Mathematics 131AH Lecture

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Functions

- Given two non-empty sets $A, B$ a function $f : A \to B$ is a way of assigning to each element $a \in A$, a unique element in $B$, denoted by $f(a)$.

- The set $A$ is called the domain of $f$, the set $B$ is called the range of $f$. If $A' \subseteq A$ then $f(A') = \{f(a) : a \in A'\} \subseteq B$ is called the image of $A'$ in $B$ under $f$ and $f(A)$ is called the image of $f$.

- If $f(A) = B$, then $f$ is surjective, or onto. If $f(a) = f(a') \Leftrightarrow a = a'$, then $f$ is injective, or one-to-one. If $f$ is injective and surjective, then $f$ is bijective.

- Two functions $f, g : A \to B$ are equal iff $\{(a, f(a)) : a \in A\} = \{(a, g(a)) : a \in A\}$.

Example. $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 2n$ is injective (just divide by 2) but not surjective because it only covers even integers. However, $g : \mathbb{R} \to \mathbb{R}, g(x) = 2x$ is injective (just plug in $\frac{2x + 1}{2}$ to get odd numbers).

Composition

Let $A, B, C \neq \emptyset$ and $f : A \to B$, $g : B \to C$ be functions. The composition of $g$ with $f$ is the function $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$.

Exercise. Let $D \neq \emptyset$, $h : C \to D$ be a function. Show composition is associative.

Proof.

\[
(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)
\]

\[
\square
\]

Remark. Composition need not be commutative. For example, let $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 2n$, $g : \mathbb{Z} \to \mathbb{Z}, g(n) = n + 1$, then

\[
(f \circ g)(n) = f(g(n)) = f(n + 1) = 2(n + 1) \neq (g \circ f)(n) = g(f(n)) = g(2n) = 2n + 1
\]

Inverses

Let $f : A \to B$ be bijective. The inverse of $f$ is $f^{-1} : B \to A$, defined as follows: if $b \in B$, then $f^{-1}(b) = a \in A$, and $a$ is the unique element in $A : f(a) = b$. In particular, $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Exercise. Let $f : A \to B$ and $g : B \to C$ be bijective. Show $g \circ f$ is also bijective, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. Assume $(g \circ f)(a) = (g \circ f)(b)$, i.e. $g(f(a)) = g(f(b))$. Since $g$ is injective, we have $f(a) = f(b)$. Since $f$ is injective, we have $a = b$. Thus $g \circ f$ is injective. Since $g$ is surjective, $\forall c \in C, \exists b \in B : g(b) = c$. And since $f$ is surjective, $\forall b \in B, \exists a \in A : f(a) = b$. So we have $g(b) = g(f(a)) = c$, so $g \circ f$ is surjective. Furthermore, since $(g \circ f)(a) = c$, we have $((g \circ f)^{-1})(c) = a$. Moreover, $(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$. Thus $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. \[\square\]
Proposition on injective functions

**Proposition.** A function \( f : B \rightarrow C \) is injective iff for any set \( A \neq \emptyset \) and any two functions \( g, h : A \rightarrow B \), we have \( f \circ g = f \circ h \implies g = h \).

**Proof.** ” \( \implies \)” Let \( a \in A \), then \( f(g(a)) = f(h(a)) \implies g(a) = h(a) \) because \( f \) is injective.

”\( \Leftarrow \)” Suppose \( f \) isn’t injective, i.e. \( \exists b_1, b_2 \in B : f(b_1) = f(b_2) \) but \( b_1 \neq b_2 \). Let \( A = \{1, 2\} \) and \( g, h : A \rightarrow B \) be functions defined as

\[
\begin{align*}
g(1) &= b_1, \quad g(2) = b_2 \\
h(1) &= h(2) = b_1.
\end{align*}
\]

Then \( g \neq h \), but notice

\[
\begin{align*}
f(g(1)) &= f(b_1) = f(h(1)) \\
f(g(2)) &= f(b_2) = f(b_1) = f(h(2))
\end{align*}
\]

so \( f \circ g = f \circ h \), contradiction. \( \Box \)

Proposition on surjective functions

**Proposition.** A function \( f : A \rightarrow B \) is surjective iff for any set \( C \neq \emptyset \) and any two functions \( g, h : B \rightarrow C \), we have \( g \circ f = h \circ f \implies g = h \).

**Proof.** ” \( \implies \)” Let \( b \in B \), then \( \exists a \in A : f(a) = b \). Then \( (g \circ f)(a) = (h \circ f)(a) \Leftrightarrow g(f(a)) = h(f(a)) \Leftrightarrow g(b) = h(b) \), so \( g = h \).

”\( \Leftarrow \)” Suppose \( f \) isn’t surjective, then \( \exists b_0 \in B : b_0 \notin f(A) \). Let \( C = \{0, 1\} \) and \( g, h : B \rightarrow C \) be functions defined as

\[
\begin{align*}
g(b) &= 0 \quad \forall b \in B \\
h(b) &= \begin{cases}
1 & \text{if } b = b_0 \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
\]

Then \( g \neq h \), but notice

\[
\begin{align*}
(g \circ f)(a) &= g(f(a)) = 0 \quad \forall a \in A \\
(h \circ f)(a) &= h(f(a)) = 0 \quad \forall a \in A
\end{align*}
\]

so \( g \circ f = h \circ f \), contradiction. \( \Box \)

**Definition.** Let \( f : A \rightarrow B \) be a function, \( B' \subseteq B \). The **preimage of \( B' \) in \( A \)** under \( f \) is \( f^{-1}(B') = \{ a \in A : f(a) \in B' \} \). The preimage of a set exists whether or not \( f \) is invertible. In particular, if \( B' \cap f(A) = \emptyset \), then \( f^{-1}(B') = \emptyset \).

