Compactness in metric spaces

Definition. Let $(X, d)$ be a metric space.

- Let $A \subseteq X$. An open cover of $A$ is any collection $\{G_i\}_{i \in I}$ of open sets such that $A \subseteq \bigcup_{i \in I} G_i$. The open cover is called finite if $I$ is finite. Otherwise, the open cover is called infinite.

- A set $K \subseteq X$ is called compact if every open cover of $K$ admits a finite subcover, that is, if $\{G_i\}_{i \in I}$ is an open cover for $K$, then $\exists \ n \geq 1$ and $i_1, \ldots, i_n \in I : K \subseteq \bigcup_{j=1}^{n} G_{i_j}$.

Proposition. Let $(X, d)$ be a metric space and let $K \subseteq X$ be compact. Then $K$ is closed and bounded.

Proof. We first show $K$ is closed, then we show $K$ is bounded.

- To show $K$ is closed, it suffices to prove $^c K$ is open. If $^c K = \emptyset$, then $K$ is closed. Thus, we may assume $^c K \neq \emptyset$. Let $x \in ^c K$. For $y \in K$, let
  $$r_y = \frac{1}{2} d(x, y).$$
  Consider the open set
  $$B_{r_y}(y) = \{z \in X : d(z, y) < r_y\}.$$ 
  Then $K \subseteq \bigcup_{y \in K} B_{r_y}(y)$. As $K$ is compact, the open cover $\{B_{r_y}(y)\}_{y \in K}$ admits a finite subcover. Thus
  $$\exists \ n \geq 1 \text{ and } y_1, \ldots, y_n \in K : K \subseteq \bigcup_{i=1}^{n} B_{r_i}(y_i),$$
  where we used the shorthand $r_i = r_{y_i}$. Let
  $$r = \min \{r_i\} \ \forall \ 1 \leq i \leq n.$$ 
  Then
  $$B_r(x) \cap B_{r_i}(y_i) = \emptyset \ \forall \ 1 \leq i \leq n.$$ 

  Otherwise we find the contradiction
  $$z \in B_r(x) \cap B_{r_i}(y_i) \implies d(x, y_i) \leq d(x, z) + d(z, y_i) < r + r_i \leq 2r_i = d(x, y_i).$$
  Thus
  $$B_r(x) \subseteq \bigcap_{i=1}^{n} B_{r_i}(y_i) = \bigcap_{i=1}^{n} B_{r_i}(y_i) \subseteq ^c K.$$ 
  By definition, $^c K$ is open and so $K$ is closed.

- We show $K$ is bounded. Clearly, $\{B_1(y)\}_{y \in K}$ is an open cover of $K$. As $K$ is compact, $\exists \ n \geq 1$ and $y_1, \ldots, y_n \in K : K \subseteq \bigcup_{i=1}^{n} B_1(y_i)$. Let $r = \max_{1 \leq k \leq n} d(y_1, y_k) + 1$. Then $K \subseteq B_r(y_1)$.

\[ \square \]

Theorem. Let $(X, d)$ be a metric space and let $K \subseteq Y \subseteq X$. Then $K$ is compact in $Y \iff K$ is compact in $X$.

Proof. We prove both ways separately.

- "$\implies"$ Let $\{G_i\}_{i \in I}$ be a collection of sets open in $X$ such that $K \subseteq \bigcup_{i \in I} G_i$. Then $V_i = G_i \cap Y$ is open in $Y \ \forall \ i \in I$. We have $K \subseteq (\bigcup_{i \in I} G_i) \cap Y = \bigcup_{i \in I} V_i$. As $K$ is compact in $Y$, $\exists \ i_1, \ldots, i_n \in I : K \subseteq \bigcup_{k=1}^{n} V_{i_k} \implies K \subseteq \bigcup_{k=1}^{n} G_{i_k}$. Thus $K$ is compact in $X$.

- "$\impliedby"$ Let $\{V_i\}_{i \in I}$ be a collection of sets open in $X$ such that $K \subseteq \bigcup_{i \in I} V_i$. Then $G_i = V_i \cap X$ is open in $X \ \forall \ i \in I$. We have $K \subseteq (\bigcup_{i \in I} V_i) \cap X = \bigcup_{i \in I} G_i$. As $K$ is compact in $X$, $\exists \ i_1, \ldots, i_n \in I : K \subseteq \bigcup_{k=1}^{n} G_{i_k} \implies K \subseteq \bigcup_{k=1}^{n} V_{i_k}$. Thus $K$ is compact in $Y$. 
• "\[\iff\]" Let \( \{V_i\}_{i \in I} \) be a collection of sets open in \( Y \) such that \( K \subseteq \bigcup_{i \in I} V_i \). Then \( \exists \{G_i\}_{i \in I} \) open in \( X : V_i = G_i \cap Y \forall i \in I \). Thus \( \{G_i\}_{i \in I} \) is an open cover for \( K \). As \( K \) is compact in \( X \), \( \exists \ i_1, \ldots, i_n \in I : K \subseteq \bigcup_{k=1}^{n} G_{i_k} \iff K \subseteq \bigcup_{k=1}^{n} V_{i_k} \). Thus \( K \) is compact in \( Y \).

\[\square\]

**Proposition.** Let \((X,d)\) be a metric space and let \( F \subseteq K \subseteq X \). If \( F \) is closed and \( K \) is compact, then \( F \) is compact.

**Proof.** Let \( \{G_i\}_{i \in I} \) be an open cover for \( F \). As \( F \subseteq \bigcup_{i \in I} G_i \implies K \subseteq \bigcup_{i \in I} G_i \cup \{F\} \implies F \subseteq \bigcup_{k=1}^{n} G_{i_k} \).

\[\square\]

**Corollary.** Let \((X,d)\) be a metric space, \( F \subseteq X \) be closed and \( K \subseteq X \) be compact. Then \( F \cap K \) is compact.

### Sequential compactness

**Definition.** Let \((X,d)\) be a metric space. A set \( K \subseteq X \) is called **sequentially compact** if every sequence in \( K \) admits a subsequence that converges in \( K \).

### Bolzano-Weierstrass

**Theorem.** Let \((X,d)\) be a metric space. An infinite set \( K \subseteq X \) is sequentially compact \(\iff\) every infinite set \( A \subseteq K \) admits an accumulation point in \( K \).

**Proof.** We prove both ways separately.

- \(\implies\) Let \( A \subseteq K \) be infinite. Then \( \exists \{a_n\}_{n \geq 1} \subseteq A : a_n \neq a_m \forall n \neq m \). As \( K \) is sequentially compact, \( \exists \{a_n\}_{n \geq 1} \subseteq A : a_n \xrightarrow{d} n \to \infty a \in K \). Clearly, \( a \in A' \) as \( \forall r > 0, B_r(a) \cap A \\setminus \{a\} \neq \emptyset \).

- \(\impliedby\) Let \( \{a_n\}_{n \geq 1} \subseteq K \). If \( \{a_n\}_{n \geq 1} \) contains a constant subsequence, then that subsequence converges to a point in \( K \). Otherwise, the set \( A = \{a_n : n \geq 1\} \) is infinite. By hypothesis, \( A' \cap K \neq \emptyset \). Let \( a \in A' \cap K \). Then \( \exists \{a_n\}_{n \geq 1} \subseteq A : a_k \xrightarrow{d} n \to \infty a \).

\[\square\]

**Proposition.** Let \((X,d)\) be a metric space. If \( K \subseteq X \) is compact, then \( K \) is sequentially compact.

**Proof.** If \( K \) is finite, then \( K \) is necessarily sequentially compact. Assume \( K \) is infinite. Let \( A \) be infinite. Then \( A' \subseteq K' \subseteq K \implies A' \cap K = A' \). We want to show \( A' \cap K \neq \emptyset \iff A' \neq \emptyset \). Assume, towards a contradiction, that \( A' = \emptyset \). Then \( \forall x \in K, \exists r_x > 0 : B_{r_x}(x) \cap A \setminus \{x\} = \emptyset \implies B_{r_x}(x) \cap A \subseteq \{x\} \).

Thus \( \{B_{r_x}(x)\}_{x \in K} \) is an open cover for \( K \) compact \(\implies\) \( \exists x_1, \ldots, x_n \in K : K \subseteq \bigcup_{i=1}^{n} B_{r_i}(x_i) \) where \( r_i = r_{x_i} \). As \( A \subseteq K \), we get the following contradiction

\[ A = A \cap \bigcup_{i=1}^{n} B_{r_i}(x_i) = \bigcup_{i=1}^{n} (A \cap B_{r_i}(x_i)) \subseteq \bigcup_{i=1}^{n} \{x_i\}. \]

Thus \( A' = A' \cap K \neq \emptyset \). By Bolzano-Weierstrass, this implies \( K \) is sequentially compact.

\[\square\]

**Proposition.** Let \((X,d)\) be a metric space and let \( K \subseteq X \) be sequentially compact. Then \( K \) is closed and bounded.

**Proof.** We first show \( K \) is closed, then we show \( K \) is bounded.

- We show \( K \) is closed \(\iff\) \( K = \overline{K} \). Fix \( x \in \overline{K} \implies \exists \{x_n\}_{n \geq 1} \subseteq K : x_n \xrightarrow{d} n \to \infty x \). As \( K \) is sequentially compact, \( \exists \{x_k\}_{n \geq 1} \subseteq K : x_k \xrightarrow{d} n \to \infty y \in K \). As \( x_n \xrightarrow{d} n \to \infty x \implies x_k \xrightarrow{d} n \to \infty x \), the limit of the convergent subsequence is unique. Thus \( x = y \in K \) and \( \overline{K} \subseteq K \implies K \) is closed.

- We show \( K \) is bounded. Assume, towards a contradiction, that \( K \) is unbounded. Let \( a_1 \in K \). Then \( K \) unbounded \(\implies\)
1. \( \exists a_2 \in K : d(a_1, a_2) \geq 1 \)
2. \( \exists a_3 \in K : d(a_1, a_3) \geq 1, d(a_2, a_3) \geq 1 \) otherwise \( K \subseteq B_1(a_1) \cup B_1(a_2) \).
3. \( \ldots \)

Proceeding inductively, we construct \( \{a_n\}_{n \geq 1} \subseteq K : d(a_n, a_m) \geq 1 \forall n \neq m \). This sequence doesn’t admit a convergent subsequence, contradicting the fact that \( K \) is sequentially compact.

\[ \square \]

**Total boundedness**

**Definition.** Let \((X, d)\) be a metric space. A set \( A \subseteq X \) is **totally bounded** if \( \forall \epsilon > 0, A \) can be covered by finitely many balls of radius \( \epsilon \).

**Remark.** 1. A totally bounded \( \implies \) \( A \) bounded
2. \( A \subseteq \mathbb{R} \) bounded \( \implies \) \( A \) totally bounded
3. \( \mathbb{N} \) endowed with the discrete metric:

\[
d(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}
\]

Then \( \mathbb{N} \) is bounded, but not totally bounded.

**Theorem.** Let \((X, d)\) be a metric space and let \( K \subseteq X \). The following statements are equivalent.

1. \( K \) is sequentially compact.
2. \( K \) is complete and totally bounded.

