = 0$, so $kerB = \{0\}$. □

Contraction mapping

Theorem. *Assume (X, d) is a complete metric space and that $\phi : X \to X$ is a contraction, that is,*

$$d(\phi(x), \phi(y)) \leq \alpha d(x, y)\ \forall\ x, y \in X, \text{ and some fixed } \alpha \in (0, 1).$$

Then ϕ admits a unique fixed point, that is,

$$\exists!\ x_0 \in X : \phi(x_0) = x_0.$$

Inverse function

Theorem. *Let G be open in $\mathbb{R}^n$ and assume $f : G \to \mathbb{R}^n$ is differentiable on G with $f' : G \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuous at $a \in G$ and $f'(a)$ invertible. Then there exists open sets U, V in $\mathbb{R}^n$ such that*

$$a \in U, \quad f(a) \in V, \quad V = f(U), \quad f : U \to V \text{ is bijective},$$

and the inverse

$$g = (f|_U)^{-1} : V \to U$$

is differentiable on V with the derivative continuous at $f(a)$.

Proof. Set

$$T = (f'(a))^{-1}.$$

Since f' is continuous at a,

$$\exists\ r > 0 : ||f'(x) - f'(a)|| \leq \frac{1}{2||T||}\ \forall\ x \in B_r(a) \subseteq G.$$

By the Lemma, this implies that $f'(x)$ is invertible for all $x \in B_r(a)$. Let $U = B_r(a)$ and $V = f(U)$. Let's first show that $f : U \to V$ is bijective. As it's clearly surjective, it suffices to check injectivity. Use the Lagrange inequality for the map h given by

$$U \ni x \mapsto f(x) - f'(a)x \in \mathbb{R}^n.$$

For $x, y \in U$,

$$\exists\, \xi \in (x, y) : ||h(x) - h(y)|| \leq ||h'(\xi)||\ ||x - y|| = ||f'(\xi) - f'(a)||\ ||x - y|| \leq \frac{1}{2||T||}||x - y||.$$

On the other hand,

$$h(x) - h(y) = [f(x) - f'(a)x] - [f(y) - f'(a)y] = f(x) - f(y) - f'(a)(x - y).$$

Then

$$\begin{aligned} &||f(x) - f(y) - f'(a)(x - y)|| \leq \frac{1}{2||T||}||x - y|| \\ \implies &||T[f(x) - f(y) - f'(a)(x - y)]|| \leq ||T||\ ||f(x) - f(y) - f'(a)(x - y)|| \leq \frac{1}{2}||x - y|| \\ \implies &||T[f(x) - f(y)] - (x - y)]|| \leq \frac{1}{2}||x - y||\ \forall\ x, y \in U. \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} ||x - y|| \leq ||T[f(x) - f(y)]|| + ||T[f(x) - f(y)] - (x - y)|| &\leq ||T||\ ||f(x) - f(y)|| + \frac{1}{2}||x - y|| \\ \implies ||f(x) - f(y)|| &\geq \frac{1}{2||T||}||x - y||\ \forall\ x, y \in U. \end{aligned}$$

This shows f is injective on U. Next, let's show $V = f(U)$ is open. Let $y_0 \in V$, then

$$\exists\ x_0 \in U : f(x_0) = y_0.$$

Let $\rho > 0$ such that

$$\overline{B_\rho(x_0)} \subseteq U.$$

Note $\overline{B_\rho(x_0)}$ is a complete metric space when endowed with the Euclidean distance on $\mathbb{R}^n$. We will show V is open, i.e.

$$B_{\frac{\rho}{2||T||}}(y_0) \subseteq V.$$

Let $y \in B_{\frac{\rho}{2||T||}}(y_0)$. We want to find

$$x_1 \in U : f(x_1) = y_1.$$

Consider the map ϕ given by

$$\overline{B_\rho(x_0)} \ni x \mapsto x + T(y_1 - f(x)).$$

We want to prove

1. $\phi : \overline{B_\rho(x_0)} \to \overline{B_\rho(x_0)}$,
2. ϕ is a contraction on $\overline{B_\rho(x_0)}$.

