Mathematics 131BH Lecture Part I

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Compactness in metric spaces

Definition. Let (X, d) be a metric space.

- Let $A \subseteq X$. An **open cover** of A is any collection $\{G_i\}_{i \in I}$ of open sets such that $A \subseteq \cup_{i \in I} G_i$. The open cover is called **finite** if I is finite. Otherwise, the open cover is called **infinite**.
- A set $K \subseteq X$ is called **compact** if every open cover of K admits a finite subcover, that is, if $\{G_i\}_{i \in I}$ is an open cover for K, then $\exists\ n \geq 1$ and $i_1, \dots, i_n \in I : K \subseteq \cup_{j=1}^n G_{i_j}$.

Proposition. *Let (X, d) be a metric space and let $K \subseteq X$ be compact. Then K is closed and bounded.*

Proof. We first show K is closed, then we show K is bounded.

- To show K is closed, it suffices to prove ${}^cK$ is open. If ${}^cK = \emptyset$, then K is closed. Thus, we may assume ${}^cK \neq \emptyset$. Let $x \in {}^cK$. For $y \in K$, let
$$r_y = \frac{1}{2} d(x, y).$$
Consider the open set
$$B_{r_y}(y) = \{z \in X : d(z, y) < r_y\}.$$
Then $K \subseteq \cup_{y \in K} B_{r_y}(y)$. As K is compact, the open cover $\{B_{r_y}(y)\}_{y \in K}$ admits a finite subcover. Thus
$$\exists\ n \geq 1 \text{ and } y_1, \dots, y_n \in K : K \subseteq \cup_{i=1}^n B_{r_i}(y_i),$$
where we used the shorthand $r_i = r_{y_i}$. Let
$$r = \min\{r_i\}\ \forall\ 1 \leq i \leq n.$$
Then
$$B_r(x) \cap B_{r_i}(y_i) = \emptyset\ \forall\ 1 \leq i \leq n.$$
Otherwise we find the contradiction
$$z \in B_r(x) \cap B_{r_i}(y_i) \implies d(x, y_i) \leq d(x, z) + d(z, y_i) < r + r_i \leq 2r_i = d(x, y_i).$$
Thus
$$B_r(x) \subseteq \cap_{i=1}^n\ {}^cB_{r_i}(y_i) =\ {}^c\left(\cup_{i=1}^n B_{r_i}(y_i)\right) \subseteq\ {}^cK.$$
By definition, ${}^cK$ is open and so K is closed.
- We show K is bounded. Clearly, $\{B_1(y)\}_{y \in K}$ is an open cover of K. As K is compact, $\exists\ n \geq 1$ and $y_1, \dots, y_n \in K : K \subseteq \cup_{i=1}^n B_1(y_i)$. Let $r = \max_{1 \leq k \leq n} d(y_1, y_k) + 1$. Then $K \subseteq B_r(y_1)$.

□

Theorem. *Let (X, d) be a metric space and let $K \subseteq Y \subseteq X$. Then K is compact in $Y \iff K$ is compact in X.*

Proof. We prove both ways separately.

- " $\implies$ " Let $\{G_i\}_{i \in I}$ be a collection of sets open in X such that $K \subseteq \cup_{i \in I} G_i$. Then $V_i = G_i \cap Y$ is open in $Y\ \forall\ i \in I$. We have $K \subseteq (\cup_{i \in I} G_i) \cap Y = \cup_{i \in I} V_i$. As K is compact in $Y, \exists\ i_1, \dots, i_n \in I : K \subseteq \cup_{k=1}^n V_{i_k} \implies K \subseteq \cup_{k=1}^n G_{i_k}$. Thus K is compact in X.

- " $\Longleftarrow$ " Let $\{V_i\}_{i\in I}$ be a collection of sets open in Y such that $K \subseteq \cup_{i\in I} V_i$. Then $\exists\ \{G_i\}_{i\in I}$ open in $X : V_i = G_i \cap Y\ \forall\ i \in I$. Thus $\{G_i\}_{i\in I}$ is an open cover for K. As K is compact in $X, \exists\ i_1, \ldots, i_n \in I : K \subseteq \cup_{k=1}^n G_{i_k} \implies K \subseteq \cup_{k=1}^n V_{i_k}$. Thus K is compact in Y.

□

Proposition. *Let (X, d) be a metric space and let $F \subseteq K \subseteq X$. If F is closed and K is compact, then F is compact.*

Proof. Let $\{G_i\}_{i\in I}$ be an open cover for F. As $F \subseteq \cup_{i\in I} G_i \implies K \subseteq \cup_{i\in I} G_i \cup\ {}^cF$. As K is compact, $\exists\ i_1, \ldots, i_n \in I : K \subseteq \cup_{k=1}^n G_{i_k} \cup\ {}^cF \implies F \subseteq \cup_{k=1}^n G_{i_k}$. □

Corollary. *Let (X, d) be a metric space, $F \subseteq X$ be closed and $K \subseteq X$ be compact. Then $F \cap K$ is compact.*

Sequential compactness

Definition. Let (X, d) be a metric space. A set $K \subseteq X$ is called **sequentially compact** if every sequence in K admits a subsequence that converges in K.

Bolzano-Weierstrass

Theorem. *Let (X, d) be a metric space. An infinite set $K \subseteq X$ is sequentially compact $\iff$ every infinite set $A \subseteq K$ admits an accumulation point in K.*

Proof. We prove both ways separately.

- " $\implies$ " Let $A \subseteq K$ be infinite. Then $\exists\ \{a_n\}_{n\geq 1} \subseteq A : a_n \neq a_m\ \forall\ n \neq m$. As K is sequentially compact, $\exists\ \{a_n\}_{n\geq 1} \subseteq A : a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$. Clearly, $a \in A'$ as $\forall\ r > 0, B_r(a) \cap A \setminus \{a\} \neq \emptyset$.
- " $\Longleftarrow$ " Let $\{a_n\}_{n\geq 1} \subseteq K$. If $\{a_n\}_{n\geq 1}$ contains a constant subsequence, then that subsequence converges to a point in K. Otherwise, the set $A = \{a_n : n \geq 1\}$ is infinite. By hypothesis, $A' \cap K \neq \emptyset$. Let $a \in A' \cap K$. Then $\exists\ \{a_n\}_{n\geq 1} \subseteq A : a_{k_n} \xrightarrow[n\to\infty]{d} a$.

□

Proposition. *Let (X, d) be a metric space. If $K \subseteq X$ is compact, then K is sequentially compact.*

Proof. If K is finite, then K is necessarily sequentially compact. Assume K is infinite. Let A be infinite. Then $A' \subseteq K' \subseteq K \implies A' \cap K = A'$. We want to show $A' \cap K \neq \emptyset \iff A' \neq \emptyset$. Assume, towards a contradiction, that $A' = \emptyset$. Then

$$\forall\ x \in K, \exists\ r_x > 0 : B_{r_x}(x) \cap A \setminus \{x\} = \emptyset \implies B_{r_x}(x) \cap A \subseteq \{x\}.$$

Thus $\{B_{r_x}(x)\}_{x\in K}$ is an open cover for K compact $\implies \exists\ x_1, \ldots, x_n \in K : K \subseteq \cup_{i=1}^n B_{r_i}(x_i)$ where $r_i = r_{x_i}$. As $A \subseteq K$, we get the following contradiction

$$A = A \cap \cup_{i=1}^n B_{r_i}(x_i) = \cup_{i=1}^n (A \cap B_{r_i}(x_i)) \subseteq \cup_{i=1}^n \{x_i\}.$$

Thus $A' = A' \cap K \neq \emptyset$. By Bolzano-Weierstrass, this implies K is sequentially compact. □

Proposition. *Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is closed and bounded.*

Proof. We first show K is closed, then we show K is bounded.

- We show K is closed $\iff K = \overline{K}$. Fix $x \in \overline{K} \implies \exists\ \{x_n\}_{n\geq 1} \subseteq K : x_n \xrightarrow[n\to\infty]{d} x$. As K is sequentially compact, $\exists\ \{x_{k_n}\}_{n\geq 1} \subseteq K : x_{k_n} \xrightarrow[n\to\infty]{d} y \in K$. As $x_n \xrightarrow[n\to\infty]{d} x \implies x_{k_n} \xrightarrow[n\to\infty]{d} x$, the limit of the convergent subsequence is unique. Thus $x = y \in K$ and $\overline{K} \subseteq K \implies K$ is closed.
- We show K is bounded. Assume, towards a contradiction, that K is unbounded. Let $a_1 \in K$. Then K unbounded $\implies$

- $\exists\ a_2 \in K : d(a_1, a_2) \geq 1$
- $\exists\ a_3 \in K : d(a_1, a_3) \geq 1, d(a_2, a_3) \geq 1$ otherwise $K \subseteq B_1(a_1) \cup B_1(a_2)$.
- ...

Proceeding inductively, we construct $\{a_n\}_{n\geq 1} \subseteq K : d(a_n, a_m) \geq 1\ \forall\ n \neq m$. This sequence doesn't admit a convergent subsequence, contradicting the fact that K is sequentially compact.

□

Total boundedness

Definition. Let (X, d) be a metric space. A set $A \subseteq X$ is **totally bounded** if $\forall\ \epsilon > 0, A$ can be covered by finitely many balls of radius ϵ.

Remark.

1. A totally bounded $\implies$ A bounded
2. $A \subseteq \mathbb{R}$ bounded $\implies$ A totally bounded
3. $\mathbb{N}$ endowed with the discrete metric :
$$d(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$
Then $\mathbb{N}$ is bounded, but not totally bounded.

Theorem. *Let (X, d) be a metric space and let $K \subseteq X$. The following statements are equivalent.*

1. *K is sequentially compact.*
2. *K is complete and totally bounded.*

Proof. We show 1 $\implies$ 2 and 2 $\implies$ 1.

- Let's show K is complete. Let $\{x_n\}_{n\geq 1} \subseteq K$ be Cauchy. As K is sequentially compact, $\exists\ \{x_{k_n}\}_{n\geq 1}$: $x_{k_n} \xrightarrow[n\to\infty]{d} x \in K \implies x_n \xrightarrow[n\to\infty]{d} x \in K$. Thus K is complete. Let's show K is totally bounded. Fix $\epsilon > 0$.
  - Let $a_1 \in K$. If $K \subseteq B_\epsilon(a_1)$, then K is totally bounded.
  - Otherwise, $\exists\ a_2 \in K : d(a_1, a_2) \geq \epsilon$. If $K \subseteq B_\epsilon(a_1) \cup B_\epsilon(a_2)$, then K is totally bounded.
  - Otherwise, $\exists\ a_3 \in K : d(a_1, a_3) \geq \epsilon, d(a_2, a_3) \geq \epsilon$
  - ...

  If this process terminates in finitely many steps, then K is totally bounded. Otherwise, we find $\{a_n\}_{n\geq 1} \subseteq K : d(a_n, a_m) \geq \epsilon\ \forall\ n \neq m$. This sequence doesn't admit a convergent subsequence, contradicting the fact that K is sequentially compact.