**Proof.** We show 1 \( \implies \) 2 and 2 \( \implies \) 1.

- Let’s show \( K \) is complete. Let \( \{x_n\}_{n \geq 1} \subseteq K \) be Cauchy. As \( K \) is sequentially compact, \( \exists \{x_{n_k}\}_{n \geq 1} : x_{n_k} \xrightarrow{d} n \to \infty x \in K \implies x_n \xrightarrow{d} n \to \infty x \in K \). Thus \( K \) is complete. Let’s show \( K \) is totally bounded. Fix \( \epsilon > 0 \).
  - Let \( a_1 \in K \). If \( K \subseteq B_\epsilon(a_1) \), then \( K \) is totally bounded.
  - Otherwise, \( \exists a_2 \in K : d(a_1, a_2) \geq \epsilon \). If \( K \subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2) \), then \( K \) is totally bounded.
  - Otherwise, \( \exists a_3 \in K : d(a_1, a_3) \geq \epsilon, d(a_2, a_3) \geq \epsilon \)
  - \( \ldots \)

If this process terminates in finitely many steps, then \( K \) is totally bounded. Otherwise, we find \( \{a_n\}_{n \geq 1} \subseteq K : d(a_n, a_m) \geq \epsilon \forall n \neq m \). This sequence doesn’t admit a convergent subsequence, contradicting the fact that \( K \) is sequentially compact.

- Let \( \{a_n\}_{n \geq 1} \subseteq K \).
  - \( K \) totally bounded \( \implies \exists J_1 \) finite and \( \{x_{j_1}^{(1)}\}_{j \in J_1} \subseteq K : K \subseteq \cup_{j \in J_1} B_1(x_{j_1}^{(1)}) \). Thus \( \exists j_1 \in J_1 : \{n \in \mathbb{N} : a_n \in B_1(x_{j_1}^{(1)})\} = \emptyset \). Let \( \{a_{n_1}^{(1)}\}_{n \geq 1} \) denote the corresponding subsequence.
  - \( K \) totally bounded \( \implies \exists J_2 \) finite and \( \{x_{j_2}^{(2)}\}_{j \in J_2} \subseteq K : K \subseteq \cup_{j \in J_2} B_2(x_{j_2}^{(2)}) \). Thus \( \exists j_2 \in J_2 : \{n \in \mathbb{N} : a_n \in B_2(x_{j_2}^{(2)})\} = \emptyset \). Let \( \{a_{n_2}^{(2)}\}_{n \geq 1} \) denote the corresponding subsequence.
  - \( \ldots \)

Proceeding inductively, we find finite sets \( J_k, \{x_{j_k}^{(k)}\}_{j \in J_k}, \{a_{n_k}^{(k)}\}_{n \geq 1} : \{a_{n_k}^{(k)}\}_{n \geq 1} \subseteq B_1(x_{j_k}^{(k)}) \). Then \( \{a_{n_k}^{(k+1)}\}_{n \geq 1} \) is a subsequence of \( \{a_{n_k}^{(k)}\}_{n \geq 1} \forall k \geq 1 \). Consider the diagonal subsequence \( \{a_{n_k}^{(k)}\}_{n \geq 1} \). Fix \( k \geq 1 \) and \( n, m \geq k \).

Then \( d(a_{n_k}^{(k)}, a_{n_k}^{(k)}) \leq d(a_{n_k}^{(k)}, x_{j_k}^{(k)}) + d(x_{j_k}^{(k)}, a_{n_k}^{(k)}) \leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \). This shows \( \{a_{n_k}^{(k)}\}_{n \geq 1} \) is Cauchy. As \( K \) is complete, \( a_{n_k}^{(k)} \xrightarrow{d} a \in K \). This proves \( K \) is sequentially compact.

3
Proof. Let $(X,d)$ be a metric space and let $K \subseteq X$ be sequentially compact. Let $\{G_i\}_{i \in I}$ be an open cover of $K$. Then $\exists \epsilon > 0$ : any ball of radius $\epsilon$ contained in $K$ is contained in at least one $G_i$.

Proof. We argue by contradiction. Then $\forall n \geq 1, \exists a_n \in K : B_{\frac{1}{n}}(a_n) \subseteq K$, but $B_{\frac{1}{n}}(a_n) \not\subseteq G_i \forall i \in I$. As $K$ is sequentially compact, $\exists \{a_{k_n}\}_{n \geq 1} : a_{k_n} \xrightarrow[n \to \infty]{} a \in K$. Thus $a \in K \subseteq \cup_{i \in I}G_i \implies \exists i_0 \in I : a \in G_{i_0} = G_{i_0} \implies \exists r > 0 : B_r(a) \subseteq G_{i_0}$. As $a_{k_n} \xrightarrow[n \to \infty]{} a$, $\exists n_r \in \mathbb{N} : d(a,a_{k_n}) < \frac{r}{2} \forall n \geq n_r$. Let $N = \max \{n_r, \lceil \frac{r}{\epsilon} \rceil \} + 1$. Notice $x \in B_{\frac{1}{N}}(a_{k_N}) \implies d(x,a) \leq d(x,a_{k_N}) + d(a_{k_N},a) < \frac{1}{N} + \frac{r}{2} \leq \frac{1}{N} + \frac{r}{2} \leq r$. Thus $B_{\frac{1}{N}}(a_{k_N}) \subseteq B_r(a) \subseteq G_{i_0}$, contradiction.

Proposition. Let $(X,d)$ be a metric space and let $K \subseteq X$ be sequentially compact. Then $K$ is compact.

Proof. Let $\{G_i\}_{i \in I}$ be an open cover of $K$. By the previous proposition, $\exists \epsilon > 0$ : any ball of radius $\epsilon$ contained in $K$ is contained in at least one $G_i$. As $K$ is totally bounded, $\exists x_1, \ldots, x_n \in K : K \subseteq \cup_{i=1}^n B_\epsilon(x_i)$. Then $\forall 1 \leq j \leq n, \exists i_j \in I : B_\epsilon(x_j) \subseteq G_{i_j} \implies K \subseteq \cup_{j=1}^n G_{i_j}$.

Heine-Borel

Collecting everything, we get the Heine-Borel theorem.

Theorem. Let $(X,d)$ be a metric space and let $K \subseteq X$. The following statements are equivalent.

1. $K$ is compact.
2. $K$ is sequentially compact.
3. $K$ is complete and totally bounded.
4. Every infinite subset of $K$ has an accumulation point in $K$.

Corollary. A set $K \subseteq \mathbb{R}$ is compact iff it’s closed and bounded.

Proof.

Exercise.

Compactness and the finite intersection property

Definition. An infinite family of closed sets $\{F_i\}_{i \in I}$ is said to have the finite intersection property if for any finite $J \subseteq I$ we have $\cap_{j \in J} F_j \neq \emptyset$.

Theorem. A metric space $(X,d)$ is compact iff for every infinite boundary of closed sets $\{F_i\}_{i \in I}$ that has the finite intersection property, we have $\cap_{i \in I} F_i \neq \emptyset$.

Proof. We prove both ways separately.

- We argue by contradiction. Assume that $\{F_i\}_{i \in I}$ is an infinite family of closed sets with the finite intersection property, but $\cap_{i \in I} F_i = \emptyset$. Then $X = \cup_{i \in I} ^c F_i$ compact $\implies \exists J \subseteq I$ finite : $X = \cup_{j \in J} ^c F_j \implies \emptyset = J \cap_{j \in J} F_j$, which contradicts the finite intersection property.

- We argue by contradiction. If $X$ isn’t compact, then $\exists \{G_i\}_{i \in I}$ open cover of $X : \{G_i\}_{i \in I}$ doesn’t admit a subcover. In particular, $I$ is infinite. Consider the family $\{^c G_i\}_{i \in I}$ of closed sets. As $X = \cup_{i \in I} G_i \implies \cap_{i \in I} ^c G_i = \emptyset$. Fix $J \subseteq I$ finite. As $\{G_i\}_{i \in I}$ doesn’t admit a finite subcover $X \neq \cup_{j \in J} G_j \implies \cap_{j \in J} ^c G_j \neq \emptyset$. Thus $\{^c G_i\}_{i \in I}$ has the finite intersection property, contradiction.

Corollary. Let $(X,d)$ be a metric space, $K \subseteq X$ be compact, and $\{F_i\}_{i \in I}$ be a family of closed sets. If $K \cap (\cap_{j \in J} F_j) = \emptyset$, then $\exists$ finite $J \subseteq I : K \cap (\cap_{j \in J} F_j) = \emptyset$. 


Proof.

Exercise.

Continuity

Definition. Let \((X, d_X), (Y, d_Y)\) be metric spaces and let \(f : X \to Y\) be a function. We say that \(f\) is continuous at \(x_0 \in X\) if for every \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(d_X(x, x_0) < \delta\) implies \(d_Y(f(x), f(x_0)) < \varepsilon\). We say \(f\) is continuous on \(X\) if \(f\) is continuous at every \(x \in X\).

Remark. A function \(f : X \to Y\) is necessarily continuous at every isolated point in \(X\). Indeed, if \(x_0 \in X\) is isolated, \(\exists \delta > 0 : \{x \in X : d_X(x, x_0) < \delta\} = \{x_0\}\). Then \(d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) = 0 < \varepsilon \forall \varepsilon > 0\).

Theorem. Let \((X, d_X), (Y, d_Y)\) be metric spaces, let \(f : X \to Y\) be a function, and \(x_0 \in X\). The following statements are equivalent.

1. \(f\) is continuous at \(x_0\).
2. For every \(\{x_n\}_{n \geq 1} \subseteq X : x_n \xrightarrow[n \to \infty]{} x_0\) we have \(f(x_n) \xrightarrow[n \to \infty]{} f(x_0)\)

Proof. We show \(1 \implies 2 \text{ and } 2 \implies 1\).

- Let \(x_n \xrightarrow[n \to \infty]{} x_0\) and \(\varepsilon > 0\). As \(f\) is continuous at \(x_0\), \(\exists \delta > 0 : d_X(x_n, x_0) < \delta \implies d_Y(f(x_n), f(x_0)) < \varepsilon\).

- As \(x_n \xrightarrow[n \to \infty]{} x_0\), \(\exists n_0 \in \mathbb{N} : d_X(x_n, x_0) < \delta \forall n \geq n_0 \implies d_Y(f(x_n), f(x_0)) < \varepsilon \forall n \geq n_0\).

- We argue by contradiction, then \(\exists \varepsilon_0 > 0 : \forall n \geq 1, \exists x_n \in X : d_X(x_n, x_0) \leq \frac{1}{n}\) but \(d_Y(f(x_n), f(x_0)) \geq \varepsilon_0\), contradiction.

Proposition. Let \((X, d_X), (Y, d_Y)\) be metric spaces and let \(f : X \to Y\) be a function. The following statements are equivalent.

1. \(f\) is continuous
2. \(G\) open in \(Y\) \(\implies f^{-1}(G)\) open in \(X\)
3. \(F\) closed in \(Y\) \(\implies f^{-1}(F)\) closed in \(X\)
4. \(B \subseteq Y\) \(\implies f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}\)
5. \(A \subseteq X\) \(\implies f(A) \subseteq \overline{f(A)}\)

Proof. We will show ".1 \implies 2 \implies 3 \implies 4 \implies 5 \implies 4 \implies 1".