**Exercise.** Let \( f : A \rightarrow B \) be a function, \( A_1, A_2 \subseteq A \), and \( B_1, B_2 \subseteq B \). Then show

1. \( f(A_1 \cup A_2) = f(A_1) \cup f(A_2) \)
2. \( f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2) \) and show \( f \) is injective iff the equality holds
3. \( f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2) \)
4. \( f^{-1}(B_1 \cap B_2) \subseteq f^{-1}(B_1) \cap f^{-1}(B_2) \)
Cardinality

Let $A, B$ be two sets. We say that $A$ and $B$ have the same cardinality (or the same cardinal number) if $\exists$ a bijection $f : A \to B$. In this case, we write $A \sim B$.

1. We say $A$ is finite if $A = \emptyset$ or $A \sim \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. If $A = \emptyset$, then the cardinality of $A$ is 0, i.e. $|A| = 0$. If $A \sim \{1, \ldots, n\}$, then the cardinality of $A$ is $n$, i.e. $|A| = n$.

2. An infinite set is a set which is not finite.

3. We say $A$ is countable if $A \sim \mathbb{N}$. In this case, $|A| = \aleph_0$.

4. We say $A$ is at most countable if $A$ is finite or countable.

5. We say $A$ is uncountable if $A$ isn’t at most countable.

Theorem. If $A$ is a finite set and $B \subseteq A$, then $B$ is a finite set.

Proof. Assume $B \neq \emptyset$ (otherwise it’s finite), then $A \neq \emptyset$. As $A$ is finite, $\exists n \in \mathbb{N}, f : A \to \{1, \ldots, n\}$ bijective. Let $b_i \in B : f(b_i) = \min \{f(b) : b \in B \setminus \{b_j : j < i\}\}$. Let $m \in \mathbb{N} : m \leq n, g : B \to \{1, \ldots, m\}$ be a function defined as $g(b_i) = i$. Then $g$ is bijective and so $B$ is finite. □

Remark. Let $A$ be a finite set and $B$ a proper subset of $A$, then $A \not\sim B$. Otherwise, there would exist a bijection between $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$ with $m \leq n$.

Example. 1. $\mathbb{N} \cup \{0, -1, \ldots, -k\} \sim \mathbb{N}$ for any $k \geq 0$

   Proof. Take the bijection $f : \mathbb{N} \cup \{0, -1, \ldots, -k\} \to \mathbb{N}$ defined as $f(n) = n + k + 1$

2. $\mathbb{Z} \sim \mathbb{N}$

   Proof. Take the bijection $f : \mathbb{Z} \to \mathbb{N}$ defined as

   $$f(n) = \begin{cases} 2(n+1) & n \geq 0 \\ -(2n+1) & n < 0. \end{cases}$$

3. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

   Proof. Take the bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as

   $$f(n, m) = \frac{(n+m-2)(n+m-1)}{2} + n$$

(a) We show $f$ is surjective.

   Proof. For $k \geq 1$, let $P(k) : \exists (n, m) \in \mathbb{N} \times \mathbb{N} : k = f(n, m)$.

   • Base step: $f(1, 1) = 1 \implies P(1)$ holds.
   • Inductive step: Let $k \geq 1 : P(k)$ holds. We want to show $P(k+1)$ holds. $\forall m \geq 2$, we have

   $$k+1 = \frac{(n+m-2)(n+m-1)}{2} + n + 1 = \frac{[(n+1) + (m-1) - 2][(n+1) + (m-1) - 1]}{2} + (n+1) = f(n+1, m-1).$$
If \( m = 1 \), then
\[
k + 1 = \frac{(n + m - 2)(n + m - 1)}{2} + n + 1 = \frac{(n + 1 - 2)(n + 1 - 1)}{2} + n + 1 = \frac{(n - 1)n}{2} + n + 1
\]
\[
= \frac{(n - 1)n + 2(n + 1)}{2} = \frac{n^2 - n + 2n + 2}{2} = \frac{n^2 + n + 2}{2} = \frac{n(n + 1)}{2} + 1 = f(1, n + 1).
\]
Thus \( \forall m \geq 1, \exists n \in \mathbb{N} : f(n, m) = k + 1 \), i.e. \( P(k + 1) \) holds. Thus \( f \) is surjective.

(b) We show \( f \) is injective.