- Let $\{a_n\}_{n\geq 1} \subseteq K$.
  - K totally bounded $\implies\ \exists\ J_1$ finite and $\{x_j^{(1)}\}_{j\in J_1} \subseteq K : K \subseteq \cup_{j\in J_1} B_1(x_j^{(1)})$. Thus $\exists\ j_1 \in J_1 : |\{n \in \mathbb{N} : a_n \in B_1(x_{j_1}^{(1)})\}| = \aleph_0$. Let $\{a_n^{(1)}\}_{n\geq 1}$ denote the corresponding subsequence.
  - K totally bounded $\implies\ \exists\ J_2$ finite and $\{x_j^{(2)}\}_{j\in J_2} \subseteq K : K \subseteq \cup_{j\in J_2} B_{\frac{1}{2}}(x_j^{(2)})$. Thus $\exists\ j_2 \in J_2 : |\{n \in \mathbb{N} : a_n \in B_{\frac{1}{2}}(x_{j_2}^{(2)})\}| = \aleph_0$. Let $\{a_n^{(2)}\}_{n\geq 1}$ denote the corresponding subsequence.
  - ...

  Proceeding inductively, we find finite sets $J_k, \{x_j^{(k)}\}_{j\in J_k}, \{a_n^{(k)}\}_{n\geq 1} : \{a_n^{(k)}\}_{n\geq 1} \subseteq B_{\frac{1}{k}}(x_{j_k}^{(k)})$. Then $\{a_n^{(k+1)}\}_{n\geq 1}$ is a subsequence of $\{a_n^{(k)}\}_{n\geq 1}\ \forall\ k \geq 1$. Consider the diagonal subsequence $\{a_n^{(n)}\}_{n\geq 1}$. Fix $k \geq 1$ and $n, m \geq k$. Then $d(a_n^n, a_m^m) \leq d(a_n^n, x_{j_k}^k) + d(x_{j_k}^k, a_m^m) \leq \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$. This shows $\{a_n^{(n)}\}_{n\geq 1}$ is Cauchy. As K is complete, $a_n^{(n)} \xrightarrow[n\to\infty]{d} a \in K$. This proves K is sequentially compact.

□

Proposition. *Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Let $\{G_i\}_{i\in I}$ be an open cover of K. Then $\exists\ \epsilon > 0 :$ any ball of radius ϵ contained in K is contained in at least one G_i.*

Proof. We argue by contradiction. Then $\forall\ n \geq 1, \exists\ a_n \in K : B_{\frac{1}{n}}(a_n) \subseteq K$, but $B_{\frac{1}{n}}(a_n) \not\subseteq G_i\ \forall\ i \in I$. As K is sequentially compact, $\exists\ \{a_{k_n}\}_{n\geq 1} : a_{k_n} \xrightarrow[n\to\infty]{d} a \in K$. Thus $a \in K \subseteq \cup_{i\in I} G_i \implies \exists\ i_0 \in I : a \in G_{i_0} = \mathring{G}_{i_0} \implies \exists\ r > 0 : B_r(a) \subseteq G_{i_0}$. As $a_{k_n} \xrightarrow[n\to\infty]{d} a, \exists\ n_r \in \mathbb{N} : d(a, a_{k_n}) < \frac{r}{2}\ \forall\ n \geq n_r$. Let $N = \max\{n_r, \lfloor \frac{2}{r} \rfloor\} + 1$. Notice $x \in B_{\frac{1}{k_N}}(a_{k_N}) \implies d(x, a) \leq d(x, a_{k_N}) + d(a_{k_N}, a) < \frac{1}{k_N} + \frac{r}{2} \leq \frac{1}{N} + \frac{r}{2} \leq r$. Thus $B_{\frac{1}{k_N}}(a_{k_N}) \subseteq B_r(a) \subseteq G_{i_0}$, contradiction. □

Proposition. *Let (X, d) be a metric space and let $K \subseteq X$ be sequentially compact. Then K is compact.*

Proof. Let $\{G_i\}_{i\in I}$ be an open cover of K. By the previous proposition, $\exists\ \epsilon > 0 :$ any ball of radius ϵ contained in K is contained in at least one G_i. As K is totally bounded, $\exists\ x_1, \ldots, x_n \in K : K \subseteq \cup_{j=1}^n B_\epsilon(x_j)$. Then $\forall\ 1 \leq j \leq n, \exists\ i_j \in I : B_\epsilon(x_j) \subseteq G_{i_j} \implies K \subseteq \cup_{j=1}^n G_{i_j}$. □

Heine-Borel

Collecting everything, we get the Heine-Borel theorem.

Theorem. *Let (X, d) be a metric space and let $K \subseteq X$. The following statements are equivalent.*

1. *K is compact.*
2. *K is sequentially compact.*
3. *K is complete and totally bounded.*
4. *Every infinite subset of K has an accumulation point in K.*

Corollary. *A set $K \subseteq \mathbb{R}$ is compact iff it's closed and bounded.*

Proof.

Exercise.

□

Compactness and the finite intersection property

Definition. An infinite family of closed sets $\{F_i\}_{i\in I}$ is said to have the **finite intersection property** if for any finite $J \subseteq I$ we have $\cap_{j\in J} F_j \neq \emptyset$.

Theorem. *A metric space (X, d) is compact iff for every infinite boundary of closed sets $\{F_i\}_{i\in I}$ that has the finite intersection property, we have $\cap_{i\in I} F_i \neq \emptyset$.*

Proof. We prove both ways separately.

- We argue by contradiction. Assume that $\{F_i\}_{i\in I}$ is an infinite family of closed sets with the finite intersection property, but $\cap_{i\in I} F_i = \emptyset$. Then $X = \cup_{i\in I}\, {}^cF_i$ compact $\implies \exists\ J \subseteq I$ finite $: X = \cup_{j\in J}\, {}^cF_j \implies \emptyset = \cap_{j\in J} F_j$, which contradicts the finite intersection property.
- We argue by contradiction. If X isn't compact, then $\exists\ \{G_i\}_{i\in I}$ open cover of $X : \{G_i\}_{i\in I}$ doesn't admit a subcover. In particular, I is infinite. Consider the family $\{{}^cG_i\}_{i\in I}$ of closed sets. As $X = \cup_{i\in I} G_i \implies \cap_{i\in I}\, {}^cG_i = \emptyset$. Fix $J \subseteq I$ finite. As $\{G_i\}_{i\in I}$ doesn't admit a finite subcover $X \neq \cup_{j\in J} G_j \implies \cap_{j\in J}\, {}^cG_j \neq \emptyset$. Thus $\{{}^cG_i\}_{i\in I}$ has the finite intersection property, contradiction.

□

Corollary. *Let (X, d) be a metric space, $K \subseteq X$ be compact, and $\{F_i\}_{i\in I}$ be a family of closed sets. If $K \cap (\cap_{i\in I} F_i) = \emptyset$, then $\exists$ finite $J \subseteq I : K \cap (\cap_{j\in J} F_j) = \emptyset$.*

Proof.

Exercise.

□

Continuity

Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a function. We say that f is **continuous** at $x_0 \in X$ if $\forall\, \epsilon > 0, \exists\, \delta > 0 : d_Y(f(x), f(x_0)) < \epsilon$ whenever $d_X(x, x_0) < \delta$. We say f is **continuous on** X if f is continuous at every $x \in X$.

Remark. A function $f : X \to Y$ is necessarily continuous at every isolated point in X. Indeed, if $x_0 \in X$ is isolated, $\exists\, \delta > 0 : \{x \in X : d_X(x, x_0) < \delta\} = \{x_0\}$. Then $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) = 0 < \epsilon\ \forall\, \epsilon > 0$.

Theorem. *Let $(X, d_X), (Y, d_Y)$ be metric spaces, let $f : X \to Y$ be a function, and $x_0 \in X$. The following statements are equivalent.*

1. *f is continuous at x_0.*
2. *for every $\{x_n\}_{n\geq 1} \subseteq X : x_n \xrightarrow[n\to\infty]{d_X} x_0$ we have $f(x_n) \xrightarrow[n\to\infty]{d_Y} f(x_0)$*

Proof. We show $1 \implies 2$ and $2 \implies 1$.

- Let $x_n \xrightarrow[n\to\infty]{d_X} x_0$ and $\epsilon > 0$. As f is continuous at $x_0, \exists\, \delta > 0 : d_X(x_n, x_0) < \delta \implies d_Y(f(x_n), f(x_0)) < \epsilon$. As $x_n \xrightarrow[n\to\infty]{d_X} x_0, \exists\, n_\delta \in \mathbb{N} : d_X(x_n, x_0) < \delta\ \forall\, n \geq n_\delta \implies d_Y(f(x_n), f(x_0)) < \epsilon\ \forall\, n \geq n_\delta$.
- We argue by contradiction, then $\exists\, \epsilon_0 > 0 : \forall\, n \geq 1, \exists\, x_n \in X : d_X(x_n, x_0) < \frac{1}{n}$ but $d_Y(f(x_n), f(x_0)) \geq \epsilon_0$, contradiction.

□

Proposition. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a function. The following statements are equivalent.*

1. *f is continuous*
2. *G open in $Y \implies f^{-1}(G)$ open in X*
3. *F closed in $Y \implies f^{-1}(F)$ closed in X*
4. *$B \subseteq Y \implies f^{-1}(\overline{B}) \supseteq \overline{f^{-1}(B)}$*
5. *$A \subseteq X \implies f(\overline{A}) \subseteq \overline{f(A)}$*

Proof. We will show "$1 \implies 2 \implies 3 \implies 4 \implies 5 \implies 4 \implies 1$".

- "$1 \implies 2$" Let $G \subseteq Y$ be open, $x_0 \in f^{-1}(G)$.

  $$\left.\begin{array}{l}\text{Then } f(x_0) \in G \text{ open} \implies \exists\, \epsilon > 0 : B^Y_\epsilon(f(x_0)) \subseteq G.\\ \text{As } f \text{ is continuous at } x_0, \exists\, \delta > 0 : f(B^X_\delta(x_0)) \subseteq B^Y_\epsilon(f(x_0)).\end{array}\right\} f(B^X_\delta(x_0)) \subseteq G \iff B^X_\delta(x_0) \subseteq f^{-1}(G).$$

  Thus $f^{-1}(G)$ is open.
- "$2 \implies 3$" Let $F \subseteq Y$ be closed $\implies\ {}^cF$ open in $Y \overset{2}{\implies}\ {}^c(f^{-1}(F)) = f^{-1}({}^cF)$ is open in $X \implies f^{-1}(F)$ is closed in X.
- "$3 \implies 4$" Let $B \subseteq Y \implies \overline{B}$ is closed $\overset{3}{\implies} f^{-1}(\overline{B})$ closed in X, $f^{-1}(B) \subseteq f^{-1}(\overline{B}) \implies \overline{f^{-1}(B)} \subseteq \overline{f^{-1}(\overline{B})} = f^{-1}(\overline{B})$.
- "$4 \implies 5$" Fix $A \subseteq X$, apply 4 to $B = f(A)$, we get $\overline{f^{-1}(f(A))} \subseteq f^{-1}(\overline{f(A)})$, $f^{-1}(f(A)) \supseteq A \implies \overline{A} \subseteq \overline{f^{-1}(f(A))} \implies \overline{A} \subseteq f^{-1}(\overline{f(A)}) \implies f(\overline{A}) \subseteq \overline{f(A)}$.