- "1 \implies 2" Let \(G \subseteq Y\) be open, \(x_0 \in f^{-1}(G)\).

  Then \(f(x_0) \in G\) open \(\implies \exists \varepsilon > 0 : B_Y^\varepsilon(f(x_0)) \subseteq G\).

  As \(f\) is continuous at \(x_0\), \(\exists \delta > 0 : f(B_X^\delta(x_0)) \subseteq B_Y^\varepsilon(f(x_0))\).

  Thus \(f^{-1}(G)\) is open.

- "2 \implies 3" Let \(F \subseteq Y\) be closed \(\implies \complement F\) open in \(Y\) \(\implies \complement(f^{-1}(F)) = f^{-1}(\complement F)\) is open in \(X\) \(\implies f^{-1}(F)\) is closed in \(X\).

- "3 \implies 4" Let \(B \subseteq Y\) \(\implies \overline{B}\) is closed \(\implies f^{-1}(\overline{B})\) closed in \(X\), \(f^{-1}(B) \subseteq f^{-1}(\overline{B})\) \(\implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})\).

- "4 \implies 5" Fix \(A \subseteq X\), apply 4 to \(B = f(A)\), we get \(f^{-1}(\overline{f(A)}) \subseteq \overline{f^{-1}(f(A))}\), \(f^{-1}(f(A)) \supseteq A \implies \complement A \subseteq \overline{f^{-1}(f(A))} \implies \complement A \subseteq f^{-1}(f(A)) \implies f(\complement A) \subseteq f(A)\).
\* "5  \implies  4" Fix \( B \subseteq Y \), apply 5 to \( A = f^{-1}(B) \), we get \( f(f^{-1}(B)) \subseteq f(f^{-1}(B)) = \overline{B} \implies f(f^{-1}(B)) \subseteq f^{-1}(B) \).

\* "4  \implies  1" Fix \( x_0 \in X \), let \( \epsilon > 0 \). Consider \( cB_e^Y(f(x_0)) \) closed in \( Y \). Let \( A = f^{-1}(cB_e^Y(f(x_0))) \). By 4, \( \overline{A} = A = f^{-1}(cB_e^Y(f(x_0))) = f^{-1}\left(\overline{cB_e^Y(f(x_0))}\right) \supseteq f^{-1}(cB_e^Y(f(x_0))) \implies A \text{ is closed. Then } cA = c(f^{-1}(cB_e^Y(f(x_0)))) = f^{-1}(B_e^Y(f(x_0))) \text{ open. We have } x \in f^{-1}(B_e^Y(f(x_0))), \text{ then } \exists \delta > 0 : B_e^Y(x_0) \subseteq f^{-1}(B_e^Y(f(x_0))) \implies f(B_e^Y(x_0)) \subseteq B_e^Y(f(x_0)). \) This shows \( f \) is continuous at \( x_0 \).

\[ \square \]

**Proposition.** Let \( (X,d_X),(Y,d_Y),(Z,d_Z) \) be metric spaces and \( f : X \to Y, g : Y \to Z \) be functions : \( f \) is continuous at \( x_0 \in X \), \( g \) is continuous at \( f(x_0) \in Y \). Then \( g \circ f : X \to Z \) is continuous at \( x_0 \).

**Proof.** Let \( \epsilon > 0 \).

\[
\begin{align*}
g \text{ continuous at } f(x_0) & \implies \exists \delta > 0 : d_Y(y, f(x_0)) < \delta \implies d_Z(g(y), g(f(x_0))) < \epsilon, \\
g \text{ continuous at } x_0 & \implies \exists \eta > 0 : d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta.
\end{align*}
\]

\[
\begin{align*}
d_X(x, x_0) < \eta & \implies d_Z(g(f(x)), g(f(x_0))) < \epsilon.
\end{align*}
\]

\[ \square \]

**Exercise.** Assume \( f, g : X \to \mathbb{R} \) are continuous at \( x_0 \in X \). Then \( f \pm g, fg \) are continuous at \( x_0 \). If in addition, \( g(x_0) \neq 0 \) then \( \frac{f}{g} \) is continuous at \( x_0 \).

#### Continuity and compactness

**Theorem.** Let \( (X,d_X),(Y,d_Y) \) be metric spaces and let \( f : X \to Y \) be continuous. If \( K \subseteq X \) is compact, then \( f(K) \) is compact.

**Proof.** Let \( \{G_i\}_{i \in I} \) be an open cover of \( f(K) \). Then \( f^{-1}(G_i) \) is open in \( X \) \( \forall i \in I \). Moreover, \( f(K) \subseteq \bigcup_{i \in I} G_i \implies K \subseteq f^{-1}\left(\bigcup_{i \in I} G_i\right) = \bigcup_{i \in I} f^{-1}(G_i) \). As \( K \) is compact, \( \exists J \subseteq I \) finite : \( K \subseteq \bigcup_{j \in J} f^{-1}(G_j) = f^{-1}\left(\bigcup_{j \in J} G_j\right) \implies f(K) \subseteq \bigcup_{j \in J} G_j \).

\[ \square \]

**Corollary.**

1. Let \( (X,d_X) \) be a compact metric space and let \( f : X \to \mathbb{R}^n \) be a continuous function. Then \( f(X) \) is closed and bounded.

2. Let \( (X,d_X) \) be a compact metric space and let \( f : X \to \mathbb{R} \) be a continuous function. Then \( \exists x_1, x_2 \in X : f(x_1) = \sup_{x \in X} f(x), f(x_2) = \inf_{x \in X} f(x) \).

**Proof.** \( f(X) \) is closed and bounded. As \( \mathbb{R} \) has the least upper bound property, \( \exists \inf_{x \in X} f(x) \in \mathbb{R}, \sup_{x \in X} f(x) \in \mathbb{R} \).

Clearly, \( \inf_{x \in X} f(x), \sup_{x \in X} f(x) \in f(X) \).

\[ \square \]

**Proposition.** Let \( (X,d_X),(Y,d_Y) \) be metric spaces with \( X \) compact and let \( f : X \to Y \) be a function that is bijective and continuous. Then the inverse \( f^{-1} : Y \to X \) is continuous.

**Proof.** Let \( F \subseteq X \) be closed. We want to show \( f(F) \) is closed in \( Y \). As \( F \) is closed and \( X \) is compact, \( f \) is continuous \( \implies f(F) \) is compact \( \implies f(F) \) is closed.

\[ \square \]

**Definition.** Let \( (X,d_X),(Y,d_Y) \) be metric spaces and let \( f : X \to Y \) be a function. We say that \( f \) is uniformly continuous if \( \forall \epsilon > 0, \exists \delta > 0 : d_X(a,b) < \delta \implies d_Y(f(a), f(b)) < \epsilon \). Compare with \( f : X \to Y \) continuous on \( X \) if \( \forall x_0 \in X, \epsilon > 0, \exists \delta_{\epsilon,x_0} : d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon \).

**Remark.**

1. Uniform continuity is a property of a function on a set. By comparison, continuity is defined pointwise.

2. Uniform continuity \( \implies \) continuity

3. A continuous function need not be uniformly continuous.

**Example.** \( f : \mathbb{R} \to \mathbb{R}, f(x) = x^2, |f(n) - f(n + \frac{1}{m})| = |2 + \frac{1}{m}| \geq 2 \).

**Proposition.** Let \( (X,d_X),(Y,d_Y) \) be metric spaces with \( X \) compact. Let \( f : X \to Y \) be a continuous function. Then \( f \) is uniformly continuous.
Proof. We want to show
\[ \forall \epsilon > 0, \exists \delta > 0 : d_Y(f(x), f(y)) < \epsilon \ \forall \ x, y \in X : d_X(x, y) < \delta. \]
We argue by contradiction. Assume
\[ \exists \epsilon_0 > 0 : \forall \delta > 0, \exists x, y \in X : d_X(x, y) < \delta \text{ but } d_Y(f(x), f(y)) \geq \epsilon_0. \]
Take \( \delta = \frac{1}{n} \) to get
\[ \exists \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \subseteq X : d_X(x_n, y_n) < \frac{1}{n} \]
but \( d_Y(f(x_n), f(y_n)) \geq \epsilon_0. \) As \( X \) is compact, \( \exists \{x_{k_n}\}_{n \in \mathbb{N}} \xrightarrow{n \to \infty} x_0. \) Note
\[ d_X(x_0, x_{k_n}) \leq d_X(x_0, x_{k_n}) + d_X(x_{k_n}, y_{k_n}) < d_X(x_0, x_{k_n}) + \frac{1}{n} \xrightarrow{n \to \infty} 0. \]
Thus \( \{y_{k_n}\}_{n \in \mathbb{N}} \xrightarrow{n \to \infty} x_0. \) As \( f \) is continuous, \( f(x_{k_n}) \xrightarrow{n \to \infty} f(x_0) \) and \( f(y_{k_n}) \xrightarrow{n \to \infty} f(x_0). \) Then we find the contradiction
\[ d_Y(f(x_{k_n}), f(y_{k_n})) \leq d_Y(f(x_{k_n}), f(x_0)) + d_Y(f(x_0), f(y_{k_n})) \xrightarrow{n \to \infty} 0. \]
\[ \square \]

### Continuity and connectedness

**Theorem.** Let \((X, d_X), (Y, d_Y)\) be metric spaces and \( f : X \to Y \) be a continuous function. If \( A \subseteq X \) is connected, then \( f(A) \) is connected.

**Proof.** Assume, towards a contradiction, that \( f(A) \) is not connected. Then \( \exists B_1, B_2 \neq \emptyset, \overline{B_1} \cap B_2 = B_1 \cap \overline{B_1} = \emptyset : f(A) = B_1 \cup B_2. \) Let \( A_1 = f^{-1}(B_1) \cap A, A_2 = f^{-1}(B_2) \cap A. \) Notice
\[ A_1 \cup A_2 = (f^{-1}(B_1) \cap A) \cup (f^{-1}(B_2) \cap A) = (f^{-1}(B_1) \cup f^{-1}(B_2)) \cap A = f^{-1}(B_1 \cup B_2) \cap A = f^{-1}(f(A)) \cap A = A \cap A = A \text{ and} \]
\[ A_1 \cap A_2 = f^{-1}(B_1) \cap A \cap (f^{-1}(B_2) \cap A) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1} \left( \overline{B_1} \cap B_2 \right) = f^{-1}(\emptyset) = \emptyset. \]
Similarly, \( A_1 \cap A_2 = \emptyset. \) So \( A_1, A_2 \) are separated \( \implies \) \( A \) is not connected, contradiction. \( \square \)

**Corollary.** Let \((X, d_X)\) be a connected metric space and \( f : X \to \mathbb{R} \) be continuous. Then \( f(X) \) is an interval. In particular, if \( X = \mathbb{R} \) and \( a, b \in \mathbb{R} : a < b \) and \( y_0 \) lies in between \( f(a) \) and \( f(b) \), then \( \exists x_0 \in (a, b) : f(x_0) = y_0. \) We say that \( f \) has the Darboux (intermediate value) property.