If both hold, then ϕ has a unique fixed point $x_1 \in \overline{B_\rho(x_0)} \subseteq U$, and

$$\phi(x_1) = x_1 \iff T(y_1 - f(x_1)) = 0 \iff f(x_1) = y.$$

We will check both hold. For $x, y \in \overline{B_\rho(x_0)} \subseteq U$,

$$||\phi(x) - \phi(y)|| = ||(x - y) - T(f(x) - f(y))|| \leq \frac{1}{2}||x - y||.$$

Thus ϕ is a contraction on $\overline{B_\rho(x_0)}$. On the other hand,

$$||\phi(x_0) - x_0|| = ||T(y_1 - f(x_0))|| \leq ||T||\ ||y_1 - y_0|| \leq ||T||\frac{\rho}{2||T||} \leq \frac{\rho}{2}.$$

By the triangle inequality, for $x \in \overline{B_\rho(x_0)}$, we have

$$||\phi(x) - x_0|| \leq ||\phi(x) - \phi(x_0)|| + ||\phi(x_0) - x_0|| \leq \frac{1}{2}||x - x_0|| + \frac{\rho}{2} \leq \frac{\rho}{2} + \frac{\rho}{2} = \rho \implies \phi(x) \in \overline{B_\rho(x_0)}.$$

We are left to show that

$$g = (f|_V)^{-1} : V \to U$$

is differentiable on V with g' continuous at $f(a)$. Let $y \in V, y + k \in V$. Then

$$\exists\, x \in U, x + h \in U : f(x) = y, f(x + h) = g + k.$$

Then

$$\begin{aligned} \frac{g(y+k) - g(y) - [f'(x)]^{-1}(y + k - y)}{||k||} &= \frac{x + h - x - [f'(x)]^{-1}[f(x+h) - f(x)]}{||k||} \\ &= -\frac{[f'(x)]^{-1}[f(x+h) - f(x) - f'(x)h]}{||h||}\frac{||h||}{||k||}. \end{aligned}$$

Recall that

$$||k|| = ||y + k - y|| = ||f(x+h)f(x)|| \leq \frac{1}{2||T||}||h|| \implies ||h|| \leq 2||T||\ ||k||.$$

So

$$\lim_{||k|| \to 0} \frac{||g(y+k) - g(y) - [f'(x)]^{-1}k||}{||k||} \leq 2||T||\ ||[f'(x)]^{-1}|| \lim_{||h|| \to 0} \frac{||f(x+h) - f(x) - f'(x)h||}{||h||} = 0.$$

This shows g is differentiable at y and

$$g'(y) = [f'(x)]^{-1} = [f'(g(y))]^{-1}.$$

Continuity at $f(a)$ follows from f' continuous at a and g continuous at $f(a)$. □

Partial derivatives in higher dimensions

Definition. Let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and let $f : G \to \mathbb{R}^m$ be a function. Assume

$$f(x_0, y_0) = 0 \text{ for some } x_0, y_0 \in G.$$

We want to find

1. open sets $U \in \mathbb{R}^n, V \in \mathbb{R}^m$ such that
$$x_0 \in U, y_0 \in V.$$
2. a unique function $\phi : U \to V$ such that
$$(x, y) \in U \times V.$$

We know $f(x, y) = 0$ and $y = f(x)$ are equivalent and define y, respectively, implicitly and explicitly.

- Let
$$G_{x_0} = \{y \in \mathbb{R}^m : (x_0, y) \in G\}.$$
If the function
$$G_{x_0} \ni y \mapsto f(x_0, y) \in \mathbb{R}^m$$
is differentiable at y_0, we denote its derivative by
$$\frac{\partial f}{\partial y}(x_0, y_0)$$
and we call it the **partial derivative of f with respect to y at** (x_0, y_0). Clearly,
$$\frac{\partial f}{\partial y}(x_0, y_0) : \mathbb{R}^m \to \mathbb{R}^m$$
is a linear transformation.

- Let

$$G_{y_0} = \{x \in \mathbb{R}^n : (x, y_0) \in G\}.$$

If the function

$$G_{y_0} \ni x \mapsto f(x, y_0) \in \mathbb{R}^n$$

is differentiable at x_0, we denote its derivative by

$$\frac{\partial f}{\partial x}(x_0, y_0)$$

and we call it the **partial derivative of f with respect to x at** (x_0, y_0). Clearly,

$$\frac{\partial f}{\partial x}(x_0, y_0) : \mathbb{R}^n \to \mathbb{R}^n$$

is a linear transformation.

Implicit function

Theorem. *Let $G \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and let $f : G \to \mathbb{R}^m$ be differentiable on G with f continuous at some point $(a, b) \in G$. Assume $f(a, b) = 0$ and $\frac{\partial f}{\partial y}(a, b)$ is invertible. Then there exists open sets $U \in \mathbb{R}^n, V \in \mathbb{R}^m$ such that*

$$a \in U, b \in V, \quad U \times V \subseteq G, \quad \exists!\ \phi : U \to V : (x, y) \in U \times V \implies f(x, y) = 0 \iff y = \phi(x).$$

Moreover, we can choose U, V such that ϕ is differentiable on U with ϕ' continuous at a and

$$\phi'(x) = -\left(\frac{\partial f}{\partial x}(x, \phi(x))\right)^{-1} \circ \frac{\partial f}{\partial x}(x, \phi(x)).$$