- "5 $\implies$ 4" Fix $B \subseteq Y$, apply 5 to $A = f^{-1}(B)$, we get $f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} = \overline{B} \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$.
- "4 $\implies$ 1" Fix $x_0 \in X$, let $\epsilon > 0$. Consider ${}^cB_\epsilon^Y(f(x_0))$ closed in Y. Let $A = f^{-1}\left({}^cB_\epsilon^Y(f(x_0))\right)$. By 4, $\overline{A} = A = f^{-1}\left({}^cB_\epsilon^Y(f(x_0))\right) = f^{-1}\left(\overline{{}^cB_\epsilon^Y(f(x_0))}\right) \supseteq \overline{f^{-1}\left({}^cB_\epsilon^Y(f(x_0))\right)} \implies A$ is closed. Then ${}^cA = {}^cf^{-1}\left({}^cB_\epsilon^Y(f(x_0))\right) = f^{-1}\left(B_\epsilon^Y(f(x_0))\right)$ open. We have $x \in f^{-1}\left(B_\epsilon^Y(f(x_0))\right)$, then $\exists\, \delta > 0 : B_\epsilon^Y(x_0) \subseteq f^{-1}\left(B_\epsilon^Y(f(x_0))\right) \implies f(B_\epsilon^Y(x_0)) \subseteq B_\epsilon^Y(f(x_0))$. This shows f is continuous at x_0.

□

Proposition. *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and $f : X \to Y, g : Y \to Z$ be functions : f is continuous at $x_0 \in X, g$ is continuous at $f(x_0) \in Y$. Then $g \circ f : X \to Z$ is continuous at x_0.*

Proof. Let $\epsilon > 0$.

$$\left.\begin{array}{l} g \text{ continuous at } f(x_0) \implies \exists\, \delta > 0 : d_Y(y, f(x_0)) < \delta \implies d_Z\left(g(y), g(f(x_0))\right) < \epsilon. \\ f \text{ continuous at } x_0 \implies \exists\, \eta > 0 : d_X(x, x_0) < \eta \implies d_Y(f(x), f(x_0)) < \delta. \end{array}\right\} d_X(x, x_0) < \eta$$

$\implies d_Z\left(g(f(x)), g(f(x_0))\right) < \epsilon$. □

Exercise. Assume $f, g : X \to \mathbb{R}$ are continuous at $x_0 \in X$. Then $f \pm g, fg$ are continuous at x_0. If in addition, $g(x_0) \neq 0$ then $\frac{f}{g}$ is continuous at x_0.

Continuity and compactness

Theorem. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be continuous. If $K \subseteq X$ is compact, then $f(K)$ is compact.*

Proof. Let $\{G_i\}_{i\in I}$ be an open cover of $f(K)$. Then $f^{-1}(G_i)$ is open in $X\ \forall\, i \in I$. Moreover, $f(K) \subseteq \cup_{i\in I} G_i \implies K \subseteq f^{-1}(\cup_{i\in I} G_i) = \cup_{i\in I} f^{-1}(G_i)$. As K is compact, $\exists\ J \subseteq I$ finite : $K \subseteq \cup_{j\in J} f^{-1}(G_j) = f^{-1}(\cup_{j\in J} G_j) \implies f(K) \subseteq \cup_{j\in J} G_j$. □

Corollary. *1. Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}^n$ be a continuous function. Then $f(X)$ is closed and bounded.*

2. Let (X, d_X) be a compact metric space and let $f : X \to \mathbb{R}$ be a continuous function. Then $\exists\ x_1, x_2 \in X$: $f(x_1) = \sup_{x\in X} f(x), f(x_2) = \inf_{x\in X} f(x)$.

Proof. $f(X)$ is closed and bounded. As $\mathbb{R}$ has the least upper bound property, $\exists \inf_{x\in X} f(x) \in \mathbb{R}, \sup_{x\in X} f(x) \in \mathbb{R}$. Clearly, $\inf_{x\in X} f(x), \sup_{x\in X} f(x) \in \overline{f(x)} = f(x)$. □

Proposition. *Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact and let $f : X \to Y$ be a function that is bijective and continuous. Then the inverse $f^{-1} : Y \to X$ is continuous.*

Proof. Let $F \subseteq X$ be closed. We want to show $f(F)$ is closed in Y. As F is closed and X is compact, F is compact, f is continuous $\implies f(F)$ is compact $\implies f(F)$ is closed. □

Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a function. We say that f is **uniformly continuous** if $\forall\ \epsilon > 0, \exists\ \delta_\epsilon > 0 : d_X(a, b) < \delta \implies d_Y(f(a), f(b)) < \epsilon$. Compare with $f : X \to Y$ continuous on X if $\forall\ x_0 \in X, \epsilon > 0, \exists\ \delta_{\epsilon, x_0} : d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \epsilon$.

Remark. 1. Uniform continuity is a property of a function on a set. By comparison, continuity is defined pointwise.

2. uniform continuity $\implies$ continuity

3. A continuous function need not be uniformly continuous.

   Example. $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2, |f(n) - f(n + \frac{1}{n})| = |2 + \frac{1}{n^2}| \geq 2$.

Proposition. *Let $(X, d_X), (Y, d_Y)$ be metric spaces with X compact. Let $f : X \to Y$ be a continuous function. Then f is uniformly continuous.*

Proof. We want to show

$$\forall\ \epsilon > 0, \exists\ \delta > 0 : d_Y(f(x), f(y)) < \epsilon\ \forall\ x, y \in X : d_X(x, y) < \delta.$$

We argue by contradiction. Assume

$$\exists\ \epsilon_0 > 0 : \forall\ \delta > 0, \exists\ x_\delta, y_\delta \in X : d_X(x_\delta, y_\delta) < \delta \text{ but } d_Y(f(x_\delta), f(y_\delta)) \geq \epsilon_0.$$

Take $\delta = \frac{1}{n}$ to get

$$\exists\ \{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \subseteq X : d_X(x_n, y_n) < \frac{1}{n}$$

but $d_Y(f(x_n), f(y_n)) \geq \epsilon_0$. As X is compact, $\exists\ \{x_{k_n}\}_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{d_X} x_0$. Note

$$d_X(x_0, y_{k_n}) \leq d_X(x_0, x_{k_n}) + d_X(x_{k_n}, y_{k_n}) < d_X(x_0, x_{k_n}) + \frac{1}{n} \xrightarrow[n\to\infty]{d_X} 0.$$

Thus $\{y_{k_n}\}_{n\in\mathbb{N}} \xrightarrow[n\to\infty]{d_X} x_0$. As f is continuous, $f(x_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0)$ and $f(y_{k_n}) \xrightarrow[n\to\infty]{d_Y} f(x_0)$. Then we find the contradiction

$$d_Y(f(x_{k_n}), f(y_{k_n})) \leq d_Y(f(x_{k_n}), f(x_0)) + d_Y(f(x_0), f(y_{k_n})) \xrightarrow[n\to\infty]{d_Y} 0.$$

□

Continuity and connectedness

Theorem. *Let $(X, d_X), (Y, d_Y)$ be metric spaces and $f : X \to Y$ be a continuous function. If $A \subseteq X$ is connected, then $f(A)$ is connected.*

Proof. Assume, towards a contradiction, that $f(A)$ is not connected. Then $\exists\ B_1, B_2 \neq \emptyset, \overline{B_1} \cap B_2 = B_1 \cap \overline{B_1} = \emptyset : f(A) = B_1 \cup B_2$. Let $A_1 = f^{-1}(B_1) \cap A, A_2 = f^{-1}(B_2) \cap A$. Notice

$$A_1 \cup A_2 = (f^{-1}(B_1) \cap A) \cup (f^{-1}(B_2) \cap A) = (f^{-1}(B_1) \cup f^{-1}(B_2)) \cap A = f^{-1}(B_1 \cup B_2) \cap A = f^{-1}(f(A)) \cap A = A \cap A = A$$

and

$$\overline{A_1} \cap A_2 = \overline{f^{-1}(B_1) \cap A} \cap (f^{-1}(B_2) \cap A) \subseteq \overline{f^{-1}(B_1)} \cap f^{-1}(B_2) \subseteq f^{-1}(\overline{B_1}) \cap f^{-1}(B_2) = f^{-1}(\overline{B_1} \cap B_2) = f^{-1}(\emptyset) = \emptyset.$$

Similarly, $A_1 \cap \overline{A_2} = \emptyset$. So A_1, A_2 are separated $\implies$ A is not connected, contradiction. □

Corollary. *Let (X, d_X) be a connected metric space and $f : X \to \mathbb{R}$ be continuous. Then $f(X)$ is an interval. In particular, if $X = \mathbb{R}$ and $a, b \in \mathbb{R} : a < b$ and y_0 lies in between $f(a)$ and $f(b)$, then $\exists\ x_0 \in (a, b) : f(x_0) = y_0$. We say that f has the Darboux (intermediate value) property.*

Remark. Functions with the Darboux property need not be continuous.

Example.

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let (X, d_X) is a metric space and $x_0 \in X$, then $f : x \to d(x, x_0)$ is continuous. Indeed, $|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y)$. Take $\delta = \epsilon$.

Proposition. *Let $(X, d_X), (Y, d_Y)$ be connected metric spaces. Then $X \times Y$ endowed with the following metric is a connected metric space:*

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}.$$

Proof. It suffices to prove that for any point in $X \times Y, \exists$ connected subset of $X \times Y$ that contains those points. Let $(x_0, y_0), (a, b) \in X \times Y$ and $f : Y \to X \times Y$ be a function defined as $f(y) = (x_0, y)$. This is continuous. Indeed, $\rho(f(y_1), f(y_2)) = d_Y(y_1, y_2)$. Take $\delta = \epsilon$. We get $f(Y)$ is connected. Let $g : X \to X \times Y$ be a function defined as $g(x) = (x, b)$. This is continuous. Indeed, $\rho(g(x_1), g(x_2)) = d_X(x_1, x_2)$. Take $\delta = \epsilon$. We get $g(X)$ is connected. Note $f(Y) \cap g(X) \neq \emptyset$. Indeed, $(x_0, b) \in f(Y) \cap g(X)$. Then $f(Y) \cap g(X)$ is connected. As $\{(x_0, y_0), (a, b)\} \subseteq f(Y) \cup g(X)$, we get the claim. □

Remark. Note one may replace the metric ρ in the proposition above by any of the equivalent metrics $\rho = d_X + d_Y$ or $\rho = \sqrt{d_X^2 + d_Y^2}$.