**Proof.** Assume \( f(n, m) = f(a, b) \), we want to show \( (n, m) = (a, b) \). Let \( r \in \mathbb{N} \) such that
\[
\frac{(n + m - 2)(n + m - 1)}{2} = \frac{(a + b - 2)(a + b - 1)}{2} + r.
\]
Suppose \( r \neq 0 \). Let \( g : x \mapsto \frac{(x - 2)(x - 1)}{2} \) and \( t \in \mathbb{N} \). Then
\[
|g(x + t) - g(x)| = \frac{(x + t - 2)(x + t - 1) - (x - 2)(x - 1)}{2} = \frac{(x + t)^2 - 3(x + t) + 2 - (x - 2)(x - 1)}{2}
\]
\[
= x^2 + 2tx + t^2 - 3x - 3t + 2 - x^2 - 3x + 2 = \frac{t(t + 2x - 3)}{2} = tx + \frac{t(t - 3)}{2}.
\]
Thus \( |g(x + t) - g(x)| \geq \max \{t, x\} - 1 \). Notice \( f(n, m) = f(a, b) \Rightarrow r = a - n \Rightarrow a = n + r \). Then
\[
r = g(n + m) - g(a + b) \geq \max \{a + b, (n + m) - (a + b)\} - 1
\]
\[
\geq a + b - 1 = (n + r) + b - 1 = r + (n + b - 1) \geq r + 1.
\]
This is a contradiction, thus \( r = 0 \). Then
\[
\frac{(n + m - 2)(n + m - 1)}{2} = \frac{(a + b - 2)(a + b - 1)}{2}.
\]
Then by hypothesis \( n = a \) and we have
\[
a^2 + a(2m - 3) + (m - 2)(m - 1) = a^2 + a(2b - 3) + (b - 2)(b - 1)
\]
\[
2a(m - b) + m^2 - 3m - b^2 + 3b = 0
\]
\[
(m - b)(2a + m + b - 3) = 0
\]
\[
m = b.
\]

**Theorem.** An infinite subset of a countable set is countable.

**Proof.** Let \( A \) be a countable set, then \( A \sim \mathbb{N} \). In particular, \( A = \{a_1, \ldots\} \). Let \( B \subseteq A : B \) is infinite. Consider \( S_1 = \{n \in \mathbb{N} : a_n \in B\} \neq \emptyset \). Let \( k_1 \in \mathbb{N} : k_1 = \min(S_1) \). Define \( g(1) = a_{k_1} \). Proceed inductively. Let \( n \in \mathbb{N} \). Assume we have defined \( g(1) = a_{k_1} \) and \( g(n) = a_{k_n} : g(i) \neq g(j) \forall 1 \leq i \neq j \leq n \). Let \( S_{n+1} = \{n \in \mathbb{N} : a_n \in B \setminus S_n\} \neq \emptyset \). Let \( k_{n+1} = \min(S_{n+1}) > k_n \). Let \( g(n + 1) = a_{k_{n+1}} \).

**Exercise.** Prove \( g \) is bijective.

**Proof.** Assume \( g(n) = g(m) \), i.e. \( a_{k_n} = a_{k_m} \), but since \( g(i) \neq g(j) \forall 1 \leq i \neq j \leq n \), we must have \( n = m \) and thus \( g \) is injective. Let \( a_{k_n} \in A \), then \( k_n = \min(S_n) \) where \( S_n = \{m \in \mathbb{N} : a_m \in B \setminus S_{n-1}\} \). Thus by definition \( \exists n \in \mathbb{N} : g(n) = a_{k_n} \) and \( g \) is surjective.

**Theorem.** An infinite set contains a countable subset.
There are three cases. If $S$ bijective composition

By definition, $f$ is surjective. Since we have proved that $A \sim B$, then $A$ is equivalent to its proper subset $A \setminus \{a_1\}$.

Claim. $f$ is injective.

Proof. Let $a, a' \in A : f(a) = f(a')$. We want to show $a = a'$.

Case 1: If $a \in A \setminus B$, then $f(a) = a$ but $f(a') = f(a)$ so $f(a') = a \in A \setminus B \implies a' \notin B \implies f(a') = a'$ but $f(a') = a$ so $a' = a$.

Case 2: If $a = a_j \in B$ then $f(a) = f(a_j) = a_{j+1}$ but $f(a') = f(a)$ so $f(a') = a_{j+1} \in B \implies a' \in B \implies \exists i \in \mathbb{N} : a' = a_i$ but then $f(a) = f(a') \implies a_{j+1} = a_{i+1}$ and $B$ is countable so $i = j \implies a = a_i = a_j = a'$.

Claim. $f$ is surjective.

Proof. By definition, $f(A \setminus B) = A \setminus B$, $f(B) = B \setminus \{a_1\} \implies f(A) = f(A \setminus B \cup B) = f(A \setminus B) \cup f(B) = A \setminus B \cup (B \setminus \{a_1\}) = A \setminus \{a_1\}$.

**Schröder-Bernstein**

**Theorem.** Assume $\exists$ two injective functions $f : A \to B$ and $g : B \to A$. Then $A \sim B$.

**Example.** $\mathbb{Q} \sim \mathbb{N}$

**Proof.** Let $f : \mathbb{N} \to \mathbb{Q}$ be a function defined as $f(n) = n$, then $f$ is injective. Let $g : \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$ be a function defined as $g(\frac{m}{n}) = (m,n)$, then $g$ is injective. Since we have proved that $\exists$ bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we can find a bijective composition $\mathbb{Q} \to \mathbb{N}$, using Schröder-Bernstein, this proves $\mathbb{Q} \sim \mathbb{N}$.