**Remark.** Functions with the Darboux property need not be continuous.

**Example.**
\[
 f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise}. \end{cases} \tag{1}
\]

Let \((X, d_X)\) is a metric space and \( x_0 \in X \), then \( f : x \to d(x, x_0) \) is continuous. Indeed, \( |f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y) \). Take \( \delta = \epsilon. \)

**Proposition.** Let \((X, d_X), (Y, d_Y)\) be connected metric spaces. Then \( X \times Y \) endowed with the following metric is a connected metric space:
\[
 \rho((x_1, y_1), (x_2, y_2)) = \max \{d_X(x_1, x_2), d_Y(y_1, y_2)\}
\]

**Proof.** It suffices to prove that for any point in \( X \times Y, \exists \) connected subset of \( X \times Y \) that contains those points. Let \((x_0, y_0), (a, b) \in X \times Y \) and \( f : Y \to X \times Y \) be a function defined as \( f(y) = (x_0, y). \) This is continuous. Indeed, \( \rho(f(y_1), f(y_2)) = d_Y(y_1, y_2). \) Take \( \delta = \epsilon. \) We get \( f(Y) \) is connected. Let \( g : X \to X \times Y \) be a function defined as \( g(x) = (x, b). \) This is continuous. Indeed, \( \rho(g(x_1), g(x_2)) = d_X(x_1, x_2). \) Take \( \delta = \epsilon. \) We get \( g(X) \) is connected. Note \( f(Y) \cap g(X) \neq \emptyset. \) Indeed, \( (x_0, b) \in f(Y) \cap g(X). \) Then \( f(Y) \cap g(X) \) is connected. As \( \{(x_0, y_0), (a, b)\} \subseteq f(Y) \cup g(X), \) we get the claim. \( \square \)
Remark. Note one may replace the metric $\rho$ in the proposition above by any of the equivalent metrics $\rho = d_X + d_Y$ or $\rho = \sqrt{d_X^2 + d_Y^2}$.

Definition. Let $(X, d_X)$ be a metric space.

- A **path** in $X$ is any continuous function $\gamma : [0, 1] \to X$. $\gamma(0)$ is called the **origin** of the path, $\gamma(1)$ is called the **end** of the path. Note $\gamma([0, 1])$ is compact and connected.

- Let $\gamma : [0, 1] \to X$ be a path in $(X, d)$. We define $\gamma : [0, 1] \to X$ via $\gamma(t) = \gamma^{-1}(t) = \gamma(1 - t)$. This is a path in $X$. For $\gamma_1, \gamma_2$ paths in $X$ with $\gamma_1(1) = \gamma_2(0)$, we define the path $\gamma_1 \vee \gamma_2 : [0, 1] \to X$ via
  $$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Theorem. Let $(X, d_X)$ be a metric space and let $\emptyset \neq A \subseteq X$, then $1 \iff 2 \iff 3$.

1. $\exists a \in A : \forall x \in A, \exists a$ path $\gamma_X : [0, 1] \to A$ with $\gamma_X(0) = a$ and $\gamma_X(1) = x$.

2. $\forall x, y \in A, \exists a$ path $\gamma_{x,y} : [0, 1] \to A$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$.

3. $A$ is connected.

Proof. "1 $\implies$ 2" Let $x, y \in A, \gamma_X, \gamma_Y : [0, 1] \to A$ as given by 1. Then $\gamma_X, \gamma_Y : [0, 1] \to A$ is the desired path.

- "2 $\implies$ 1" Take $a$ to be any point in $A$.

- "1 $\implies$ 3" For $x \in A$, let $A_x = \gamma_X([0, 1])$ connected. Moreover, $\{a\} \in \cap_{x \in A} A_x$. Therefore, $\cup_{x \in A} A_x$ is connected. But $\cup_{x \in A} A_x = A$.

Definition. If either 1 or 2 hold, we say $A$ is **path connected**.

Exercise. Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected, and hence connected.

Proof. We will show that any point in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined to $(\sqrt{2}, \sqrt{2})$ via a path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. Then $x \notin \mathbb{Q}$ or $y \notin \mathbb{Q}$. Say $x \notin \mathbb{Q}$. Then $\gamma_1 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ defined as $\gamma_1(t) = ((1 - t)\sqrt{2} + tx, \sqrt{2})$ is a path, and $\gamma_2 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ defined as $\gamma_2(t) = (x, (1 - t)\sqrt{2} + ty)$ is a path. Then $\gamma_1 \vee \gamma_2 : [0, 1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ is a path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ joining $(\sqrt{2}, \sqrt{2})$ to $(x, y)$.

Remark. Connected sets are not necessarily path connected.

Example. See equation 1.

- Let $G_f = \{(x, f(x)) : x \in [0, \infty)\}$. Then $G_f$ is connected, but not path connected. Let $g : [0, \infty) \to \mathbb{R}^2$ be a function defined as $g(x) = (x, f(x))$. Then $g$ is continuous on $[0, \infty)$ because $f$ is, so $G_f \setminus \{(0, 0)\} = g([0, \infty))$ is connected. Consider $G_f = \{(0, 0)\} \cup g([0, \infty))$. Note $\{(0, 0)\} \subseteq \overline{g([0, \infty))}$. Indeed, for $x_n = \frac{1}{n\pi}$, we get $g(x_n) = (\frac{1}{n\pi}, 0) \to (0, 0)$. Therefore, $\overline{\{(0, 0)\}} = \overline{g([0, \infty))}$ is connected.

- To see that $G_f$ isn’t path connected, it suffices to see that there is no path connecting $(0, 0)$ to $\left(\frac{1}{\pi}, 0\right)$. Indeed, any such path would be discontinuous at $t = 0$, because $\lim_{n \to \infty} \frac{1}{\pi n} = 0 \not\in (0, 1)$.

Proposition. Let $\emptyset \neq A \subseteq X$, then $A$ is connected iff any two points in $A$ can be joined by a polygonal arc lying in $A$.

Proof. " $\iff$ " is immediate since path connectedness $\implies$ connectedness. We show " $\implies$ " . Fix $a \in A$ and let $A_1 = \{x \in A : x$ can be joined to $a$ by a polygonal arc in $A\} \neq \emptyset$ because $a \in A_1$. We will show that $A_1$ is both open and closed in $A$, then $A$ connected $\implies A_1 = A$.

- Let’s show $A_1$ is open in $A$. Pick $x \in A_1 \subseteq A$ open, $\exists r > 0 : B_r(x) \subseteq A$. As any point in $B_r(x)$ can be joined by a segment to $x$ lying in the ball and $x$ is joined by a polygonal arc to $a$, any point in $B_r(x)$ can be joined by a polygonal arc to $a$. Thus $B_r(x) \subseteq A$. This proves $A_1$ is open.

- Let’s show $A_1$ is closed in $A$. If $A_2 = A \setminus A_1 = \emptyset$, then we’re done. So assume $\exists y \in A_2 \subseteq A$ open $\implies \exists r > 0 : B_r(y) \subseteq A$. If $B_r(y) \subseteq A_2$, then we’re done. So assume $B_r(y) \subseteq A_1 \neq \emptyset$. Proof by picture, contradiction.
Convergent sequences of functions

Definition. Let \((X, d_X), (Y, d_Y)\) be metric spaces. For \(n \geq 1\), let \(f_n : X \to Y\) be functions. We say the sequence \(\{f_n\}_{n \geq 1}\) converges pointwise if \(\forall x \in X\), the sequence \(\{f_n(x)\}_{n \geq 1} \subseteq Y\) converges. Thus, we say \(\{f_n\}_{n \geq 1}\) converges pointwise to \(f\) if \(\forall x \in X, \epsilon > 0, \exists n(\epsilon, x) \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon \forall n \geq n(\epsilon, x)\).

Remark. For \(\epsilon > 0\), the function \(n(\epsilon, x) : X \to \mathbb{N}\) can be bounded or unbounded. If it’s bounded, we get the following definition.

Definition. Let \((X, d_X), (Y, d_Y)\) be metric spaces, \(f_n : X \to Y, f : X \to Y\) be functions. We say that \(\{f_n\}_{n \geq 1}\) converges uniformly to \(f\) and write \(f_n \xrightarrow{u} f\) if \(\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon \forall n \geq n_\epsilon, x \in X\).

Remark. • uniform convergence \(\iff\) pointwise convergence
  - For \((X, d_X), (Y, d_Y)\) metric spaces, let \(B(X, Y) = \{f : X \to Y \mid f\text{ is bounded}\}\). We define \(d : B(X, Y) \times B(X, Y) \to \mathbb{R}\) via \(d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))\). Then \((B(X, Y), d)\) is a metric space. Moreover, \(f_n \xrightarrow{u} f\) \iff \(\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon \forall n \geq n_\epsilon, x \in X\).
  - \(\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : \sup_{x \in X} d_Y(f(x), f_n(x)) \leq \epsilon \forall n \geq n_\epsilon, x \in X\).
  - \(\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d(f, f_n) \leq \epsilon \forall n \geq n_\epsilon, x \in X\).
  - \(d(f, f_n) \xrightarrow{n \to \infty} 0\).

• pointwise convergence \(\not\iff\) uniform convergence

Example. for \(n \geq 1\), let \(f_n : [0, 1] \to \mathbb{R}, f_n(x) = x^n\), then

\[
\lim_{n \to \infty} f_n(x) = \begin{cases} 
0 & \text{if } x \in [0, 1) \\
1 & \text{if } x = 1.
\end{cases}
\]

Let \(f : [0, 1] \to \mathbb{R}\) be a function defined as

\[
f(x) = \begin{cases} 
0 & \text{if } x \in [0, 1) \\
1 & \text{if } x = 1.
\end{cases}
\]

We have \(\{f_n\}_{n \geq 1}\) converges pointwise to \(f\). However, \(\{f_n\}_{n \geq 1}\) doesn’t converge uniformly to \(f\). Indeed,

\[d(f_n, f) = \sup_{x \in [0, 1]} |f_n(x) - f(x)| = \sup_{x \in [0, 1]} |x^n| = 1 \not\to 0.\]

Weierstrass

Theorem. Let \((X, d_X), (Y, d_Y)\) be metric spaces. Assume that the sequence of functions \(f_n : X \to Y\) converges uniformly to the function \(f : X \to Y\). If \(f_n\) is continuous at \(x_0 \in X\) for all \(n \geq 1\), then \(f\) is continuous at \(x_0\). In particular, a uniform limit of continuous functions is continuous.

Proof. Fix \(\epsilon > 0\). Then

\[f_n \xrightarrow{u} f \implies \exists n_\epsilon \in \mathbb{N} : d_Y(f_n(x), f(x)) < \frac{\epsilon}{3} \forall n \geq n_\epsilon, x \in X.\]

Fix \(n_0 \geq n_\epsilon\). As \(f_{n_0}\) is continuous at \(x_0\),

\[\exists \delta(\epsilon, x_0) > 0 : d_X(f_{n_0}(x), f_{n_0}(x_0)) < \frac{\epsilon}{3} \forall x \in B_\delta^X(x_0).\]

For \(x \in B_\delta^X(x_0)\), we have

\[d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_{n_0}(x)) + d_Y(f_{n_0}(x), f_{n_0}(x_0)) + d_Y(f_{n_0}(x_0), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.\]

Remark. The converse isn’t true. \(\exists\) a sequence of continuous functions that converges pointwise to a continuous function, but the convergence isn’t uniform.