Definition. Let (X, d_X) be a metric space.

- A **path** in X is any continuous function $\gamma : [0,1] \to X$. $\gamma(0)$ is called the **origin** of the path, $\gamma(1)$ is called the **end** of the path. Note $\gamma([0,1])$ is compact and connected.
- Let $\gamma : [0,1] \to X$ be a path in (X, d). We define $\gamma : [0,1] \to X$ via $\gamma^{-1}(t) = \gamma(1-t)$. This is a path in X. For γ_1, γ_2 paths in X with $\gamma_1(1) = \gamma_2(0)$, we define the path $\gamma_1 \vee \gamma_2 : [0,1] \to X$ via

$$\gamma_1 \vee \gamma_2(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \gamma_2(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

Theorem. *Let (X, d_X) be a metric space and let $\emptyset \neq A \subseteq X$, then $1 \iff 2 \implies 3$.*

1. *$\exists\, a \in A : \forall\, x \in A, \exists$ a path $\gamma_X : [0,1] \to A$ with $\gamma_X(0) = a$ and $\gamma_X(1) = x$.*
2. *$\forall\, x, y \in A, \exists$ a path $\gamma_{x,y} : [0,1] \to A$ with $\gamma_{x,y}(0) = x$ and $\gamma_{x,y}(1) = y$.*
3. *A is connected.*

Proof.
- "$1 \implies 2$" Let $x, y \in A, \gamma_X, \gamma_Y : [0,1] \to A$ as given by 1. Then $\gamma_{X,Y} = \gamma_X^{-1} \vee \gamma_Y : [0,1] \to A$ is the desired path.
- "$2 \implies 1$" Take a to be any point in A.
- "$1 \implies 3$" For $x \in A$, let $A_X = \gamma_X([0,1])$ connected. Moreover, $\{a\} \in \cap_{x \in A} A_x$. Therefore, $\cup_{x \in A} A_x$ is connected. But $\cup_{x \in A} A_x = A$.

□

Definition. If either 1 or 2 hold, we say A is **path connected**.

Exercise. Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected, and hence connected.

Proof. We will show that any point in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ can be joined to $(\sqrt{2}, \sqrt{2})$ via a path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$. Let $(x, y) \in \mathbb{R}^2 \setminus \mathbb{Q}^2$. Then $x \notin \mathbb{Q}$ or $y \notin \mathbb{Q}$. Say $x \notin \mathbb{Q}$. Then $\gamma_1 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ defined as $\gamma_1(t) = ((1-t)\sqrt{2} + tx, \sqrt{2})$ is a path, and $\gamma_2 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ defined as $\gamma_2(t) = (x, (1-t)\sqrt{2} + ty)$ is a path. Then $\gamma_1 \vee \gamma_2 : [0,1] \to \mathbb{R}^2 \setminus \mathbb{Q}^2$ is a path in $\mathbb{R}^2 \setminus \mathbb{Q}^2$ joining $(\sqrt{2}, \sqrt{2})$ to (x, y). □

Remark. Connected sets are not necessarily path connected.

Example. See equation 1.

- Let $G_f = \{(x, f(x)) : x \in [0, \infty)\}$. Then G_f is connected, but not path connected. Let $g : [0, \infty) \to \mathbb{R}^2$ be a function defined as $g(x) = (x, f(x))$. Then g is continuous on $[0, \infty)$ because f is, so $G_f \setminus \{(0,0)\} = g((0, \infty))$ is connected. Consider $G_f = \{(0,0)\} \cup g((0, \infty))$. Note $\{(0,0)\} \subseteq \overline{g((0, \infty))}$. Indeed, for $x_n = \frac{1}{\pi n}$, we get $g(x_n) = (\frac{1}{\pi n}, 0) \to (0,0)$. Therefore, $\{(0,0)\} = g((0, \infty))$ is connected.
- To see that G_f isn't path connected, it suffices to see that there is no path connecting $(0,0)$ to $(\frac{1}{\pi}, 0)$. Indeed, any such path would be discontinuous at $t = 0$, because $(\frac{1}{2n\pi + \frac{\pi}{2}}, 1) \xrightarrow[n \to \infty]{} (0,1) \neq (0,0)$.

Proposition. *Let $\emptyset \neq A \subseteq X$, then A is connected iff any two points in A can be joined by a polygonal arc lying in A.*

Proof. " $\Longleftarrow$ " is immediate since path connectedness $\implies$ connectedness. We show " $\implies$ ". Fix $a \in A$ and let $A_1 = \{x \in A : x \text{ can be joined to } a \text{ by a polygonal arc in } A\} \neq \emptyset$ because $a \in A_1$. We will show that A_1 is both opened and closed in A, then A connected $\implies A_1 = A$.

- Let's show A_1 is open in A. Pick $x \in A_1 \subseteq A$ open, $\exists\, r > 0 : B_r(x) \subseteq A$. As any point in $B_r(x)$ can be joined by a segment to x lying in the ball and x is joined by a polygonal arc to a, then any point in $B_r(x)$ can be joined by a polygonal arc to a. Thus $B_r(x) \subseteq A$. This proves A_1 is open.
- Let's show A_1 is closed in A. If $A_2 = A \setminus A_1 = \emptyset$, then we're done. So assume $\exists\, y \in A_2 \subseteq A$ open $\implies \exists\, r > 0 : B_r(y) \subseteq A$. If $B_r(y) \subseteq A_2$, then we're done. So assume $B_r(y) \subseteq A_1 \neq \emptyset$. Proof by picture, contradiction.

□

Convergent sequences of functions

Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces. For $n \geq 1$, let $f_n : X \to Y$ be functions. We say the sequence $\{f_n\}_{n\geq 1}$ **converges pointwise** if $\forall\ x \in X$, the sequence $\{f_n(x)\}_{n\geq 1} \subseteq Y$ converges. Thus, we say $\{f_n\}_{n\geq 1}$ converges pointwise to f if $\forall\ x \in X, \epsilon > 0, \exists\ n(\epsilon, x) \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon\ \forall\ n \geq n(\epsilon, x)$.

Remark. For $\epsilon > 0$, the function $n(\epsilon, x) : X \to \mathbb{N}$ can be bounded or unbounded. If it's bounded, we get the following definition.

Definition. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f_n : X \to Y, f : X \to Y$ be functions. We say that $\{f_n\}_{n\geq 1}$ **converges uniformly to** f and write $f_n \xrightarrow[n\to\infty]{u} f$ if $\forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon\ \forall\ n \geq n_\epsilon, x \in X$.

Remark.

- uniform convergence $\implies$ pointwise convergence
- For $(X, d_X), (Y, d_Y)$ metric spaces, let $B(X, Y) = \{f : X \to Y \mid f \text{ is bounded}\}$. We define $d : B(X, Y) \times B(X, Y) \to \mathbb{R}$ via $d(f, g) = \sup_{x\in X} d_Y(f(x), g(x))$. Then $(B(X, Y), d)$ is a metric space. Moreover,

$$\begin{aligned} f_n \xrightarrow[n\to\infty]{u} f &\iff \forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : d_Y(f(x), f_n(x)) < \epsilon\ \forall\ n \geq n_\epsilon, x \in X \\ &\iff \forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : \sup_{x\in X} d_Y(f(x), f_n(x)) \leq \epsilon\ \forall\ n \geq n_\epsilon, x \in X \\ &\iff \forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : d(f, f_n) \leq \epsilon\ \forall\ n \geq n_\epsilon, x \in X \\ &\iff d(f, f_n) \xrightarrow[n\to\infty]{} 0 \end{aligned}$$

- pointwise convergence $\not\Longrightarrow$ uniform convergence

  Example. for $n \geq 1$, let $f_n : [0, 1] \to \mathbb{R}, f_n(x) = x^n$, then

$$f_n(x) \xrightarrow[n\to\infty]{} \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases}$$

  Let $f : [0, 1] \to \mathbb{R}$ be a function defined as

$$f(x) = \begin{cases} 0 & \text{if } x \in [0, 1) \\ 1 & \text{if } x = 1. \end{cases}$$

  We have $\{f_n\}_{n\geq 1}$ converges pointwise to f. However, $\{f_n\}_{n\geq 1}$ doesn't converge uniformly to f. Indeed,

$$d(f_n, f) = \sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup_{x\in[0,1)} |x^n| = 1 \not\to 0.$$

Weierstrass

Theorem. *Let $(X, d_X), (Y, d_Y)$ be metric spaces. Assume that the sequence of functions $f_n : X \to Y$ converges uniformly to the function $f : X \to Y$. If f_n is continuous at $x_0 \in X$ for all $n \geq 1$, then f is continuous at x_0. In particular, a uniform limit of continuous functions is continuous.*

Proof. Fix $\epsilon > 0$. Then

$$f_n \xrightarrow[n\to\infty]{u} f \implies \exists\ n_\epsilon \in \mathbb{N} : d_Y(f_n(x), f(x)) < \frac{\epsilon}{3}\ \forall\ n \geq n_\epsilon, x \in X.$$

Fix $n_0 \geq n_\epsilon$. As f_{n_0} is continuous at x_0,

$$\exists\ \delta(\epsilon, x_0) > 0 : d_Y(f_{n_0}(x), f_{n_0}(x_0)) < \frac{\epsilon}{3}\ \forall\ x \in B^X_\delta(x_0).$$

For $x \in B^X_\delta(x_0)$, we have

$$d_Y(f(x), f(x_0)) \leq d_Y(f(x), f_{n_0}(x)) + d_Y(f_{n_0}(x), f_{n_0}(x_0)) + d_Y(f_{n_0}(x_0), f(x_0)) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Remark. The converse isn't true. $\exists$ a sequence of continuous functions that converges pointwise to a continuous function, but the convergence isn't uniform.

Example. $f_n : (0, 1) \to \mathbb{R}, f_n(x) = x^n, \{f_n\}_{n\geq 1}$ converges pointwise to $f : (0, 1) \to \mathbb{R}, f \equiv 0$. But the convergence is not uniform : $d(f_n, f) = \sup_{x\in(0,1)} |x^n| = 1 \not\to 0$.