Proof. Let $F : G \to \mathbb{R}^n \times \mathbb{R}^n$ given by

$$F(x, y) = (x, f(x, y)).$$

By hypothesis, F is differentiable on G with F' continuous at (a, b). In fact,

$$F'(x, y) = \begin{bmatrix} 1|_{\mathbb{R}^n} & 0 \\ \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \end{bmatrix}.$$

Note that

$$F'(x, y) \text{ is invertible } \iff ker(F'(x, y)) = \{0\} \subseteq \mathbb{R}^{n+m}.$$

Let $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ such that

$$F'(x, y)(u, v) = 0.$$

Then

$$F'(x, y)(u, v) = \begin{bmatrix} u \\ \frac{\partial f}{\partial x}(x, y)u + \frac{\partial f}{\partial y}(x, y)v \end{bmatrix} = [0] \implies u = 0, \quad \frac{\partial f}{\partial y}(x, y)v = 0 \implies v = 0$$

and

$$F'(x, y) \text{ is invertible } \iff ker(F'(x, y)) = \{0\} \subseteq \mathbb{R}^{n+m}$$
$$\iff ker(\frac{\partial f}{\partial y}(x, y)) = \{0\} \subseteq \mathbb{R}^n \iff \frac{\partial f}{\partial y}(x, y) \text{ is invertible.}$$

As $\frac{\partial f}{\partial y}(a, b)$ is invertible, $F'(a, b)$ is invertible. By the inverse function theorem, $\exists\ W_0 \supseteq G$ open such that

$$(a, b) \in W_0, \quad F(W_0) = D_0 \text{ open } \subseteq \mathbb{R}^n \times \mathbb{R}^m.$$

The function $F : W_0 \to D_0$ is bijective and its inverse $(F|_{W_0})^{-1}$ is differentiable on D_0 with continuous derivative and

$$F(a, b) = (a, f(a, b)) = (a, 0).$$

As $(a, b) \subseteq W_0$ is open, there exists open sets $a \in U_0 \subseteq \mathbb{R}^n$ and $b \in V \subseteq \mathbb{R}^m$ such that

$$U_0 \times V \subseteq W_0.$$

Let

$$U = \{x \in U_0 \mid \exists\, y \in V : f(x, y) = 0\}.$$

Note $x \in U \iff (x, 0) \in F(U_0 \times V)$.

Claim. U is open.

We have the function ψ given by

$$u_0 \ni x \mapsto (x, 0)$$

and continuous. Then

$$F(u_0 \times v) = [(F|_{W_0})^{-1}]^{-1}(U_0, V)$$

is open because (U_0, V) is open and $(F|_{W_0})^{-1}$ is continuous. Then

$$U = \psi^{-1}(F(u_0 \times v))$$

is open.

Claim.

$$\forall\, x \in U, \exists!\; y \in V : f(x, y) = 0.$$

The existence of such y is given by the definition of U. Let's prove uniqueness. Assume $y_1, y_2 \in V$ such that

$$f(x, y_1) = f(x, y_2) = 0.$$

But then

$$\left.\begin{array}{l} F(x, y_1) = (x, f(x, y_1) = (x, 0) \\ F(x, y_1) = (x, f(x, y_1) = (x, 0) \end{array}\right\} \implies F(x, y_1) = F(x, y_2), \quad F \text{ injective } \implies y_1 = y_2.$$

Let $\phi : U \to V : \phi(x) = y$ where y is the unique point in V for which $f(x, y) = 0$. In particular,

$$F(x, \phi(x)) = (x, f(x, \phi(x))) = (x, f(x, y)) = (x, 0).$$

Let $\pi_2 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m, \pi_2(x, y) = y$. Then

$$\begin{array}{ccc} U \ni & \xrightarrow{\;\psi\;} & (x, 0) \\ \phi \downarrow & & \uparrow F \\ V \ni \phi(x) & \xleftarrow{\;\pi_2\;} & (x, \phi(x)) \end{array}$$

Thus

$$\phi(x) = (\pi_2 \circ F^{-1} \circ \psi)(x)$$

is differentiable on U. Moreover, it's continuous at a. From the inverse function theorem, we know that $F'(x)$ is invertible on $U \times V \iff \frac{\partial f}{\partial y}(x, \phi(x))$ is invertible on U. As $f(x, \phi(x)) = 0$, we can use the chain rule to get

$$\frac{\partial f}{\partial x}(x, \phi(x)) + \frac{\partial f}{\partial y}(x, \phi(x))\phi'(y) = 0 \implies \phi'(x) = -\left[\frac{\partial f}{\partial y}(x, \phi(x))\right]^{-1} \circ \frac{\partial f}{\partial x}(x, \phi(x)).$$

□