Example. \(f_n : (0, 1) \to \mathbb{R}, f_n(x) = x^n\), \(\{f_n\}_{n \geq 1}\) converges pointwise to \(f : (0, 1) \to \mathbb{R}, f \equiv 0\). But the convergence is not uniform : \(d(f_n, f) = \sup_{x \in (0, 1)} |x^n| = 1 \not\to 0.\)
Dini

**Theorem.** Let \((X,d_X)\) be a compact metric space, \(f_n : X \to \mathbb{R}\) be continuous functions : \(\{f_n\}_{n \geq 1}\) converges pointwise to a continuous function \(f : X \to \mathbb{R}\). If \(\{f_n\}_{n \geq 1}\) is monotone, then \(\{f_n\}_{n \geq 1}\) converges uniformly to \(f\).

**Proof.** Assume, without loss of generality, \(\{f_n\}_{n \geq 1}\) is increasing, i.e.

\[
 f_n(x) \leq f_{n+1}(x) \quad \forall \ n \geq 1, x \in X.
\]

Then \(\{f - f_n\}\) is decreasing so

\[
 \forall \ x > 0, \lim_{n \to \infty} (f(x) - f_n(x)) = \inf_{n \geq 1} \{f(x) - f_n(x)\} = 0.
\]

Fix \(\epsilon > 0, x_0 \in X\). We have

\[
 \inf_{n \geq 1} \{f(x_0) - f_n(x_0)\} = 0 < \epsilon \implies \exists n(\epsilon, x_0) \in \mathbb{N} : |f(x_0) - f_n(\epsilon, x_0)(x_0)| < \epsilon.
\]

Notice \(f - f_{n(\epsilon, x_0)}\) is continuous at \(x_0\). Thus

\[
 \exists \delta(\epsilon, x_0) > 0 : |(f(x) - f_n(\epsilon, x_0)(x)) - (f(x_0) - f_n(\epsilon, x_0)(x_0))| < \epsilon \quad \forall \ x \in B_\delta(\epsilon, x_0)(x_0).
\]

So for \(x \in B_\delta(\epsilon, x_0)(x_0)\), we have

\[
 f(x) - f_n(\epsilon, x_0)(x) < \epsilon + f(x_0) - f_n(\epsilon, x_0)(x_0) < 2\epsilon.
\]

Note that \(\{B_\delta(\epsilon, x)(x)\}\) form an open cover of \(X\) compact, thus

\[
 \exists x_1, \ldots, x_N \in X : X \subseteq \bigcup_{k=1}^N B_\delta(\epsilon, x_k(x_k))
\]

Let \(n_\epsilon = \max_{1 \leq k \leq N} n(\epsilon, x_k)\) and \(n \geq n_\epsilon\). For \(x \in X\),

\[
 \exists 1 \leq k \leq N : x \in B_\delta(\epsilon, x_k(x_k))
\]

Then \(f(x) - f_n(x) \leq f(x) - f_n(\epsilon, x_0)(x) < 2\epsilon\). By definition, \(f_n \xrightarrow{n \to \infty} f\).

**Remark.** The compactness of \(X\) is essential. Consider \(f_n : (0,1) \to \mathbb{R}, f_n(x) = \frac{1}{n+x^2}\), continuous, \(f_n(x) \geq f_{n+1}(x) \forall x \in (0,1), n \geq 1\). Then \(\{f_n\}_{n \geq 1}\) converges pointwise to \(f : (0,1) \to \mathbb{R}, f \equiv 0\). But the convergence isn’t uniform: \(d(f_n, f) = \sup_{x \in (0,1)} \frac{1}{n+x^2} = 1 \neq 0\).

**Theorem.** Let \((X,d_X)\) be a metric space and \(C(X) = \{f : X \to \mathbb{R} : f \text{ is bounded and continuous}\}\). For \(f, g \in C(X)\), let \(d(f, g) = \sup_{x \in X} |f(x) - g(x)|\). Then \((C(X), d)\) is a metric space.

**Exercise.** \((C(X), d)\) is complete, connected, but not compact because unbounded.

**Definition.** Let \(F \subseteq C(X)\).

- We say \(F\) is **uniformly bounded** if \(\exists M > 0 : |f(x)| \leq M \forall f \in F, x \in X\).
- We say \(F\) is **equicontinuous** if \(\forall \epsilon > 0, \exists \delta > 0 : d(f(x), f(y)) < \epsilon \forall f \in F, x, y \in X : d(x, y) < \delta\).

**Arzela-Ascoli**

**Theorem.** Let \([a, b]\) be a compact interval in \(\mathbb{R}\). Let \(F \subseteq C([a, b])\). The following statements are equivalent.

1. Every sequence in \(F\) admits a (necessarily) uniformly convergent subsequence.
2. \(F\) is uniformly bounded and equicontinuous.

**Proof.** \(1 \implies 2\)
Let’s show $F$ is uniformly bounded, that is, $F$ is bounded with respect to the uniform metric. Indeed, if $F$ were not uniformly bounded, then we would be able to construct a sequence

$$\{f_n\}_{n \geq 1} \subseteq F : d(f_1, f_{n+1}) > 1 + d(f_1, f_n) \forall n \geq 1.$$ 

Then $d(f_n, f_m) \geq |d(f_1, f_n) - d(f_1, f_m)| > |n - m|$. So $\{f_n\}_{n \geq 1}$ cannot have a convergent subsequence.

Let’s show $F$ is totally bounded. Let $\varepsilon > 0, f_1 \in F$.

* If $F \subseteq B_\varepsilon(f_1)$, then $F$ is totally bounded. Otherwise, $\exists f_2 \in F : d(f_2, f_1) \geq \varepsilon$.
* If $F \subseteq B_\varepsilon(f_1) \cup B_\varepsilon(f_2)$, then $F$ is totally bounded. Otherwise, $\exists f_3 \in F : d(f_1, f_3) \geq \varepsilon, d(f_2, f_3) \geq \varepsilon$.
* ... If this process terminates in finitely many steps, then $F$ is totally bounded. Otherwise, we find a sequence

$$\{f_n\}_{n \geq 1} \subseteq F : d(f_n, f_m) \geq \varepsilon \forall n \neq m.$$ 

This sequence doesn’t admit a convergent subsequence.

Let’s show $F$ is equicontinuous. Let $\varepsilon > 0$. As $F$ is totally bounded,

$$\exists f_1, \ldots, f_n \in F : F \subseteq \bigcup_{k=1}^n B_{\varepsilon/10}(f_k).$$

Fix $1 \leq k \leq n$. As $f_k : [a, b] \rightarrow \mathbb{R}$ is continuous, it is uniformly continuous. So

$$\exists \delta_k(\varepsilon) > 0 : |f_k(x) - f_k(y)| < \frac{\varepsilon}{10} \forall x, y \in [a, b], |x - y| < \delta_k.$$ 

Let $\delta(\varepsilon) = \min_{1 \leq k \leq n} \delta_k(\varepsilon)$. Then $\forall x, y \in [a, b]$ with $|x - y| < \delta$ and all $1 \leq k \leq n$, we have

$$|f_k(x) - f_k(y)| < \frac{\varepsilon}{10}.$$ 

Let $f \in F$, then

$$\exists 1 \leq k \leq n : f \in B_{\varepsilon/10}(f_k).$$

For $x, y \in [a, b]$ with $|x - y| < \delta$ we get

$$|f(x) - f(y)| \leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)|$$

$$\leq 2d(f, f_k) + |f_k(x) - f_k(y)| < 2\frac{\varepsilon}{10} + \frac{\varepsilon}{10} < \varepsilon.$$ 

By definition, $F$ is equicontinuous.

2 $\implies$ 1 Assume $F$ is uniformly bounded and equicontinuous. Let $\{f_n\}_{n \geq 1} \subseteq F$. If $q \in [a, b] \cap \mathbb{Q}$, then $\{f_n(q)\}_{n \geq 1}$ is a bounded sequence of real numbers. In particular, $\{f_n(q)\}_{n \geq 1}$ has a convergent subsequence. Passing to a subsequence for every $q \in [a, b] \cap \mathbb{Q}$ (using the fact that $[a, b] \cap \mathbb{Q}$ is countable) and using a diagonal argument, we find a subsequence $\{f_{k_n}(j_q)\}_{n \geq 1}$ that converges at every rational $q \in [a, b]$. Let $\varepsilon > 0$. As $F$ is equicontinuous,

$$\exists \delta > 0 : |f(x) - f(y)| < \frac{\varepsilon}{10} \forall f \in F, |x - y| < \delta.$$ 

As $[a, b]$ is compact,

$$\exists q_1, \ldots, q_N \in [a, b] \cap \mathbb{Q} : [a, b] \subseteq \bigcup_{j=1}^N (q_j - \delta, q_j + \delta).$$

Now $\{f_{k_n}(q_j)\}$ is convergent so

$$\exists n_j(\varepsilon) \in \mathbb{N} : |f_{k_n}(q_j) - f_{k_m}(q_j)| < \frac{\varepsilon}{10} \forall n, m \geq n_j(\varepsilon).$$

Let $x \in [a, b]$, then $\exists 1 \leq j \leq N : |x - q_j| < \delta$. Now

$$|f_{k_n}(x) - f_{k_m}(x)| < |f_{k_n}(x) - f_{k_n}(q_j)| + |f_{k_n}(q_j) - f_{k_m}(q_j)| + |f_{k_m}(q_j) - f_{k_m}(x)| < \varepsilon \forall n, m \geq n(\varepsilon).$$ 

We proved that $\{f_{k_n}\}_{n \geq 1}$ is Cauchy with respect to the uniform metric. Let $f(x) = \lim_{n \to \infty} f_{k_n}(x)$. We have by Weierstrass

$$f_{k_n} \xrightarrow{n \to \infty} f \in C([a, b]).$$
Corollary. Let $F \subseteq C([a,b])$. Then $F$ is compact iff $F$ is closed, uniformly bounded, and equicontinuous.

Remark. 
- The compactness of $[a, b]$ is essential. Let 
  
  \[
  F = \{ f : \mathbb{R} \to \mathbb{R}, |f(x) - f(y)| \leq |x - y|, \sup_{x \in \mathbb{R}} |f(x)| \leq 1 \}.
  \]

  Then $F$ is uniformly bounded and equicontinuous. Consider $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{1+|x|^2}$. Clearly, $\sup_{x \in \mathbb{R}} |f(x)| = \sup_{x \in \mathbb{R}} \frac{1}{1+|x|^2} = 1$. For $x, y \in \mathbb{R}$,
  
  \[
  |f(x) - f(y)| = \frac{1}{1+x^2} - \frac{1}{1+y^2} = \frac{|x - y||x + y|}{(1+x^2)(1+y^2)} \leq |x - y| \left( \frac{|x|}{1 + x^2} + \frac{|y|}{1 + y^2} \right) \leq |x - y| \left( \frac{1}{2} + \frac{1}{2} \right) = |x - y|.
  \]

  For $n \geq 1$, let $f_n(x) = f(x - n) = \frac{1}{1+(x-n)^2}$. For $x \in \mathbb{R}, f_n(x) \xrightarrow{n \to \infty} 0$. So $\{f_n\}_{n \geq 1}$ converges pointwise to the function $g : \mathbb{R} \to \mathbb{R}, g \equiv 0$. However, $\{f_n\}_{n \geq 1}$ doesn’t admit a uniformly convergent subsequence because $\sup_{x \in \mathbb{R}} f_n(x) = 1 \forall n \geq 1$.