Dini

Theorem. *Let (X, d_X) be a compact metric space, $f_n : X \to \mathbb{R}$ be continuous functions : $\{f_n\}_{n\geq 1}$ converges pointwise to a continuous function $f : X \to \mathbb{R}$. If $\{f_n\}_{n\geq 1}$ is monotone, then $\{f_n\}_{n\geq 1}$ converges uniformly to f.*

Proof. Assume, without loss of generality, $\{f_n\}_{n\geq 1}$ is increasing, i.e.

$$f_n(x) \leq f_{n+1}(x) \;\forall\; n \geq 1, x \in X.$$

Then $\{f - f_n\}$ is decreasing so

$$\forall\; x > 0, \lim_{n\to\infty} (f(x) - f_n(x)) = \inf_{n\geq 1} \{f(x) - f_n(x)\} = 0.$$

Fix $\epsilon > 0, x_0 \in X$. We have

$$\inf_{n\geq 1} \{f(x_0) - f_n(x_0)\} = 0 < \epsilon \implies \exists\; n(\epsilon, x_0) \in \mathbb{N} : |f(x_0) - f_{n(\epsilon,x_0)}(x_0)| < \epsilon.$$

Notice $f - f_{n(\epsilon,x_0)}$ is continuous at x_0. Thus

$$\exists\; \delta(\epsilon, x_0) > 0 : |(f(x) - f_{n(\epsilon,x_0)}(x)) - (f(x_0) - f_{n(\epsilon,x_0)}(x_0))| < \epsilon \;\forall\; x \in B_{\delta(\epsilon,x_0)}(x_0).$$

So for $x \in B_{\delta(\epsilon,x_0)}(x_0)$, we have

$$f(x) - f_{n(\epsilon,n_0)}(x) < \epsilon + f(x_0) - f_{n(\epsilon,n_0)}(x_0) < 2\epsilon.$$

Note that $\{B_{\delta(\epsilon,x)}(x)\}$ form an open cover of X compact, thus

$$\exists\; x_1, \ldots, x_N \in X : X \subseteq \cup_{k=1}^N B_{\delta(\epsilon,x_k)}(x_k).$$

Let $n_\epsilon = \max_{1\leq k\leq N} n(\epsilon, x_k)$ and $n \geq n_\epsilon$. For $x \in X$,

$$\exists\; 1 \leq k \leq N : x \in B_{\delta(\epsilon,x_k)}(x_k).$$

Then $f(x) - f_n(x) \leq f(x) - f_{n(\epsilon,x_0)}(x) < 2\epsilon$. By definition, $f_n \xrightarrow[n\to\infty]{u} f$. □

Remark. The compactness of X is essential. Consider $f_n : (0, 1) \to \mathbb{R}, f_n(x) = \frac{1}{nx+1}$, continuous, $f_n(x) \geq f_{n+1}(x) \;\forall\; x \in (0, 1), n \geq 1$. Then $\{f_n\}_{n\geq 1}$ converges pointwise to $f : (0, 1) \to \mathbb{R}, f \equiv 0$. But the convergence isn't uniform: $d(f_n, f) = \sup_{x\in(0,1)} |\frac{1}{nx+1}| = 1 \not\to 0$.

Theorem. *Let (X, d_X) be a metric space and $C(X) = \{f : X \to \mathbb{R} : f$ is bounded and continuous$\}$. For $f, g \in C(X)$, let $d(f, g) = \sup_{x\in X} |f(x) - g(x)|$. Then $(C(X), d)$ is a metric space.*

Exercise. $(C(X), d)$ is complete, connected, but not compact because unbounded.

Definition. Let $F \subseteq C(X)$.

- We say F is **uniformly bounded** if $\exists\; M > 0 : |f(x)| \leq M \;\forall\; f \in F, x \in X$.
- We say F is **equicontinuous** if $\forall\; \epsilon > 0, \exists\; \delta > 0 : d(f(x), f(y)) < \epsilon \;\forall\; f \in F, x, y \in X : d(x, y) < \delta$.

Arzela-Ascoli

Theorem. *Let $[a, b]$ be a compact interval in $\mathbb{R}$. Let $F \subseteq C([a, b])$. The following statements are equivalent.*

1. *Every sequence in F admits a (necessarily) uniformly convergent subsequence.*
2. *F is uniformly bounded and equicontinuous.*

Proof.
- $1 \implies 2$

- Let's show F is uniformly bounded, that is, F is bounded with respect to the uniform metric. Indeed, if F were not uniformly bounded, then we would be able to construct a sequence

$$\{f_n\}_{n\geq 1} \subseteq F : d(f_1, f_{n+1}) > 1 + d(f_1, f_n)\ \forall\ n \geq 1.$$

Then $d(f_n, f_m) \geq |d(f_1, f_n) - d(f_1, f_m)| > |n - m|$. So $\{f_n\}_{n\geq 1}$ cannot have a convergent subsequence.

- Let's show F is totally bounded. Let $\epsilon > 0, f_1 \in F$.
  * If $F \subseteq B_\epsilon(f_1)$, then F is totally bounded. Otherwise, $\exists\ f_2 \in F : d(f_2, f_1) \geq \epsilon$.
  * If $F \subseteq B_\epsilon(f_1) \cup B_\epsilon(f_2)$. then F is totally bounded. Otherwise, $\exists\ f_3 \in F : d(f_1, f_3) \geq \epsilon, d(f_2, f_3) \geq \epsilon$.
  * ...

If this process terminates in finitely many steps, then F is totally bounded. Otherwise, we find a sequence

$$\{f_n\}_{n\geq 1} \subseteq F : d(f_n, f_m) \geq \epsilon\ \forall\ n \neq m.$$

This sequence doesn't admit a convergent subsequence.

- Let's show F is equicontinuous. Let $\epsilon > 0$. As F is totally bounded,

$$\exists\ f_1, \ldots, f_n \in F : F \subseteq \cup_{k=1}^n B_{\frac{\epsilon}{10}}(f_k).$$

Fix $1 \leq k \leq n$. As $f_k : [a, b] \to \mathbb{R}$ is continuous, it is uniformly continuous. So

$$\exists\ \delta_k(\epsilon) > 0 : |f_k(x) - f_k(y)| < \frac{\epsilon}{10}\ \forall\ x, y \in [a, b], |x, y| < \delta_k.$$

Let $\delta(\epsilon) = \min_{1\leq k\leq n} \delta_k(\epsilon)$. Then $\forall\ x, y \in [a, b]$ with $|x - y| < \delta$ and all $1 \leq k \leq n$, we have

$$|f_k(x) - f_k(y)| < \frac{\epsilon}{10}.$$

Let $f \in F$, then

$$\exists\ 1 \leq k \leq n : f \in B_{\frac{\epsilon}{10}}(f_k).$$

For $x, y \in [a, b]$ with $|x - y| < \delta$ we get

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f_k(y) - f(y)| \\ &\leq 2d(f, f_k) + |f_k(x) - f_k(y)| < 2\frac{\epsilon}{10} + \frac{\epsilon}{10} < \epsilon.\end{aligned}$$

By definition, F is equicontinuous.

- $2 \implies 1$ Assume F is uniformly bounded and equicontinuous. Let $\{f_n\}_{n\geq 1} \subseteq F$. If $q \in [a, b] \cap \mathbb{Q}$, then $\{f_n(q)\}_{n\geq 1}$ is a bounded sequence of real numbers. In particular, $\{f_n(q)\}_{n\geq 1}$ has a convergent subsequence. Passing to a subsequence for every $q \in [a, b] \cap \mathbb{Q}$ (using the fact that $[a, b] \cap \mathbb{Q}$ is countable) and using a diagonal argument, we find a subsequence $\{f_{k_n}\}_{n\geq 1}$ that converges at every rational $q \in [a, b]$. Let $\epsilon > 0$. As F is equicontinuous,

$$\exists\ \delta > 0 : |f(x) - f(y)| < \frac{\epsilon}{10}\ \forall\ f \in F, |x - y| < \delta.$$

As $[a, b]$ is compact,

$$\exists\ q_1, \ldots, q_N \in [a, b] \cap \mathbb{Q} : [a, b] \subseteq \cup_{j=1}^N (q_j - \delta, q_j + \delta).$$

Now $\{f_{k_n}(q_j)\}$ is convergent so

$$\exists\ n_j(\epsilon) \in \mathbb{N} : |f_{k_n}(q_j) - f_{k_m}(q_j)| < \frac{\epsilon}{10}\ \forall\ n, m \geq n_j(\epsilon).$$

Let $x \in [a, b]$, then $\exists\ 1 \leq j \leq N : |x - q_j| < \delta$. Now

$$|f_{k_n}(x) - f_{k_m}(x)| < |f_{k_n}(x) - f_{k_n}(q_j)| + |f_{k_n}(q_j) - f_{k_m}(q_j)| + |f_{k_m}(q_j) - f_{k_m}(x)| < \epsilon\ \forall\ n, m \geq n(\epsilon).$$

We proved that $\{f_{k_n}\}_{n\geq 1}$ is Cauchy with respect to the uniform metric. Let $f(x) = \lim_{n\to\infty} f_{k_n}(x)$. We have by Weierstrass

$$f_{k_n} \xrightarrow[n\to\infty]{u} f \in C([a, b]).$$

□

Corollary. *Let $F \subseteq C([a,b])$. Then F is compact iff F is closed, uniformly bounded, and equicontinuous.*

Remark.

- The compactness of $[a,b]$ is essential. Let
$$F = \{f : \mathbb{R} \to \mathbb{R}, |f(x) - f(y)| \leq |x-y|, \sup_{x\in\mathbb{R}} |f(x)| \leq 1\}.$$
Then F is uniformly bounded and equicontinuous. Consider $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$. Clearly, $\sup_{x\in\mathbb{R}} |f(x)| = \sup_{x\in\mathbb{R}} \frac{1}{1+x^2} = 1$. For $x, y \in \mathbb{R}$,
$$\begin{aligned} |f(x) - f(y)| = |\frac{1}{1+x^2} - \frac{1}{1+y^2}| &= \frac{|x-y||x+y|}{(1+x^2)(1+y^2)} \\ &\leq |x-y| \left(\frac{|x|}{|1+x^2||1+y^2|} + \frac{|y|}{|1+x^2||1+y^2|} \right) \\ &\leq |x-y|(\frac{1}{2} + \frac{1}{2}) = |x-y|. \end{aligned}$$
For $n \geq 1$, let $f_n(x) = f(x-n) = \frac{1}{1+(x-n)^2}$. For $x \in \mathbb{R}, f_n(x) \xrightarrow[n\to\infty]{} 0$. So $\{f_n\}_{n\geq 1}$ converges pointwise to the function $g : \mathbb{R} \to \mathbb{R}, g \equiv 0$. However, $\{f_n\}_{n\geq1}$ doesn't admit a uniformly convergent subsequence because $\sup_{x\in\mathbb{R}} f_n(x) = 1 \ \forall \ n \geq 1$.

- The uniform boundedness of F is essential. Take $F = \{f : [0,1] \to \mathbb{R} : f \text{ is constant}\}$. This is equicontinuous but not uniformly bounded. Indeed, $f_n(x) \equiv n$ doesn't admit a convergent subsequence.