**Proof.** We will decompose $A, B$ into disjoint sets

$$A = A_1 \cup A_2 \cup A_3 : A_i \cap A_j = \emptyset$$

$$B = B_1 \cup B_2 \cup B_3 : B_i \cap B_j = \emptyset$$

and we will show $A_i \sim B_j \forall 1 \leq i \leq 3$. Let $a \in A$ and consider

$$S_a = \{a, g^{-1}(a), (f^{-1} \circ g^{-1})(a), (g^{-1} \circ f^{-1} \circ g^{-1})(a), \ldots \}.$$ 

There are three cases. If $S_a$ is finite, let $x$ be its last element.

1. $S_a$ is infinite.
2. $S_a$ terminates in $A$, i.e. $x = a$ or $x = (f^{-1} \circ g^{-1} \circ \cdots \circ g^{-1})(a)$ and $g^{-1}(x) = \emptyset$.
3. $S_a$ terminates in $B$, i.e. $x = g^{-1}(a)$ or $x = (g^{-1} \circ f^{-1} \circ \cdots \circ f^{-1})(a)$ and $f^{-1}(x) = \emptyset$. 

$\square$
Let

\[ A_1 = \{ a \in A : S_a \text{ is infinite} \} \]
\[ A_2 = \{ a \in A : S_a \text{ ends in } A \} \]
\[ A_3 = \{ a \in A : S_a \text{ ends in } B \} \]

By construction, \( A = A_1 \cup A_2 \cup A_3 \) and \( A_i \cap A_j = \emptyset \forall i \neq j \).

For \( b \in B \) let \( T_b = \{ b, f^{-1}(b), (g^{-1} \circ f^{-1})(b), \ldots \} \). Similarly, let

\[ B_1 = \{ b \in B : T_b \text{ is infinite} \} \]
\[ B_2 = \{ b \in B : T_b \text{ ends in } B \} \]
\[ B_3 = \{ b \in B : T_b \text{ ends in } A \} \]

By construction, \( B = B_1 \cup B_2 \cup B_3 \) and \( B_i \cap B_j = \emptyset \forall i \neq j \).

Let \( f : A_1 \to B_1, f : A_2 \to B_2, g : B_3 \to A_3 \) be functions defined as bijections. Let \( h : A \to B \) be a function defined as

\[ h = \begin{cases} 
  f & \text{on } A_1 \cup A_2 \\
  (g|_{B_3})^{-1} & \text{on } A_3,
\end{cases} \]

is bijective.

**Claim.** \( h \) is bijective.

\[ \square \]

**Theorem.** If \( A \) is any set, then \( A \) is not equivalent to its power set \( P(A) = \{ B : B \subseteq A \} \).

**Proof.** If \( A = \emptyset \) then \( |A| = 0 \) but \( P(A) = \{ \emptyset \} \) so \( |P(A)| = 1 \) and so \( A \not\sim P(A) \). Assume \( A \neq \emptyset \). Suppose towards a contradiction that \( A \sim P(A) \). Then \( \exists f : A \to P(A) \) a surjective function. Consider \( B = \{ a \in A : a \notin f(a) \} \in P(A) \). As \( f \) is surjective, \( \exists b \in A : f(b) = B \). If \( b \in f(b) \), then since \( f(b) = B \), we have \( b \in B \) and by definition \( b \notin f(b) \). If \( b \notin f(b) \), then by the definition of \( B \), \( b \in B \). But since \( f(b) = B \), we have \( b \in f(b) \). This is a circular contradiction, thus \( A \not\sim P(A) \).

\[ \square \]

**Remark.** \( b \) is like the barber who shaved all people who didn’t shave themselves. Who shaved the barber?

**Theorem.** The interval \( [0, 1] \subseteq \mathbb{R} \) has cardinality \( 2^\aleph_0 \).

**Proof.** Last time we identified \( [0, 1) \) with the set of functions

\[ F = \{ f : \mathbb{N} \to \{0, 1\} : \forall n \geq 1, \exists m > n : f(m) = 0 \} \]

We will show \( F \sim 2^\mathbb{N} = \{ f : \mathbb{N} \to \{0, 1\} \} \). Let \( F_0 : F \to 2^\mathbb{N} \) be a function defined as \( F_0(f) = f \). Then \( F_0 \) is injective but not surjective because the image of \( F_0 \) doesn’t contain functions with finitely many zeroes.

Define \( G : 2^\mathbb{N} \to [0, 1) \) via the following procedure: for \( f \in 2^\mathbb{N} \), define the binary expansion \( G(f) = 0.0f(1)0 \cdots = \sum_{n \geq 1} 2^{-2n} f(n) \).