- The uniform boundedness of $F$ is essential. Take $F = \{ f : [0,1] \to \mathbb{R} : f$ is constant $\}$. This is equicontinuous but not uniformly bounded. Indeed, $f_n(x) \equiv n$ doesn’t admit a convergent subsequence.

- The equicontinuity of $F$ is essential. Consider 
  
  \[
  F = \{ f : [0,1] \to \mathbb{R}, f$ is continuous, $\sup_{x \in [0,1]} |f(x)| \leq 1 \}.
  \]

  This set is not equicontinuous. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}, f_n(x) = \sin(nx)$. Let $x_n = \frac{\pi}{2n}, y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \xrightarrow{n \to \infty} 0$. But $|f_n(x_n) - f_n(y_n)| = 2$. The sequence $\{f_n\}_{n \geq 1}$ doesn’t admit a uniformly convergent subsequence. Assume, towards a contradiction, that $\exists \{f_n\}_{n \geq 1}$ that converges uniformly. By Weierstrass, the limit function $f : [0,1] \to \mathbb{R}$ is continuous. As $f_n(0) = 0$, we must have $f(0) = 0$. Then $f$ continuous at $x = 0$ yields
  
  \[
  \implies \forall \epsilon > 0, \exists \delta > 0 : |f(x)| < \epsilon \forall 0 \leq x < \delta.
  \]

  Moreover,
  
  \[
  f_{k_n} \xrightarrow{u}{\sup}_{n \to \infty} f, \exists N \in \mathbb{N} : |f_{k_n}| < 2\epsilon \forall n \geq N, 0 \leq x < \delta.
  \]

  But $f_{k_n}(\frac{x}{2\pi k_n}) = 1$. Take $n$ sufficiently large: $\frac{x}{2\pi k_n} < \delta$ to get a contradiction.

The oscillation of a function

Definition. Let $(X,d)$ be a metric space, $\emptyset \neq A \subseteq X, f : X \to \mathbb{R}$. The oscillation of a function on $A$ is $\omega(f,A) = \sup_{x \in A} f(x) - \inf_{x \in A} f(x) = \sup_{x,y \in A} (f(x) - f(y)) \geq 0$. The oscillation of $f$ at $x_0 \in X$ is $\omega(f,x_0) = \inf_{r>0} \omega(f,B_r(x_0))$.

Proposition. Let $(X,d)$ be a metric space, $f : X \to \mathbb{R}$. Then $f$ is continuous at $x_0 \in X$ iff $\omega(f,x_0) = 0$.

Proof. Let $\epsilon > 0$.

\[ \implies \quad \text{As } f \text{ is continuous at } x_0, \exists \delta > 0 : |f(x) - f(y)| < \frac{\epsilon}{2} \forall x \in B_\delta(x_0). \text{ Then for } x \in B_\delta(x_0), f(x) - \frac{\epsilon}{2} < f(x) < f(x_0) + \frac{\epsilon}{2} \implies f(x_0) - \frac{\epsilon}{2} \leq \inf_{x \in B_\delta(x_0)} f(x) \leq \sup_{x \in B_\delta(x_0)} f(x) \leq f(x) + \frac{\epsilon}{2} \implies \omega(f,B_\delta(x_0)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \implies \omega(f,x_0) \leq \epsilon. \]

\[ \iff \quad \text{We have } \omega(f,x_0) = \inf_{\delta>0} \omega(f,B_\delta(x_0)) = 0 < \epsilon \implies \exists \delta > 0 : \omega(f,B_\delta(x_0)) < \epsilon \implies \sup_{x,y \in B_\delta(x_0)} (f(x) - f(y)) < \epsilon \implies \sup_{x \in B_\delta(x_0)} |f(x) - f(x_0)| < \epsilon \implies |f(x) - f(x_0)| < \epsilon \forall x \in B_\delta(x_0). \]

This shows $f$ is continuous at $x_0$.

\[ \Box \]
Proposition. Let \((X, d)\) be a metric space, \(f : X \to \mathbb{R}\) be a function, and \(\alpha > 0\). Then \(A = \{x \in X : \omega(f, x) < \alpha\}\) is open.

Proof. Let \(x_0 \in A \implies \omega(f, x_0) = \inf_{\delta > 0} \omega(f, B_\delta(x_0)) < \alpha \implies \exists \delta > 0 : \omega(f, B_\delta(x_0)) < \alpha.

Claim. \(B_\delta(x_0) \subseteq A\)

Let \(y \in B_\delta(x_0)\). Then \(B_{\delta - d(x_0, y)}(y) \subseteq B_\delta(x_0)\) and \(\omega(f, y) \leq \omega(f, B_{\delta - d(x_0, y)}(y)) \leq \omega(f, B_\delta(x_0)) < \alpha\). So \(y \in A\).

Remark. Let \((X, d)\) is a metric space, \(f : X \to \mathbb{R}\). Then \(\{x \in X : f\text{ is continuous at }x\} = \{x \in X : \omega(f, x) = 0\} = \bigcap_{n \geq 1} \{x \in X : \omega(f, x) < \frac{1}{n}\} = \bigcap_{n \geq 1} \{G_n\} \text{ open.}\) Note \(G_{n+1} \subseteq G_n \forall n \geq 1\).

Exercise. Show that there are no functions \(f : \mathbb{R} \to \mathbb{R}\) such that \(f\) is continuous at every rational point and discontinuous at every irrational point.

Proof. By contradiction. Assume \(f : \mathbb{R} \to \mathbb{R}\) is continuous on \(\mathbb{R}\) and discontinuous on \(\mathbb{R} \setminus \mathbb{Q}\). Then \(Q = \bigcap_{n \geq 1} G_n\) with \(G_n = \bar{G}_n\). As \(Q\) is dense in \(\mathbb{R}\), we get \(\bar{G}_n = \mathbb{R} \forall n \geq 1 \implies \bigcap_{n \geq 1} \bar{G}_n = \mathbb{R}\). Let \(\{q_n\}_{n \geq 1}\) denote an enumeration of \(Q\). For \(n \geq 1\), let \(H_n = \mathbb{R} \setminus \{q_n\} = (-\infty, q_n) \cup (q_n, \infty)\) open. Moreover, \(\overline{H_n} = \mathbb{R}\). As \(\mathbb{R}\) is complete, if satisfied the Baire property, that is, if \(\{A_n\}_{n \geq 1}\) is a sequence of open dens sets, then \(\bigcap_{n \geq 1} A_n = \mathbb{R}\). Then we must have that \((\bigcap_{n \geq 1} G_n) \cup (\bigcap_{n \geq 1} H_n) = \emptyset\), contradiction.

Lemma. Let \((X, d)\) be a metric space with the Baire property and \(\emptyset \neq W = \overline{W} \subseteq X\). Then \(W\) has the Baire property.

Proof. Let \(\{D_n\}_{n \geq 1}\) be a sequence of open, dense sets in \(W\). As \(D_n\) is open in \(W\), \(\exists G_n\) open in \(X\) : \(D_n = G_n \cap W\) open in \(X\). Also \(\overline{D_n} \cap W = W \implies W \subseteq \overline{D_n} \implies W \subseteq \overline{D_n} \forall n \geq 1\).

For \(n \geq 1\), let \(B_n = D_n \cup \omega(W)\) open in \(X\). Then \(\overline{B_n} = D_n \cup \omega(W) \supseteq W \cup \omega(W) = X \implies B_n\) is dense in \(X\). As \(X\) has the Baire property,

\[
X = \bigcap_{n \geq 1} B_n = \bigcap_{n \geq 1} (D_n \cup \omega(W)) = (\bigcap_{n \geq 1} D_n) \cup \omega(W) = \bigcap_{n \geq 1} D_n \cup \omega(W) = \bigcap_{n \geq 1} D_n \cup \omega(W) \cap W = \left(\bigcap_{n \geq 1} D_n \cap W\right) \cup (\omega(W) \cap W).
\]

But \(W = \overline{W} \subseteq \overline{W} \implies \omega(W) \cap W = \emptyset\). So \(\bigcap_{n \geq 1} D_n\) is dense in \(W\).

Theorem. Let \((X, d)\) be a metric space with the Baire property. If \(f_n : X \to \mathbb{R}\) are continuous functions converging pointwise to \(f : X \to \mathbb{R}\), then the set of points at which \(f\) is continuous is dense in \(X\).

Claim. It suffices to prove the theorem under the additional hypothesis that \(|f_n(x)| \leq 1 \forall n \geq 1, x \in X\).

Proof. Indeed, assume that the theorem holds for this restricted set of functions and let \(\{f_n\}_{n \geq 1}\) be as in the theorem. Consider \(\phi : \mathbb{R} \to (-1, 1), \phi(x) = \frac{x}{1 + |x|}\) bijective continuous with inverse \(\phi^{-1} : (-1, 1) \to \mathbb{R}, \phi(y) = \frac{y}{1 - |y|}\) continuous. Then \(\phi \circ f_n : X \to (-1, 1)\) is continuous, \(\{\phi \circ f_n\}_{n \geq 1}\) is uniformly bounded by 1, and \(\phi \circ f_n \xrightarrow{n \to \infty} \phi \circ f\) pointwise. Then \(\{x \in X : \phi \circ f \text{ is continuous at }x\}\) is dense in \(X\). As \(\phi \circ f\) is continuous at \(x \iff f\) is continuous at \(x\), we get the claim.
We have $X = \cup_{n\geq 1} F_n$. Since
\[
\forall x \in X, \exists n(x) \in \mathbb{N} : x \in F_{n(x)}, F_n = \{ x \in X : \sup_{k \geq n} f_k(x) - \inf_{l \geq n} f_l(x) \leq \frac{1}{4N} \} = \{ x \in X : \sup_{k,l \geq n} (f_k(x) - f_l(x)) \leq \frac{1}{4N} \} = \cap_{k,l \geq n} \{ x \in X : f_k(x) - f_l(x) \leq \frac{1}{4N} \}.
\]