- The equicontinuity of F is essential. Consider
$$F = \{f : [0,1] \to \mathbb{R}, f \text{ is continuous}, \sup_{x\in[0,1]} |f(x)| \leq 1\}.$$
This set is not equicontinuous. For $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}, f_n(x) = \sin(nx)$. Let $x_n = \frac{3\pi}{2n}, y_n = \frac{\pi}{2n}$. Then $|x_n - y_n| = \frac{\pi}{n} \xrightarrow[n\to\infty]{} 0$. But $|f_n(x_n) - f_n(y_n)| = 2$. The sequence $\{f_n\}_{n\geq1}$ doesn't admit a uniformly convergent subsequence. Assume, towards a contradiction, that $\exists \ \{f_n\}_{n\geq1}$ that converges uniformly. By Weierstrass, the limit function $f : [0,1] \to \mathbb{R}$ is continuous. As $f_n(0) = 0$, we must have $f(0) = 0$. Then f continuous at $x = 0$ yields
$$\implies \forall \ \epsilon > 0, \exists \ \delta > 0 : |f(x)| < \epsilon \ \forall \ 0 \leq x < \delta.$$
Moreover,
$$f_{k_n} \xrightarrow[n\to\infty]{u} f, \exists \ N \in \mathbb{N} : |f_{k_n}| < 2\epsilon \ \forall \ n \geq N, 0 \leq x < \delta.$$
But $f_{k_n}(\frac{\pi}{2k_n}) = 1$. Take n sufficiently large : $\frac{\pi}{2k_n} < \delta$ to get a contradiction.

The oscillation of a function

Definition. Let (X,d) be a metric space, $\emptyset \neq A \subseteq X, f : X \to \mathbb{R}$. The **oscillation of a function on** A is $\omega(f,A) = \sup_{x\in A} f(x) - \inf_{x\in A} f(x) = \sup_{x,y\in A} (f(x) - f(y)) \geq 0$. The **oscillation of** f **at** $x_0 \in X$ is $\omega(f,x_0) = \inf_{\epsilon>0} \omega(f, B_\epsilon(x_0))$.

Proposition. *Let (X,d) be a metric space, $f : X \to \mathbb{R}$. Then f is continuous at $x_0 \in X$ iff $\omega(f,x_0) = 0$.*

Proof. Let $\epsilon > 0$.

- " $\implies$ " As f is continuous at $x_0, \exists \ \delta > 0 : |f(x) - f(x_0)| < \frac{\epsilon}{2} \ \forall \ x \in B_\delta(x_0)$. Then for $x \in B_\delta(x_0), f(x_0) - \frac{\epsilon}{2} < f(x) < f(x_0) + \frac{\epsilon}{2} \implies f(x_0) - \frac{\epsilon}{2} \leq \inf_{x\in B_\delta(x_0)} f(x) \leq \sup_{x\in B_\delta(x_0)} f(x) \leq f(x_0) + \frac{\epsilon}{2} \implies \omega(f, B_\delta(x_0)) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \implies \omega(f,x_0) \leq \epsilon$.

- " $\impliedby$ " We have $\omega(f,x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0)) = 0 < \epsilon \implies \exists \ \delta > 0 : \omega(f, B_\delta(x_0)) < \epsilon \implies \sup_{x,y\in B_\delta(x_0)} (f(x) - f(y)) < \epsilon \implies \sup_{x\in B_\delta(x_0)} |f(x) - f(x_0)| < \epsilon \implies |f(x) - f(x_0)| < \epsilon \ \forall \ x \in B_\delta(x_0)$. This shows f is continuous at x_0.

□

Proposition. *Let (X,d) be a metric space, $f : X \to \mathbb{R}$ be a function, and $\alpha > 0$. Then $A = \{x \in X : \omega(f,x) < \alpha\}$ is open.*

Proof. Let $x_0 \in A \implies \omega(f,x_0) = \inf_{\delta>0} \omega(f, B_\delta(x_0)) < \alpha \implies \exists\, \delta > 0 : \omega(f, B_\delta(x_0)) < \alpha$.
Claim. $B_\delta(x_0) \subseteq A$

Let $y \in B_\delta(x_0)$. Then $B_{\delta - d(x_0,y)}(y) \subseteq B_\delta(x_0)$ and $\omega(f,y) \leq \omega(f, B_{\delta-d(x_0,y)}(y)) \leq \omega(f, B_\delta(x_0)) < \alpha$. So $y \in A$. ☐

Remark. Let (X,d) is a metric space, $f : X \to \mathbb{R}$. Then $\{x \in X : f \text{ is continuous at } x\} = \{x \in X : \omega(f,x) = 0\} = \cap_{n\geq 1}\{x \in X : \omega(f,x) < \frac{1}{n}\} = \cap_{n\geq 1}\{G_n\}$ open. Note $G_{n+1} \subseteq G_n \ \forall\ n \geq 1$.

Exercise. Show that there are no functions $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at every rational point and discontinuous at every irrational point.

Proof. By contradiction. Assume $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and discontinuous on $\mathbb{R} \setminus \mathbb{Q}$. Then $\mathbb{Q} = \cap_{n\geq 1} G_n$ with $G_n = \mathring{G}_n$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, we get $\overline{G_n} = \mathbb{R}\ \forall\ n \geq 1 \implies \overline{\cap_{n\geq 1} G_n} = \mathbb{R}$. Let $\{q_n\}_{n\geq 1}$ denote an enumeration of $\mathbb{Q}$. For $n \geq 1$, let $H_n = \mathbb{R} \setminus \{q_n\} = (-\infty, q) \cup (q, \infty)$ open. Moreover, $\overline{H_n} = \mathbb{R}$. As $\mathbb{R}$ is complete, if satisfied the Baire property, that is, if $\{A_n\}_{n\geq 1}$ is a sequence of open dens sets, then $\overline{\cap_{n\geq 1} A_n} = \mathbb{R}$. Then we must have that $(\cap_{n\geq 1} G_n) \cup (\cap_{n\geq 1} H_n) = \emptyset$, contradiction. ☐

Lemma. *Let (X,d) be a metric space with the Baire property and $\emptyset \neq W = \mathring{W} \subseteq X$. Then W has the Baire property.*

Proof. Let $\{D_n\}_{n\geq 1}$ be a sequence of open, dense sets in W. As D_n is open in $W, \exists\, G_n$ open in $X : D_n = G_n \cap W$ open in X. Also

$$\overline{D_n} \cap W = W \implies W \subseteq \overline{D_n} \implies \overline{W} \subseteq \overline{D_n}\ \forall\ n \geq 1.$$

For $n \geq 1$, let $B_n = D_n \cup\ {}^c(\overline{W})$ open in X. Then $\overline{B_n} = \overline{D_n \cup\ {}^c(\overline{W})} = \overline{D_n} \cup \overline{{}^c(\overline{W})} \supseteq \overline{W} \cup\ {}^c\overline{W} = X \implies B_n$ is dense in X. As X has the Baire property,

$$X = \overline{\cap_{n\geq 1} B_n} = \overline{\cap_{n\geq 1}(D_n \cup\ {}^c(\overline{W}))} = \overline{(\cap_{n\geq 1} D_n) \cup\ {}^c(\overline{W})} = \overline{\cap_{n\geq 1} D_n} \cup \overline{{}^c(\overline{W})} = \overline{\cap_{n\geq 1} D_n} \cup\ {}^c(\mathring{\overline{W}})$$
$$\implies W = \left(\overline{\cap_{n\geq 1} D_n} \cup\ {}^c(\mathring{\overline{W}})\right) \cap W = \left(\overline{\cap_{n\geq 1} D_n} \cap W\right) \cup \left({}^c(\mathring{\overline{W}}) \cap W\right).$$

But $W = \mathring{W} \subseteq \mathring{\overline{W}} \implies {}^c(\mathring{\overline{W}}) \cap W = \emptyset$. So $\cap_{n\geq 1} D_n$ is dense in W. ☐

Theorem. *Let (X,d) be a metric space with the Baire property. If $f_n : X \to \mathbb{R}$ are continuous functions converging pointwise to $f : X \to \mathbb{R}$, then the set of points at which f is continuous is dense in X.*

Claim. It suffices to prove the theorem under the additional hypothesis that $|f_n(x)| \leq 1\ \forall\ n \geq 1, x \in X$.

Proof. Indeed, assume that the theorem holds for this restricted set of functions and let $\{f_n\}_{n\geq 1}$ be as in the theorem. Consider $\phi : \mathbb{R} \to (-1,1), \phi(x) = \frac{x}{1+|x|}$ bijective continuous with inverse $\phi^{-1} : (-1,1) \to \mathbb{R}, \phi(y) = \frac{y}{1-|y|}$ continuous. Then $\phi \circ f_n : X \to (-1,1)$ is continuous, $\{\phi \circ f_n\}_{n\geq 1}$ is uniformly bounded by 1, and $\phi \circ f_n \xrightarrow[n\to\infty]{} \phi \circ f$ pointwise. Then $\{x \in X : \phi \circ f \text{ is continuous at } x\}$ is dense in X. As $\phi \circ f$ is continuous at $x \iff f$ is continuous at x, we get the claim. ☐

Proof. From now on, assume $|f_n(x)| \leq 1\ \forall\ n \geq 1, x \in X$. We want to show $\overline{\cap_{n\geq 1} G_n} = X$. As X has the Baire property, it suffices to show that $\overline{G_n} = X\ \forall\ n \geq 1$. Fix $N \geq 1$. To show $\overline{G_N} = X$, it suffices to show that $G_N \cap W \neq \emptyset\ \forall\ \emptyset = W = \mathring{W} \subseteq X$. Fix $\emptyset = W = \mathring{W} \subseteq X$. We want to show $G_N \cap W \neq \emptyset$. For $n \geq 1$, let $u_n(x) = \inf_{m\geq n} f_m(x)$ and $v_n(x) = \sup_{m\geq n} f_m(x)$. Then $\{u_n\}_{n\geq 1}$ is an increasing sequence of functions, and $\{v_n\}_{n\geq 1}$ is a decreasing sequence of functions. As $f_n(x) \xrightarrow[n\to\infty]{} f(x)\ \forall\ x \in X$, we have $\lim_{n\to\infty} u_n(x) = f(x) = \lim_{n\to\infty} v_n(x)\ \forall\ x \in X$. Let

$$F_n = \{x \in X : v_n(x) - u_n(x) \leq \frac{1}{4N}\}.$$

We have $X = \cup_{n\geq 1} F_n$. Since

$$\forall\, x \in X, \exists\, n(x) \in \mathbb{N} : x \in F_{n(x)}, F_n = \{x \in X : \sup_{k\geq n} f_k(x) - \inf_{l\geq n} f_l(x) \leq \frac{1}{4N}\}$$
$$= \{x \in X : \sup_{k,l\geq n} (f_k(x) - f_l(x)) \leq \frac{1}{4N}\}$$
$$= \cap_{k,l\geq n}\{x \in X : f_k(x) - f_l(x) \leq \frac{1}{4N}\}.$$

But

$$\{x \in X : (f_k - f_l)(x) \leq \frac{1}{4N}\} = (f_k - f_l)^{-1}([-2, \frac{1}{4N}])$$

is the continuous preimage of a closed set, hence it's closed. Thus F_n is closed so

$$X = \cup_{n\geq 1} F_n \implies W = \cup_{n\geq 1}(F_n \cap W),$$

which is the union of closed sets in W. By the previous lemma, W has the Baire property, and $\mathring{W} \neq \emptyset$ so $\exists\, n_1 \geq 1 : F_{n_1} \mathring{\cap} W \neq \emptyset$. Let

$$x_0 \in F_{n_1} \mathring{\cap} W \implies \exists\, \delta > 0 : B_\delta(x_0) \subseteq F_{n_1 \cap W}.$$