**Claim.** \( G \) is injective.

**Proof.** Assume \( G(f) = G(g) \) for \( f, g \in 2^\mathbb{N} \). We want to show \( f = g \). Consider \( A = \{ n \geq 1 : f(n) \neq g(n) \} \). If \( A = \emptyset \implies f = g \). Assume towards a contradiction that \( A \neq \emptyset \). Let \( n_0 = \min A \), we have

\[ 0 = G(f) - G(g) = \sum_{n \geq 1} 2^{-2n} f(n) - \sum_{n \geq 1} 2^{-2n} g(n) = \sum_{n \geq 1} 2^{-2n} [f(n) - g(n)] \]
\[ -2^{-2n_0} [f(n_0) - g(n_0)] = \sum_{n \geq n_0 + 1} 2^{-2n} [f(n) - g(n)] \]
\[ 2^{-2n_0} = \left| \sum_{n \geq n_0 + 1} 2^{-2n} [f(n) - g(n)] \right| \leq \sum_{n \geq n_0 + 1} 2^{-2n} |f(n) + |g(n)|| \]
\[ 2^{-2n_0} \leq 2 \cdot 2^{-2(n_0+1)} \sum_{k \geq 0} 2^{-2k} \leq 2 \cdot 2^{-2(n_0+1)} \frac{1}{1 - \frac{1}{4}} = \frac{2}{3} 2^{-2n_0} < 2^{-2n_0} \]
Lemma. If $\delta$ is unbounded then it is called an unbounded metric space.

Proof. Consider $\sim \subseteq \mathbb{R}^n$. Let $(\mathbb{R}^n, d_2)$ be a metric space. Then $(\mathbb{R}^n, d_2)$ is a metric space with Euclidian metric

$$d_2(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

Definition. A metric space $(X, d)$ is called bounded if $\exists M > 0 : d(x, y) \leq M \ \forall x, y \in X$. If $(X, d)$ is not bounded then it is called an unbounded metric space.

Lemma. Let $(X, d)$ be an unbounded metric space. Then $\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$ is a bounded metric on $X$.

Proof. Clearly $\tilde{d}(x, y) \leq 1 \ \forall x, y \in X$. We only need to show $\tilde{d}$ is a metric. Properties 1,2,3 of the metric are easily verified, we will show property 4. The key observation is that $x \mapsto \frac{x}{1 + x} = 1 - \frac{x}{1 + x}$ is an increasing function. Thus, since $d(x, y) \leq d(x, z) + d(z, y)$, we get

$$\tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, z) + d(z, y)}{1 + d(x, z) + d(z, y)} \leq \frac{d(x, z)}{1 + d(x, z)} + \frac{d(z, y)}{1 + d(z, y)} = \tilde{d}(x, z) + \tilde{d}(z, y).$$

Definition. Let $(X, d)$ be a metric space and $\emptyset \neq A \subseteq X$. Consider $D_A = \{d(x, y) : x, y \in A\} \subseteq \mathbb{R}$. If $D_A$ is bounded, then $\sup D_A = \delta(A)$ is called the diameter of $A$. If $D_A$ is unbounded, we define the diameter of $A$ to be $\delta(A) = \infty$.

Example. Let $(\mathbb{R}^n, d_2)$, $B_R(0) = \{x \in \mathbb{R}^n : d_2(x, 0) < R\}$. Then $\delta(B_R(0)) = 2R$.

Definition. Let $(X, d)$ be a metric space and let $\emptyset \neq A, B \subseteq X$. Then the distance between $A$ and $B$ $d(A, B)$ is defined as $\inf \{d(a, b) : a \in A, b \in B\}$.

Remark. The distance between sets is not a metric, i.e. $d(A, B) = 0 \iff A \cap B \neq \emptyset$.

Example. Let $A = (-1, 0)$ and $B = (0, 1)$. Then $d(A, B) = 0$ but $A \cap B = \emptyset$.

Definition. Let $(X, d)$ be a metric space, $\emptyset \neq A \subseteq X$. For all $x \in X$, the distance of $x$ to $A$ is $d(x, A) = \inf \{d(x, a) : a \in A\}$.

Remark. $d(x, A) = 0 \iff x \in A$.

Example. Let $A = (0, 1)$ and $x = 0$. 

This is a contradiction, proving that $G$ is injective. As $[0, 1) \sim F$, $G$ induces an injection from $2^n$ into $F$. By Schröder-Bernstein, $2^n \sim F \sim [0, 1)$. □

Metric spaces

Definition. Let $X$ be a non-empty subset. A metric on $X$ is a map $d : X \times X \to \mathbb{R}$ that satisfies

1. $d(x, y) \geq 0 \ \forall x, y \in X$
2. $d(x, y) = 0$ iff $x = y$
3. $d(x, y) = d(y, x) \ \forall x, y \in X$
4. $d(x, y) \leq d(x, z) + d(z, y) \ \forall x, y, z \in X$

Then $(X, d)$ is called a metric space.

Example. 1. The discrete metric: if $X \neq \emptyset$, let

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

2. $(\mathbb{R}^n, d_2)$ is a metric space with Euclidian metric

$$d_2(x, y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$$

Definition. Let $(X, d)$ be a metric space and $\emptyset \neq A \subseteq X$. Consider $D_A = \{d(x, y) : x, y \in A\} \subseteq \mathbb{R}$. If $D_A$ is bounded, then $\sup D_A = \delta(A)$ is called the diameter of $A$. If $D_A$ is unbounded, we define the diameter of $A$ to be $\delta(A) = \infty$.

Example. Let $(\mathbb{R}^n, d_2)$, $B_R(0) = \{x \in \mathbb{R}^n : d_2(x, 0) < R\}$. Then $\delta(B_R(0)) = 2R$.

Definition. Let $(X, d)$ be a metric space and let $\emptyset \neq A, B \subseteq X$. Then the distance between $A$ and $B$ $d(A, B)$ is defined as $\inf \{d(a, b) : a \in A, b \in B\}$.