But
\[
\{ x \in X : (f_k - f_l)(x) \leq \frac{1}{4N} \} = (f_k - f_l)^{-1}([-2, \frac{1}{4N}])
\]
is the continuous preimage of a closed set, hence it’s closed. Thus $F_n$ is closed so
\[
X = \cup_{n\geq 1} F_n \implies W = \cup_{n\geq 1} (F_n \cap W),
\]
which is the union of closed sets in $W$. By the previous lemma, $W$ has the Baire property, and $\hat{W} \neq \emptyset$ so $\exists n_1 \geq 1 : F_{n_1} \cap W \neq \emptyset$. Let
\[
x_0 \in F_{n_1} \cap W \implies \exists \delta > 0 : B_\delta(x_0) \subseteq F_{n_1} \cap \hat{W}.
\]
Since $f_{n_1}$ is continuous at $x_0$, shrinking $\delta$ if necessary, we may assume $\omega(f_{n_1}, B_\delta(x_0)) < \frac{1}{2N}$. We will show $x_0 \in G_N$. In particular, $x_0 \in G_N \cap W \neq \emptyset$. Then
\[
\omega(f, B_\delta(x_0)) = \sup_{x \in \overline{B}_\delta(x_0)} |f(x) - f(y)| \leq \sup_{x \in \overline{B}_\delta(x_0)} |v_n(x) - u_n(y)|
\]
\[
= \sup_{x \in \overline{B}_\delta(x_0)} |v_n(x) - u_n(x) + u_n(x) - v_n(y) + v_n(y) - u_n(y)|
\]
\[
\leq \sup_{x \in \overline{B}_\delta(x_0)} |v_n(x) - u_n(x)| + \sup_{x \in \overline{B}_\delta(x_0)} |u_n(x) - v_n(y)| + \sup_{x \in \overline{B}_\delta(x_0)} |v_n(y) - u_n(y)|
\]
\[
\leq \frac{1}{4N} + \sup_{x \in \overline{B}_\delta(x_0)} |f_n(x) - f_n(y)| + \frac{1}{4N} < \frac{1}{2N} \implies x_0 \in G_N.
\]

\[\Box\]

Weierstrass approximation

**Theorem.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists polynomials $P_n : [a, b] \to \mathbb{R}$ of degree at most $n$ such that $P_n \xrightarrow{n \to \infty} f$ on $[a, b]$.

**Proof.** We may assume $[a, b] = [0, 1]$. Indeed, the function $\phi : [0, 1] \to [a, b], \phi(t) = (1-t)a + tb$ is bijective and continuous. Then $f \circ \phi : [0, 1] \to \mathbb{R}$ is continuous. If we find polynomials $P_n$ of degree at most $n : P_n \xrightarrow{n \to \infty} f \circ \phi$ on $[0, 1]$, then $P_n \circ \phi^{-1} \xrightarrow{n \to \infty} f$ on $[a, b]$. From now on, $[a, b] = [0, 1]$. For $n \geq 0$, let $P_n(x) = \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k}$ be Bernstein polynomials. We will show $P_n \xrightarrow{n \to \infty} f$ on $[0, 1]$. Let $\epsilon > 0$. As $f : [0, 1] \to \mathbb{R}$ is continuous, $f$ is uniformly continuous. Thus $\exists \delta > 0 : |f(x) - f(y)| < \epsilon \forall x, y \in [0, 1], |x - y| < \delta$. Fix $x \in [0, 1]$. Then
\[
|P_n(x) - f(x)| = \left| \sum_{k=0}^n \binom{n}{k} \frac{f_k}{n} x^k (1-x)^{n-k} - f(x) \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \right|
\]
\[
\leq \sum_{k=0}^n |f_k - f(x)| \binom{n}{k} x^k (1-x)^{n-k}
\]
\[
\leq \sum_{0 \leq k \leq n, |x - \frac{k}{n}| < \delta} |f_k - f(x)| \binom{n}{k} x^k (1-x)^{n-k} + \sum_{0 \leq k \leq n, |nx - k| \geq \delta} |f_k - f(x)| \binom{n}{k} x^k (1-x)^{n-k}
\]
\[
\leq \epsilon \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} + 2 \sup_{y \in [0,1]} |f(y)| \sum_{k=0}^n \binom{n}{k} \frac{(nx - k)^2}{n^2 \delta^2} \binom{n}{k} x^k (1-x)^{n-k}
\]
\[
\leq \epsilon + 2 \sup_{y \in [0,1]} |f(y)| \frac{1}{n^2 \delta^2} \sum_{k=0}^n (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}.
\]
Now compute
\[\sum_{k=0}^{n} (nx - k)^2 \binom{n}{k} x^k (1-x)^{n-k}\]
\[= n^2 x^2 \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} - 2n x \sum_{k=0}^{n} k \binom{n}{k} x^k (1-x)^{n-k} + \sum_{k=0}^{n} k^2 \binom{n}{k} x^k (1-x)^{n-k}\]
\[= n^2 x^2 - 2n^2 x^2 \sum_{k=1}^{n-1} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k} + \sum_{k=1}^{n} k(n-1) \frac{n!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k}\]
\[= -n^2 x^2 + n(n-1)x^2 \sum_{k=1}^{n-1} \frac{(n-2)!}{(k-2)!(n-k)!} x^{k-2} (1-x)^{n-k} + n x \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} x^{k-1} (1-x)^{n-k}\]
\[= -n^2 x^2 + n^2 x^2 - n x^2 + nx = nx(1-x) \leq \frac{n}{4}.
\]
Thus \(|P_n(x) - f(x)| \leq \epsilon + 2 \sup_{y \in [0,1]} |f(y)| \frac{1}{\pi^2} \frac{\pi^2}{2} < 2\epsilon\) for \(n\) large depending only on \(\delta\) and \(\sup_{y \in [0,1]} |f(y)|\). This proves uniform convergence. \(\square\)

**Exercise.** Let \(a > 0\). Show that there are polynomials \(P_n : [-a, a] \to \mathbb{R}\) of degree \(\leq n\) such that \(P_n(0) = 0\) and \(P_n \xrightarrow{n \to \infty} |x|\) on \([-a, a]\).

**Proof.** Note \(f : [-a, a] \to \mathbb{R}, f(x) = |x|\) is continuous, thus by the Weierstrass approximation theorem, \(\exists Q_n : [-a, a] \to \mathbb{R}\) polynomial of degree \(\leq n\) such that \(Q_n \xrightarrow{n \to \infty} |x|\) on \([-a, a]\). Note \(Q_n(0) \xrightarrow{n \to \infty} 0\). Let \(P_n(x) = Q_n(x) - Q_n(0)\) polynomial of degree \(\leq n\) and \(P_n(0) = 0\).

**Claim.** \(P_n \xrightarrow{n \to \infty} |x|\) on \([-a, a]\).

Notice \(|P_n(x) - |x|| \leq |Q_n(x) - |x|| + |Q_n(0)|\). Given \(\epsilon > 0\),
\[\exists n_1(\epsilon) \in \mathbb{N} : \sup_{x \in [-a, a]} |Q_n(x) - |x|| < \frac{\epsilon}{2} \quad \forall \ n \geq n_1(\epsilon)\]
\[\exists n_2(\epsilon) \in \mathbb{N} : |Q_n(0)| < \frac{\epsilon}{2} \quad \forall \ n \geq n_2(\epsilon)\]
For \(n \geq n(\epsilon) = \max\{n_1(\epsilon), n_2(\epsilon)\}\), we have \(|P_n(x) - |x|| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall \ x \in [-a, a]\). Thus \(P_n \xrightarrow{n \to \infty} |x|\) on \([-a, a]\). \(\square\)

**Definition.** Let \((X, d)\) be a metric space. A set of real-valued functions \(A \subseteq \{f : X \to \mathbb{R}\}\) is called an **algebra** if
1. If \(f, g \in A\), then \(f + g \in A\).
2. If \(f, g \in A\), then \(fg \in A\).
3. If \(f \in A\) and \(\lambda \in \mathbb{R}\), then \(\lambda f \in A\).

**Stone-Weierstrass**

**Theorem.** Let \((X, d)\) be a compact metric space and \(A \subseteq C(X)\) be an algebra. Assume that \(A\) satisfies the following two properties:
1. \(A\) separates points in \(X\), that is, if \(x, y \in X\) with \(x \neq y\), then \(\exists f \in A : f(x) \neq f(y)\).
2. \(A\) vanishes at no point in \(X\), that is, if \(x \in X\), then \(\exists f \in A : f(x) \neq 0\).

Then \(A\) is dense in \(C(X)\).

**Example.** \(A = \{P : X \to \mathbb{R}\text{ polynomials}\}\) is an algebra, it separates points, and vanishes at no point.

**Definition.** Let \(f : \mathbb{R} \to \mathbb{R}\) be a function. For \(a \in \mathbb{R}\), we write \(\lim_{x \to a} f(x) = L \in \mathbb{R}\) if for any sequence \(\{x_n\}_{n \geq 1} \subseteq \mathbb{R} \setminus \{a\} : x_n \xrightarrow{n \to \infty} a\), we have \(f(x_n) \xrightarrow{n \to \infty} L\). Equivalently, if \(\forall \epsilon > 0, \exists \delta > 0 : x \in (a - \delta, a + \delta) \setminus \{a\} \implies |f(x) - L| < \epsilon\).
Exercise. Extend this definition to cover $L = \pm \infty, a = \pm \infty$.

Remark. $f$ is continuous at $a \in \mathbb{R}$ iff $\lim_{x \to a} f(x) = f(a)$. Similarly, one defines the left-limit $f(a^-) = \lim_{x \to a^-} f(x)$ and the right-limit $f(a^+) = \lim_{x \to a^+} f(x)$.

**Differentiation**

**Definition.** Let $I$ be an open interval and $f : I \to \mathbb{R}$ a function. We say $f$ is differentiable at $a \in I$ if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. In this case we write $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ and we call it the derivative of $f$ at $a$.

**Example.** Fix $n \geq 1, f : \mathbb{R} \to \mathbb{R}, f(x) = x^n$. For $a \in \mathbb{R}, x \neq a, \frac{f(x) - f(a)}{x - a} = x^n - a^n = x^{n-1} + x^{n-2}a + \ldots + a^{n-1} \xrightarrow{n \to \infty} na^{n-1}$. So $f'(a) = na^{n-1}$.

**Lemma.** Let $I$ be an open interval, $f : I \to \mathbb{R}$ be a differentiable at $a \in I$. Then $f$ is continuous at $a$.

**Proof.** For $x \in I \setminus \{a\}$ we write $f(x) = \frac{f(x) - f(a)}{x - a}(x-a) + f(a)$. Then $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$ and $\lim_{x \to a} (x-a) = 0$. Thus $\lim_{x \to a} f(x) = f(a)$.

**Theorem.** Let $I$ be an open interval and $f, g : I \to \mathbb{R}$ be differentiable at $a \in I$. Then $g$ is differentiable at $g(a)$ for

1. for any $\lambda \in \mathbb{R}, \lambda f$ is differentiable at $a$ and $(\lambda f)' = \lambda f'$.
2. $f + g$ is differentiable at $a$ and $(f + g)'(a) = f'(a) + g'(a)$.
3. $fg$ is differentiable at $a$ and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.
4. if $g(a) \neq 0$ then $\frac{f}{g}$ is differentiable at $a$ then $(\frac{f}{g})'(a) = \frac{f(a)g(a) - f(a)g'(a)}{g(a)^2}$.

**Proof.** 1. For $x \neq a$, $(\frac{\lambda f(x) - \lambda f(a)}{x - a}) = \frac{\lambda(f(x) - f(a))}{x - a} \xrightarrow{x \to a} \lambda f'(a).