Since f_{n_1} is continuous at x_0, shrinking δ if necessary, we may assume $\omega(f_{n_1}, B_\delta(x_0)) < \frac{1}{2N}$. We will show $x_0 \in G_N$. In particular, $x_0 \in G_N \cap W \neq \emptyset$. Then

$$\begin{aligned}
\omega(f, B_\delta(x_0)) &= \sup_{x,y\in B_\delta(x_0)} |f(x) - f(y)| \leq \sup_{x,y\in B_\delta(x_0)} |v_{n_1}(x) - u_{n_1}(y)| \\
&= \sup_{x,y\in B_\delta(x_0)} |v_{n_1}(x) - u_{n_1}(x) + u_{n_1}(x) - v_{n_1}(y) + v_{n_1}(y) - u_{n_1}(y)| \\
&\leq \sup_{x,y\in B_\delta(x_0)} |v_{n_1}(x) - u_{n_1}(x)| + \sup_{x,y\in B_\delta(x_0)} |u_{n_1}(x) - v_{n_1}(y)| + \sup_{x,y\in B_\delta(x_0)} |v_{n_1}(y) - u_{n_1}(y)| \\
&\leq \frac{1}{4N} + \sup_{x,y\in B_\delta(x_0)} |f_{n_1}(x) - f_{n_1}(y)| + \frac{1}{4N} = \frac{1}{2N} + \omega(f_n, B_\delta(x_0)) < \frac{1}{N} \implies x_0 \in G_N.
\end{aligned}$$

□

Weierstrass approximation

Theorem. *Let $f : [a,b] \to \mathbb{R}$ be continuous. Then there exists polynomials $P_n : [a,b] \to \mathbb{R}$ of degree at most n such that $P_n \xrightarrow[n\to\infty]{u} f$ on $[a,b]$.*

Proof. We may assume $[a,b] = [0,1]$. Indeed, the function $\phi : [0,1] \to [a,b], \phi(t) = (1-t)a + tb$ is bijective and continuous. Then $f \circ \phi : [0,1] \to \mathbb{R}$ is continuous. If we find polynomials P_n of degree at most $n : P_n \xrightarrow[n\to\infty]{u} f \circ \phi$ on $[0,1]$, then $P_n \circ \phi^{-1} \xrightarrow[n\to\infty]{u} f$ on $[a,b]$. From now on, $[a,b] = [0,1]$. For $n \geq 0$, let $P_n(x) = \sum_{k=0}^n f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k}$ be Bernstein polynomials. We will show $P_n \xrightarrow[n\to\infty]{u} f$ on $[0,1]$. Let $\epsilon > 0$. As $f : [0,1] \to \mathbb{R}$ is continuous, f is uniformly continuous. Thus $\exists\, \delta > 0 : |f(x) - f(y)| < \epsilon\ \forall\, x,y \in [0,1], |x-y| < \delta$. Fix $x \in [0,1]$. Then

$$\begin{aligned}
|P_n(x) - f(x)| &= |\sum_{k=0}^n f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} - f(x)\sum_{k=0}^n \binom{n}{k}x^k(1-x)^{n-k}| \\
&\leq \sum_{k=0}^n |f(\frac{k}{n}) - f(x)|\binom{n}{k}x^k(1-x)^{n-k} \\
&\leq \sum_{0\leq k\leq n, |x-\frac{k}{n}|<\delta} |f(\frac{k}{n}) - f(x)|\binom{n}{k}x^k(1-x)^{n-k} + \sum_{0\leq k\leq n, |nx-k|\geq n\delta} |f(\frac{k}{n}) - f(x)|\binom{n}{k}x^k(1-x)^{n-k} \\
&\leq \epsilon\sum_{k=0}^n \binom{n}{k}x^k(1-x)^{n-k} + 2\sup_{y\in[0,1]}|f(y)|\sum_{k=0}^n \left(\frac{nx-k}{n\delta}\right)^2\binom{n}{k}x^k(1-x)^{n-k} \\
&\leq \epsilon + 2\sup_{y\in[0,1]}|f(y)|\frac{1}{n^2\delta^2}\sum_{k=0}^n (nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}.
\end{aligned}$$

Now compute

$$\sum_{k=0}^{n}(nx-k)^2\binom{n}{k}x^k(1-x)^{n-k}$$

$$= n^2x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} - 2nx\sum_{k=0}^{n}k\binom{n}{k}x^k(1-x)^{n-k} + \sum_{k=0}^{n}k^2\binom{n}{k}x^k(1-x)^{n-k}$$

$$= n^2x^2 - 2n^2x^2\sum_{k=1}^{n-1}\frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} + \sum_{k=1}^{n}(k-1+1)\frac{n!}{(k-1)!(n-k)!}x^k(1-x)^{n-k}$$

$$= -n^2x^2 + n(n-1)x^2\sum_{k=2}^{n-1}\frac{(n-2)!}{(k-2)!(n-k)!}x^{k-2}(1-x)^{n-k} + nx\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}$$

$$= -n^2x^2 + n^2x^2 - nx^2 + nx = nx(1-x) \le \frac{n}{4}.$$

Thus $|P_n(x)-f(x)| \le \epsilon + 2\sup_{y\in[0,1]}|f(y)|\frac{1}{n^2\delta^2}\frac{n}{4} < 2\epsilon$ for n large depending only on δ and $\sup_{y\in[0,1]}|f(y)|$. This proves uniform convergence. $\square$

Exercise. Let $a>0$. Show that there are polynomials $P_n : [-a,a]\to\mathbb{R}$ of degree $\le n : P_n(0)=0$ and $P_n \xrightarrow[n\to\infty]{u} |x|$ on $[-a,a]$.

Proof. Note $f:[-a,a]\to\mathbb{R}, f(x)=|x|$ is continuous, thus by the Weierstrass approximation theorem, $\exists\ Q_n : [-a,a]\to\mathbb{R}$ polynomial of degree $\le n : Q_n \xrightarrow[n\to\infty]{u} |x|$ on $[-a,a]$. Note $Q_n(0) \xrightarrow[n\to\infty]{} 0$. Let $P_n(x) = Q_n(x) - Q_n(0)$ polynomial of degree $\le n$ and $P_n(0)=0$.

Claim. $P_n \xrightarrow[n\to\infty]{u} |x|$ on $[-a,a]$.

Notice $|P_n(x)-|x|| \le |Q_n(x)-|x|| + |Q_n(0)|$. Given $\epsilon>0$,

$$\exists\ n_1(\epsilon)\in\mathbb{N} : \sup_{x\in[-a,a]}|Q_n(x)-|x|| < \frac{\epsilon}{2}\ \forall\ n\ge n_1(\epsilon)$$

$$\exists\ n_2(\epsilon)\in\mathbb{N} : |Q_n(0)| < \frac{\epsilon}{2}\ \forall\ n\ge n_2(\epsilon)$$

For $n \ge n(\epsilon) = \max\{n_1(\epsilon), n_2(\epsilon)\}$, we have $|P_n(x)-|x|| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon\ \forall\ x\in[-a,a]$. Thus $P_n \xrightarrow[n\to\infty]{u} |x|$ on $[-a,a]$. $\square$

Definition. Let (X,d) be a metric space. A set of real-valued functions $A\subseteq\{f:X\to\mathbb{R}\}$ is called an **algebra** if

1. If $f,g\in A$, then $f+g\in A$.
2. If $f,g\in A$, then $fg\in A$.
3. If $f\in A$ and $\lambda\in\mathbb{R}$, then $\lambda f\in A$.

Stone-Weierstrass

Theorem. *Let (X,d) be a compact metric space and $A\subseteq C(X)$ be an algebra. Assume that A satisfies the following two properties:*

1. *A separates points in X, that is, if $x,y\in X$ with $x\ne y$, then $\exists\ f\in A : f(x)\ne f(y)$.*
2. *A vanishes at no point in X, that is, if $x\in X$, then $\exists\ f\in A : f(x)\ne 0$.*

Then A is dense in $C(X)$.

Example. $A=\{P:X\to\mathbb{R}$ polynomials$\}$ is an algebra, it separates points, and vanishes at no point.

Definition. Let $f:\mathbb{R}\to\mathbb{R}$ be a function. For $a\in\mathbb{R}$, we write $\lim_{x\to a}f(x) = L\in\mathbb{R}$ if for any sequence $\{x_n\}_{n\ge1}\subseteq\mathbb{R}\backslash\{a\} : x_n \xrightarrow[n\to\infty]{} a$, we have $f(x_n)\xrightarrow[n\to\infty]{} L$. Equivalently, if $\forall\ \epsilon>0, \exists\ \delta>0 : x\in(a-\delta,a+\delta)\backslash\{a\} \implies |f(x)-L|<\epsilon$.

Exercise. Extend this definition to cover $L = \pm\infty, a = \pm\infty$.

Remark. f is continuous at $a \in \mathbb{R}$ iff $\lim_{x\to a} f(x) = f(a)$. Similarly, one defines the left-limit $f(a^-) = \lim_{x\to a} f(x)$ and the right-limit $f(a^+) = \lim_{x\to a} f(x)$.

Differentiation

Definition. Let I be an open interval and $f : I \to \mathbb{R}$ a function. We say f is **differentiable** at $a \in I$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and is finite. In this case we write $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ and we call it the **derivative** of f at a.

Example. Fix $n \geq 1, f : \mathbb{R} \to \mathbb{R}, f(x) = x^n$. For $a \in \mathbb{R}, x \neq a$, $\frac{f(x)-f(a)}{x-a} = \frac{x^n - a^n}{x-a} = x^{n-1} + x^{n-2}a + \cdots + a^{n-1} \xrightarrow[n\to\infty]{} na^{n-1}$. So $f'(a) = na^{n-1}$.

Lemma. *Let I be an open interval, $f : I \to \mathbb{R}$ be a differentiable at $a \in I$. Then f is continuous at a.*

Proof. For $x \in I\backslash\{a\}$ we write $f(x) = \frac{f(x)-f(a)}{x-a}(x-a) + f(a)$. Then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ and $\lim_{x\to a}(x-a) = 0$. Thus $\lim_{x\to a} f(x) = f(a)$. ☐

Theorem. *Let I be an open interval and $f, g : I \to \mathbb{R}$ be differentiable at $a \in I$. Then*

1. *for any $\lambda \in \mathbb{R}, \lambda f$ is differentiable at a and $(\lambda f)' = \lambda f'$.*
2. *$f + g$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$.*
3. *fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$.*
4. *if $g(a) \neq 0$ then $\frac{f}{g}$ is differentiable at a then $(\frac{f}{g})'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$.*

Proof.