Remark. The distance between sets is not a metric, i.e. $d(A, B) = 0 \iff A \cap B \neq \emptyset$.

Example. Let $A = (-1, 0)$ and $B = (0, 1)$. Then $d(A, B) = 0$ but $A \cap B = \emptyset$.

Definition. Let $(X, d)$ be a metric space, $\emptyset \neq A \subseteq X$. For all $x \in X$, the distance of $x$ to $A$ is $d(x, A) = \inf \{d(x, a) : a \in A\}$.

Remark. $d(x, A) = 0 \iff x \in A$.

Example. Let $A = (0, 1)$ and $x = 0$. 

This is a contradiction, proving that $G$ is injective. As $[0, 1) \sim F$, $G$ induces an injection from $2^n$ into $F$. By Schröder-Bernstein, $2^n \sim F \sim [0, 1)$. □
Holder’s inequality

**Theorem.** Let $1 \leq p \leq \infty$ and $q$ be its dual, that is $\frac{1}{p} + \frac{1}{q} = 1$. Note that if $p = 1$ then $q = \infty$ and vice-versa. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then

$$\sum_{k=1}^{n} |x_k y_k| \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}$$

with the convention that if $r = \infty$ then

$$\left( \sum_{k=1}^{n} |x_k|^r \right)^{\frac{1}{r}} = \max_{1 \leq k \leq n} |x_k|.$$

If $p = q = 2$, then this is called the Cauchy-Schwarz inequality.

**Proof.** Assume $p = 1$, then

$$\sum_{k=1}^{n} |x_k y_k| \leq \sum_{k=1}^{n} |x_k| \max_{1 \leq i \leq n} |y_i| \sum_{k=1}^{n} |x_k|$$

Equality holds iff $|y_k|$ is constant. Similarly, one can prove Holder’s inequality if $p = \infty$. Let $1 < p < \infty$. Recall $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \log x$ is concave, that is

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b) \ \forall \ a, b > 0, t \in (0,1)$$

with equality iff $a = b$. This gives

$$\log(ta + (1-t)b) \geq t \log a + (1-t) \log b = \log a^t b^{1-t} \implies a^t b^{1-t} \leq ta + (1-t)b$$

Fix $k \in [1,n]$ and apply the previous inequality with

$$a = \frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p}, \quad b = \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q}, \quad t = \frac{1}{p} \in (0,1)$$

Note $1-t = \frac{1}{q}$. We get

$$a^{\frac{1}{q}} b^{\frac{1}{p}} = \frac{|x_k|}{\left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}}} \frac{|y_k|^q}{\left( \sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}} \leq \frac{1}{p} a + \frac{1}{q} b = \frac{1}{p} \sum_{k=1}^{n} |x_k|^p \frac{1}{q} \sum_{k=1}^{n} |y_k|^q = 1 = \frac{p}{q}$$

Sum over $1 \leq k \leq n$

$$\sum_{k=1}^{n} \frac{|x_k||y_k|}{\left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}} \leq \sum_{k=1}^{n} \frac{|x_k|^p}{p} \frac{1}{q} \sum_{k=1}^{n} |y_k|^q = \frac{p}{q} = 1$$

$$\implies \sum_{k=1}^{n} |x_k y_k| \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}.$$

We know equality holds iff $a = b$, i.e. $\forall \ 1 \leq k \leq n$

$$\frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p} = \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q} \iff |x_k|^p = \frac{\sum_{k=1}^{n} |x_k|^p}{\sum_{k=1}^{n} |y_k|^q} |y_k|^q \iff \exists c \in \mathbb{R} : |x_k|^p = c |y_k|^q.$$  

\textbf{Remark.} The proof extends to sequences of real numbers. More precisely, if $\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \subseteq \mathbb{R}$, then

$$\sum_{n \geq 1} |x_n y_n| \leq \left( \sum_{k \geq 1} |x_k|^p \right)^{\frac{1}{p}} \left( \sum_{k \geq 1} |y_k|^q \right)^{\frac{1}{q}} \forall \ 1 \leq p, q \leq \infty : \frac{1}{p} + \frac{1}{q} = 1.$$
Minkowski

Corollary. Let $1 \leq p \leq \infty$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then

$$
\left( \sum_{k=1}^{n} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}}
$$

with the convention that for $p = \infty$,

$$
\left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} = \max \{|x_k| : 1 \leq k \leq n\}.
$$

Proof. For all $1 \leq k \leq n$ we have $|x_k + y_k| \leq |x_k| + |y_k|$. In particular,

$$
\max_{1 \leq k \leq n} |x_k + y_k| \leq \max_{1 \leq k \leq n} |x_k| + \max_{1 \leq k \leq n} |y_k| \quad p = \infty
$$

$$
\sum_{k=1}^{n} \max |x_k + y_k| \leq \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k| \quad p = 1.
$$

The dual of $p$ is $\frac{p}{p-1}$. Then

$$
\sum_{k=1}^{n} |x_k + y_k|^{p-1} |x_k + y_k| \leq \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}
$$

$$
\sum_{k=1}^{n} |x_k + y_k|^p \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |x_k + y_k|^{p-1} \right)^{\frac{p-1}{p}} + \left( \sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^{n} |x_k + y_k|^{p-1} \right)^{\frac{p-1}{p}}
$$

$$
\Rightarrow \left( \sum_{k=1}^{n} |x_k + y_k|^p \right)^{\frac{1-p}{p}} = \left( \sum_{k=1}^{n} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k=1}^{n} |y_k|^p \right)^{\frac{1}{p}}
$$

Remark. Minkowski for sequences of real numbers becomes

$$
\left( \sum_{k \geq 1} |x_k + y_k|^p \right)^{\frac{1}{p}} \leq \left( \sum_{k \geq 1} |x_k|^p \right)^{\frac{1}{p}} + \left( \sum_{k \geq 1} |y_k|^p \right)^{\frac{1}{p}}
$$

with the obvious modification if $p = \infty$.