2. For $x \neq a$, $(\frac{f(x) + g(x) - f(a) - g(a)}{x - a}) = \frac{(f(x) - f(a))}{x - a} + \frac{(g(x) - g(a))}{x - a} \xrightarrow{x \to a} f'(a) + g'(a).

3. For $x \neq a$, $(\frac{f(a)g(x) - f(x)g(a)}{x - a}) = \frac{(f(x) - f(a))g(x)}{x - a} \xrightarrow{x \to a} f'(a)g(a) + f(a)g'(a).

4. For $x \neq a$, $(\frac{f(x) - f(a)}{x - a}g(x) - f(a)\frac{g(x)}{x - a}) = \frac{f(x) - f(a)}{g(x)} + \frac{f(a)}{g(x)}\frac{g(x) - g(a)}{x - a} \xrightarrow{x \to a} f'(a)\frac{1}{g(a)} + f(a)\frac{1}{g(a)}g'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.

**Theorem.** Let $I$ and $J$ be open intervals and assume $f : I \to \mathbb{R}$ is differentiable at $a \in I$ and $g : J \to \mathbb{R}$ is differentiable at $f(a) \in J$. Then $g \circ f$ is well-defined on a neighbourhood of $a$, is differentiable at $a$ and $(g \circ f)'(a) = g'(f(a))f'(a)$.

**Proof.** As $f(a) \in J$ open, $\exists \epsilon > 0 : (f(a) - \epsilon, f(a) + \epsilon) \subseteq J$. Since $f$ is continuous at $a, \exists \delta > 0 : \forall x \in I$ then $|f(x) - f(a)| < \epsilon$. As $I$ is open, choosing $\delta$ even smaller (if necessary), we may ensure $(a - \delta, a + \delta) \subseteq I$. So $g \circ f$ is well-defined on $(a - \delta, a + \delta)$. Let

$$h(y) = \begin{cases} \frac{g(y) - g(f(a))}{y - f(a)} & y \in J \setminus \{f(a)\} \\ \frac{g'(f(a))}{y - f(a)} & y = f(a) \end{cases}$$

As $g$ is differentiable at $f(a)$, we have $\lim_{y \to f(a)} h(y) = \lim_{y \to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a)) = h(f(a))$. So $h$ is continuous at $f(a)$. So for $y \in J \setminus \{f(a)\}$, we write $g(y) - g(f(a)) = h(y)(y - f(a))$. For $x \in (a - \epsilon, a + \epsilon) \setminus \{a\}$ we have $g(f(x)) - g(f(a)) = h(f(x))(f(x) - f(a)) \iff \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} = h(f(x)) \xrightarrow{x \to a} g'(f(a))f'(a) \iff (g \circ f)'(a) = g'(f(a))f'(a)$.
Proof. Assume \( f \) attains its maximum at \( x_0 \). Otherwise, replace \( f \) by \( -f \).

- For \( x_n \in (a, x_0) \) with \( x_n \xrightarrow{n \to \infty} x_0 \) we have \( f'(x_0) = \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0 \).
- For \( x_n \in (x_0, b) \) with \( x_n \xrightarrow{n \to \infty} x_0 \) we have \( f'(x_0) = \lim_{n \to \infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \leq 0 \).

Combining the two, we get \( f'(x_0) = 0 \). \( \square \)

### Rolle

**Theorem.** Assume \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If \( f(a) = f(b) \), then \( \exists \ x_0 \in (a, b) : f'(x_0) = 0 \).

Proof. As \( f \) is continuous on the compact interval \([a, b]\), it attains its maximum and minimum on \([a, b]\). Thus \( \exists \ y_0, z_0 \in [a, b] : f(y_0) \leq f(x) \leq f(z_0) \ \forall \ x \in [a, b]. \)

1. Suppose \( \{y_0, z_0\} = \{a, b\}. \) As \( f(a) = f(b) \) we get that \( f \) is constant on \([a, b]\). Then \( \forall \ x \in (a, b) \) we have

\[
\frac{f'(x)}{f(y)-f(x)} = 0.
\]

2. Suppose either \( y_0 \notin \{a, b\} \) or \( z_0 \notin \{a, b\} \). If \( y_0 \in \{a, b\} \), then by the previous theorem, \( f'(y_0) = 0 \). Likewise for \( z_0 \).

\( \square \)

### Mean value theorem

**Theorem.** Assume \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then \( \exists \ x_0 \in (a, b) : f'(x_0) = \frac{f(b)-f(a)}{b-a} \).

**Remark.** If \( f(a) = f(b) \), we recover Rolle’s theorem.

Proof. Let \( l : [a, b] \to \mathbb{R} \) given by \( l(x) = f(a) + \frac{l(b)-f(a)}{b-a} (x-a) \). Then \( l \) is continuous on \([a, b]\), differentiable on \((a, b)\) with \( l'(x) = \frac{l(b)-f(a)}{b-a} \ \forall \ x \in (a, b) \) and \( l(a) = f(a), l(b) = f(b) \). Let \( g : [a, b] \to \mathbb{R}, g(x) = f(x) - l(x) \) continuous on \([a, b]\), differentiable on \((a, b)\) and \( g(a) = g(b) = 0 \). By Rolle’s theorem, \( \exists \ x_0 \in (a, b) : g'(x_0) = 0 = f'(x_0) - \frac{l(b)-f(a)}{b-a} \iff f'(x_0) = \frac{l(b)-f(a)}{b-a}. \)

**Corollary.** If \( f : (a, b) \to \mathbb{R} \) is differentiable with \( f'(x_0) = 0 \ \forall \ x \in (a, b) \), then \( f \) is constant.

Proof. Assume, towards a contradiction, that \( \exists \ a < x_1 < x_2 < b : f(x_1) \neq f(x_2). \) As \( f \) is differentiable on \((a, b)\), it’s continuous on \([a, b]\). Thus \( f \) is continuous on \([x_1, x_2]\) and differentiable on \((x_1, x_2)\). By the mean value theorem, \( \exists \ x_0 \in (x_1, x_2) : 0 = f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}. \) But \( f'(x_0) = 0 \ \forall \ x \in (a, b) \) by hypothesis, so \( f(x_2) - f(x_1) = 0 \iff f(x) = f(x_2) \), contradiction.

**Corollary.** If \( f, g : (a, b) \to \mathbb{R} \) are differentiable with \( f'(x) = g'(x) \ \forall \ x \in (a, b) \), then \( \exists \ c \in \mathbb{R} : f(x) = g(x) + c \ \forall \ x \in (a, b) \).

**Corollary.** Let \( f : (a, b) \to \mathbb{R} \) be differentiable.

1. If \( f'(x_0) \geq 0 \ \forall \ x \in (a, b) \), then \( f \) is increasing.
2. If \( f'(x_0) > 0 \ \forall \ x \in (a, b) \), then \( f \) is strictly increasing.
3. If \( f'(x_0) \leq 0 \ \forall \ x \in (a, b) \), then \( f \) is decreasing.
4. If \( f'(x_0) < 0 \ \forall \ x \in (a, b) \), then \( f \) is strictly decreasing.

Proof. 1. Let \( a < x_1 < x_2 < b \). Then \( f \) is continuous on \([x_1, x_2]\) because it’s continuous on \((a, b)\) and differentiable on \((x_1, x_2)\). By the mean value theorem, \( \exists \ x_0 \in (x_1, x_2) : 0 \leq f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}. \) Thus \( f(x_2) - f(x_1) \geq 0 \).

**Exercise.** Prove the remaining.
Intermediate value properties for derivatives

**Theorem.** Let \( f : (a, b) \to \mathbb{R} \) be differentiable. If \( a < x_1 < x_2 < b \) and \( \lambda \) lies between \( f'(x_1) \) and \( f'(x_2) \), then \( \exists \, x_0 \in (x_1, x_2) : f'(x_0) = \lambda \).

**Proof.** Assume WLOG that \( f'(x_1) < \lambda < f'(x_2) \). Let \( g : (a, b) \to \mathbb{R}, g(x) = f(x) - \lambda x \) be differentiable on \((a, b)\). Then \( g \) is continuous on \((a, b)\). We want to find \( x_0 \in (x_1, x_2) : g'(x_0) = 0 \). As \( g \) is continuous on \([x_1, x_2]\) compact, it attains its maximum at a point \( x_0 \in [x_1, x_2] \). If we can show that \( x_0 \notin \{x_1, x_2\} \), then \( x_0 \in (x_1, x_2) \) and \( g'(x_0) = 0 \). Let’s show \( x_0 \neq x_1 \). We have \( \lim_{x \to x_1} \frac{g(x) - g(x_1)}{x - x_1} = g'(x_1) = f'(x_1) - \lambda < 0 \). Thus \( \exists \, \delta > 0 : 0 < |x - x_1| < \delta \), then \( \frac{g(x) - g(x_1)}{x - x_1} < 0 \). For \( x_1 - \delta < x < x_1 \), we get \( g(x_1) < g(x) \) so \( g(x_1) \) is not a maximum and \( x_0 \neq x_1 \). Similarly, \( x_0 \neq x_2 \). \( \square \)

**Theorem.** Let \( I \) be an open interval and \( f : I \to \mathbb{R} \) be continuous and injective. Then \( J = f(I) \) is an interval. If \( f \) is differentiable at \( x_0 \in I \) and \( f'(x_0) \neq 0 \), then the inverse \( f^{-1} : J \to I \) is differentiable at \( y_0 = f(x_0) \) and \( (f^{-1})'(y_0) = \frac{1}{f'(x_0)} \).

**Proof.** As \( f \) is injective and continuous, it’s strictly monotone. Therefore \( f^{-1} : J \to I \) is strictly monotone. As \( f^{-1}(J) = I \) is an interval, we have \( f^{-1} \) is continuous. Assume WLOG that \( f \) is increasing.

**Claim.** \( J \) is open.

Assume, towards a contradiction, that \( J \) is not open. Suppose \( \inf J \in J \). Then, as \( J = f(I), \exists \, a \in I : f(a) = \inf J \). As \( I \) is open, \( \exists \, \varepsilon > 0 : (a - \varepsilon, a + \varepsilon) \subseteq I \). As \( f \) is strictly increasing, we get \( \inf J = f(a) > f\left(\frac{a}{2}\right) \in J \), contradiction.

**Exercise.** Consider \( \sup J \in J \).

This shows \( J \) is open. We know \( \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \neq 0 \implies \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} \). Let \( \epsilon > 0 \). Then \( \exists \, \delta > 0 : 0 < |x - x_0| < \delta \implies |\frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)}| < \varepsilon, f^{-1} \) is continuous at \( y_0 \in J \). Then \( \exists \, \eta > 0 : 0 < |y - y_0| < \eta \implies 0 < |f^{-1}(y) - f^{-1}(y_0)| < \delta \). Putting these together, we get \( 0 < |y - y_0| < \eta \implies |\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} - \frac{1}{f'(x_0)}| < \varepsilon \). Thus \( \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{1}{f'(x_0)} = (f^{-1})'(y_0) \). \( \square \)