1. For $x \neq a$, $\frac{(\lambda f)(x) - (\lambda f)(a)}{x-a} = \lambda \frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} \lambda f'(a)$.
2. For $x \neq a$, $\frac{(f+g)(x)-(f+g)(a)}{x-a} = \frac{f(x)-f(a)}{x-a} + \frac{g(x)-g(a)}{x-a} \xrightarrow[x\to a]{} f'(a) + g'(a)$.
3. For $x \neq a$, $\frac{(fg)(x)-(fg)(a)}{x-a} = \frac{f(x)-f(a)}{x-a}g(x) + f(a)\frac{g(x)-g(a)}{x-a}$. Then g continuous $a \implies \lim_{x\to a} g(x) = g(a)$ so $\lim_{x\to a} \frac{(fg)(x)-(fg)(a)}{x-a} = f'(a)g(a) + f(a)g'(a)$.
4. For $x \neq a$, $\frac{(\frac{f}{g})(x)-(\frac{f}{g})(a)}{x-a} = \frac{f(x)-f(a)}{x-a}\frac{1}{g(x)} + f(a)\frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} = \frac{f(x)-f(a)}{x-a}\frac{1}{g(x)} + \frac{f(a)}{g(a)}\frac{1}{g(x)}\frac{g(x)-g(a)}{x-a}$. Then $g(a) \neq 0$ and g continuous $a \implies \lim_{x\to a} \frac{1}{g(x)} = \frac{1}{g(a)}$ so $\frac{(\frac{f}{g})(x)-(\frac{f}{g})(a)}{x-a} = f'(a)\frac{1}{g(a)} + \frac{f(a)}{g(a)}\frac{1}{g(a)}g'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}$. ☐

Theorem. *Let I and J be open intervals and assume $f : I \to \mathbb{R}$ is differentiable at $a \in I$ and $g : J \to \mathbb{R}$ is differentiable at $f(a) \in J$. Then $f \circ g$ is well-defined on a neighbourhood of a, is differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$.*

Proof. As $f(a) \in J$ open, $\exists\, \epsilon > 0 : (f(a) - \epsilon, f(a) + \epsilon) \subseteq J$. Since f is continuous at $a, \exists\, \delta > 0$: if $|x - a| < \delta$ with $x \in I$ then $|f(x) - f(a)| < \epsilon$. As I is open, choosing δ even smaller (if necessary), we may ensure $(a - \delta, a + \delta) \subseteq I$. So $g \circ f$ is well-defined on $(a - \delta, a + \delta)$. Let

$$h(y) = \begin{cases} \frac{g(y)-g(f(a))}{y-f(a)} & y \in J \setminus \{f(a)\} \\ g'(f(a)) & y = f(a) \end{cases}$$

As g is differentiable at $f(a)$, we have $\lim_{y\to f(a)} h(y) = \lim_{y\to f(a)} \frac{g(y)-g(f(a))}{y-f(a)} = g'(f(a)) = h(f(a))$. So h is continuous at $f(a)$. So for $y \in J \setminus \{f(a)\}$, we write $g(y) - g(f(a)) = h(y)(y - f(a))$. For $x \in (a - \epsilon, a + \epsilon) \setminus \{a\}$ we have $g(f(x)) - g(f(a)) = h(f(x))(f(x) - f(a)) \iff \frac{g(f(x))-g(f(a))}{x-a} = h(f(x))\frac{f(x)-f(a)}{x-a} \xrightarrow[x\to a]{} g'(f(a))f'(a) \implies (g \circ f)'(a) = g'(f(a))f'(a)$. ☐

Theorem. *Let $f : (a, b) \to \mathbb{R}$. If f attains its maximum or minimum at $x_0 \in (a, b)$ and f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Assume f attains its maximum at x_0. Otherwise, replace f by $-f$.

- For $x_n \in (a, x_0)$ with $x_n \xrightarrow[n\to\infty]{} x_0$ we have $f'(x_0) = \lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n-x_0} \geq 0$.
- For $x_n \in (x_0, b)$ with $x_n \xrightarrow[n\to\infty]{} x_0$ we have $f'(x_0) = \lim_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n-x_0} \leq 0$.

Combining the two, we get $f'(x_0) = 0$. □

Rolle

Theorem. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then $\exists\ x_0 \in (a, b) : f'(x_0) = 0$.*

Proof. As f is continuous on the compact interval $[a, b]$, it attains its maximum and minimum on $[a, b]$. Thus $\exists\ y_0, z_0 \in [a, b] : f(y_0) \leq f(x) \leq f(z_0)\ \forall\ x \in [a, b]$.

1. Suppose $\{y_0, z_0\} = \{a, b\}$. As $f(a) = f(b)$ we get that f is constant on $[a, b]$. Then $\forall\ x \in (a, b)$ we have $f'(x) = \lim_{y\to x} \frac{f(y)-f(x)}{y-x} = 0$.
2. Suppose either $y_0 \notin \{a, b\}$ or $z_0 \notin \{a, b\}$. If $y_0 \in \{a, b\}$, then by the previous theorem, $f'(y_0) = 0$. Likewise for z_0.

□

Mean value theorem

Theorem. *Assume $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b). Then $\exists\ x_0 \in (a, b) : f'(x_0) = \frac{f(b)-f(a)}{b-a}$.*

Remark. If $f(a) = f(b)$, we recover Rolle's theorem.

Proof. Let $l : [a, b] \to \mathbb{R}$ given by $l(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x - a)$. Then l is continuous on $[a, b]$, differentiable on (a, b) with $l'(x) = \frac{f(b)-f(a)}{b-a}\ \forall\ x \in (a, b)$ and $l(a) = f(a), l(b) = f(b)$. Let $g : [a, b] \to \mathbb{R}, g(x) = f(x) - l(x)$ continuous on $[a, b]$, differentiable on (a, b) and $g(a) = g(b) = 0$. By Rolle's theorem, $\exists\ x_0 \in (a, b) : g'(x_0) = 0 = f'(x_0) - \frac{f(b)-f(a)}{b-a} \implies f'(x_0) = \frac{f(b)-f(a)}{b-a}$. □

Corollary. *If $f : (a, b) \to \mathbb{R}$ is differentiable with $f'(x_0) = 0\ \forall\ x \in (a, b)$, then f is constant.*

Proof. Assume, towards a contradiction, that $\exists\ a < x_1 < x_2 < b : f(x_1) \neq f(x_2)$. As f is differentiable on (a, b), it's continuous on (a, b). Thus f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2). By the mean value theorem, $\exists\ x_0 \in (x_1, x_2) : 0 = f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. But $f'(x_0) = 0\ \forall\ x \in (a, b)$ by hypothesis, so $f(x_2) - f(x_1) = 0 \iff f(x_1) = f(x_2)$, contradiction. □

Corollary. *If $f, g : (a, b) \to \mathbb{R}$ are differentiable with $f'(x) = g'(x)\ \forall\ x \in (a, b)$, then $\exists\ c \in \mathbb{R} : f(x) = g(x)+c\ \forall\ x \in (a, b)$.*

Corollary. *Let $f : (a, b) \to \mathbb{R}$ be differentiable.*

1. *If $f'(x_0) \geq 0\ \forall\ x \in (a, b)$, then f is increasing.*
2. *If $f'(x_0) > 0\ \forall\ x \in (a, b)$, then f is strictly increasing.*
3. *If $f'(x_0) \leq 0\ \forall\ x \in (a, b)$, then f is decreasing.*
4. *If $f'(x_0) < 0\ \forall\ x \in (a, b)$, then f is strictly decreasing.*

Proof. 1. Let $a < x_1 < x_2 < b$. Then f is continuous on $[x_1, x_2]$ because it's continuous on (a, b) and differentiable on (x_1, x_2). By the mean value theorem, $\exists\ x_0 \in (x_1, x_2) : 0 \leq f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Thus $f(x_2) - f(x_1) \geq 0$.

□

Exercise. Prove the remaining.

Intermediate value properties for derivatives

Theorem. *Let $f : (a,b) \to \mathbb{R}$ be differentiable. If $a < x_1 < x_2 < b$ and λ lies between $f'(x_1)$ and $f'(x_2)$, then $\exists\, x_0 \in (x_1, x_2) : f'(x_0) = \lambda$.*

Proof. Assume WLOG that $f'(x_1) < \lambda < f'(x_2)$. Let $g : (a,b) \to \mathbb{R}, g(x) = f(x) - \lambda x$ be differentiable on (a,b). Then g is continuous on (a,b). We want to find $x_0 \in (x_1, x_2) : g'(x_0) = 0$. As g is continuous on $[x_1, x_2]$ compact, it attains its maximum at a point $x_0 \in [x_1, x_2]$. If we can show that $x_0 \notin \{x_1, x_2\}$, then $x_0 \in (x_1, x_2)$ and $g'(x_0) = 0$. Let's show $x_0 \neq x_1$. We have $\lim_{x \to x_1} \frac{g(x)-g(x_1)}{x-x_1} = g'(x_1) = f'(x_1) - \lambda < 0$. Thus $\exists\, \delta > 0$: if $0 < |x - x_1| < \delta$, then $\frac{g(x)-g(x_1)}{x-x_1} < 0$. For $x_1 - \delta < x < x_1$, we get $g(x_1) < g(x)$ so $g(x_1)$ is not a maximum and $x_0 \neq x_1$. Similarly, $x_0 \neq x_2$. □

Theorem. *Let I be an open interval and $f : I \to \mathbb{R}$ be continuous and injective. Then $J = f(I)$ is an interval. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then the inverse $f^{-1} : J \to I$ is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$.*

Proof. As f is injective and continuous, it's strictly monotone. Therefore $f^{-1} : J \to I$ is strictly monotone. As $f^{-1}(J) = I$ is an interval, we have f^{-1} is continuous. Assume WLOG that f is increasing.

Claim. J is open.

Assume, towards a contradiction, that J is not open. Suppose $\inf J \in J$. Then, as $J = f(I), \exists\, a \in I : f(a) = \inf J$. As I is open, $\exists\, \epsilon > 0 : (a - \epsilon, a + \epsilon) \subseteq I$. As f is strictly increasing, we get $\inf J = f(a) > f(a - \frac{\epsilon}{2}) \in J$, contradiction.

Exercise. Consider $\sup J \in J$.

This shows J is open. We know $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = f'(x_0) \neq 0 \implies \lim_{x \to x_0} \frac{x-x_0}{f(x)-f(x_0)} = \frac{1}{f'(x_0)}$. Let $\epsilon > 0$. Then $\exists\, \delta > 0 : 0 < |x - x_0| < \delta \implies |\frac{x-x_0}{f(x)-f(x_0)} - \frac{1}{f'(x_0)}| < \epsilon, f^{-1}$ is continuous at $y_0 \in J$. Then $\exists\, y > 0 : 0 < |y - y_0| < \eta \implies 0 < |f^{-1}(y) - f^{-1}(y_0)| < \delta$. Putting these together, we get $0 < |y - y_0| < \eta \implies |\frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} - \frac{1}{f'(x_0)}| < \epsilon$. Thus $\lim_{y \to y_0} \frac{f^{-1}(y)-f^{-1}(y_0)}{y-y_0} = \frac{1}{f'(x_0)} = (f^{-1})'(y_0)$ □