Example. Fix $1 \leq p \leq \infty$ and define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ via $d_p(x, y) = (\sum_{k=1}^{n} |x_k - y_k|^p)^{\frac{1}{p}}$ with the convention that $d_\infty(x, y) = \max_{1 \leq k \leq n} |x_k - y_k|$. Then $(\mathbb{R}^n, d_p)$ is a metric space, the triangle inequality following from Minkowski.

Topology

Definition. Let $(X, d)$ be a metric space. A neighbourhhood of a point $a \in X$ is $B_r(a) = \{x \in X : d(a, x) < r\}$ for some $r > 0$.

Example. 1. $(\mathbb{R}^2, d_2)$, $B_1(0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\}$ (circle)

2. $(\mathbb{R}^2, d_1)$, $B_1(0) = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ (rombus)

3. $(\mathbb{R}^2, d_\infty)$, $B_1(0) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}$ (square)

Definition. Let $(X, d)$ be a metric space, $\emptyset \neq A \subseteq X$. A point $a \in A$ is called an interior point of $A$ if $\exists r > 0 : B_r(a) \subseteq A$. The set of all interior points of $A$ is called the interior of $A$ and is denoted by $A$. A set $A$ is open iff $A = A$. 
Example. 1. $\emptyset$

2. $X$

3. $B_r(a) \forall a \in X, r > 0$

Exercise. Let $(X, d)$ be a metric space, $\emptyset \neq A, B \subseteq X$. Then

1. If $A \subseteq B$, then $\hat{A} \subseteq \hat{B}$
2. $\hat{A} \cup \hat{B} \subseteq A \cup B$
3. $\hat{A} \cap \hat{B} = A \cap B$
4. $\hat{A} = \hat{A}$
5. An infinite union of open sets is open.

Proof. 1. Assume $A \subseteq B$. Let $a \in \hat{A}$, then $\exists r > 0 : B_r(a) \subseteq A \subseteq B$. By definition, $a \in \hat{B}$ and thus $\hat{A} \subseteq \hat{B}$.

2. Let $x \in \hat{A} \cup \hat{B}$, then exactly one of the following must be true:
   - $x \in \hat{A} \cap \hat{B}$, then $\exists r_a > 0 : B_{r_a}(x) \subseteq A$ and $\exists r_b > 0 : B_{r_b}(x) \subseteq B$. Let $r = \min \{r_a, r_b\}$. Then $B_r(x) \subseteq A \cap B$. But $A \cap B \subseteq A \cup B$ so $B_r(x) \subseteq A \cup B$.
   - $x \notin \hat{A} \cup \hat{B}$ but $x \in \hat{A} \setminus \hat{B}$, then $\exists r_a > 0 : B_{r_a}(x) \subseteq A \setminus B$. But $A \setminus B \subseteq A \cup B$ so $B_{r_a}(x) \subseteq A \cup B$.
   - $x \notin \hat{A} \cup \hat{B}$ but $x \in \hat{B} \setminus \hat{A}$, then $\exists r_b > 0 : B_{r_b}(x) \subseteq B \setminus A$. But $B \setminus A \subseteq A \cup B$ so $B_{r_b}(x) \subseteq A \cup B$.

   In all cases $\exists r > 0 : B_r(x) \subseteq A \cup B \iff x \in A \cup B$.

3. Let $x \in \hat{A} \cap \hat{B}$, then $\exists r_a > 0 : B_{r_a}(x) \subseteq A$ and $\exists r_b > 0 : B_{r_b}(x) \subseteq B$. Let $r = \min \{r_a, r_b\}$, then $B_r(x) \subseteq A \cap B \implies x \in A \cap B$.

4. $\hat{A} \subseteq \hat{A} \subseteq A$

   - Suppose $A \subseteq \hat{A}$, i.e. $\exists a \in A : a \notin \hat{A}$. Then $\forall r_a > 0$, we would have $B_{r_a}(a) \cap \hat{A} = \emptyset$. Then $\forall b \in B_{r_a}(a)$, we would also have $b \notin \hat{A}$. Thus $\forall r_b > 0$, we would have $B_{r_b}(b) \cap A = \emptyset$. And since $b \in B_{r_b}(b)$, we would have $b \notin A$. But notice $a \in B_{r_a}(a)$ so we can pick $b = a$. But we chose $a$ so that $a \in A$, thus we have a contradiction so $\hat{A} \subseteq \hat{A}$.

   So $\hat{A} = A$.

5. Let $U = \cup_{n \geq 1} A_n : A_n = \hat{A}_n$, then
   $$u \in U = \cup_{n \geq 1} A_n \iff \exists m \geq 1 : u \in A_m = \hat{A}_m \iff \exists r > 0 : u \in B_r(u) \subseteq \cup_{n \geq 1} A_n \iff u \in \cup_{n \geq 1} A_n = \hat{U}$$

Remark. An infinite intersection of open sets needs not be open. Consider the open set $A_n = \left( -\frac{1}{n}, \frac{1}{n} \right)$ and its infinite intersection $\cap_{n \geq 1} A_n = \{0\}$ which is not open.

Exercise. Let $(X, d)$ be a metric space and let $A \subseteq X$. Show that $\hat{A}$ is the largest open set contained in $A$.

Proof. Let $S \subseteq A$ be an open subset of $A$, i.e. an arbitrary point $s \in S$ is an interior point. Since $S$ is a subset of $A$, $s \in A$. Since $s$ is an interior point, $s \in \hat{A}$, by definition. Since $s$ was arbitrary, $S \subseteq \hat{A}$. But since $S$ was an arbitrary open set, $\hat{A}$ must be the largest open set in $A$.

Definition. Let $(X, d)$ be a metric space. A set $A \subseteq X$ is closed if $^cA$ is open.

Example. $\emptyset, X, ^cB_r(x) = \{y \in X : d(x, y) \geq r\}$ are all closed sets.

Proposition. 1. An infinite intersection of closed sets is closed.

2. A finite union of closed sets is closed.
Proof. 1. Let $I$ be an infinite set and $\{A_i\}_{i \in I}$ a collection of closed sets. Then $^c(\cap_{i \in I} A_i) = \cup_{i \in I} ^c A_i$. Since $^c A_i$ is open for all $i$, we showed that the infinite union is also open.

2. Let $A_1, \ldots, A_n$ be closed sets. Then $^c(\cup_{k=1}^n A_k) = \cap_{k=1}^n ^c A_k$. Since $^c A_k$ is open for all $i$, we showed that the finite intersection is also open.

Definition. Let $(X,d)$ be a metric space, $A \subseteq X$.

- A point $a \in X$ is called an adherent point of $A$ if $\forall r > 0$, we have $B_r(a) \cap A \neq \emptyset$.
- The collection of all adherent points of $A$ is called the closure of $A$ and is denoted by $\overline{A}$.
- An adherent point of $A$ is called isolated if $\exists r > 0 : B_r(a) \cap A = \{a\}$.
- If every point in $A$ is isolated, then $A$ is called an isolated set.
- An adherent point $a$ of $A$ that is not isolated is called an accumulation point of $A$.

Remark. 1. $A \subseteq \overline{A}$

2. $\overline{A} = A \cup A'$

3. In $(\mathbb{R},|\cdot|)$, $\mathbb{R}\setminus[-1,1] = (-\infty,-1] \cup [1,\infty)$

4. In $(\mathbb{R}^2,d_2)$, $\mathbb{R}^2\setminus([-1,1] \times \{0\}) = \mathbb{R}^2$

Exercise. Make it rigorous.

Proof. 1. Let $a \in A$, then $\forall r > 0, a \in B_r(a) \Rightarrow a \in B_r(a) \cap A$ so $B_r(a) \cap A \neq \emptyset \Rightarrow a \in \overline{A}$.

2. Let $x \in \overline{A}$, then $\forall r > 0, B_r(x) \cap A \neq \emptyset$. Since $x \in B_r(x)$, either $x \in A$ or $\exists x \neq y \in B_r(x) \cap A$. Then $x \in A \Rightarrow x \in A \cup A'$, and if $\exists x \neq y \in B_r(x) \cap A$, then $B_r(x) \cap (A \setminus \{x\}) \neq \emptyset \Rightarrow x \in A' \Rightarrow x \in A \cup A'$.

Let $x \in A \cup A'$, then either $x \in A \subseteq \overline{A}$ or $x \in A' \subseteq \overline{A} \Rightarrow \forall r > 0, B_r(x) \cap (A \setminus \{x\}) \neq \emptyset \Rightarrow B_r(x) \cap A \neq \emptyset \Rightarrow x \in \overline{A}$.

3. Incomplete

$$x \in \mathbb{R}\setminus[-1,1] \iff \forall r > 0, B_r(x) \cap \mathbb{R}\setminus[-1,1] \neq \emptyset$$

$$\iff B_r(x) \cap \mathbb{R} \cap ^c[-1,1] \neq \emptyset$$

$$\iff B_r(x) \cap \mathbb{R} \cap ((-\infty,-1) \cup (1,\infty)) \neq \emptyset$$

$$\iff B_r(x) \cap ((-\infty,-1) \cup (1,\infty)) \neq \emptyset$$

$$\iff (B_r(x) \cap (-\infty,-1)) \cup (B_r(x) \cap (1,\infty)) \neq \emptyset$$

Then either $B_r(x) \cap (-\infty,-1) \neq \emptyset$ or $B_r(x) \cap (1,\infty) \neq \emptyset$ or both. Then $B_r(x) \cap (-\infty,-1) \neq \emptyset \Rightarrow x \in (-\infty,-1)$

Proposition. Let $(X,d)$ be a metric space, $A \subseteq X$. The following are equivalent:

1. The point $a \in X$ is an accumulation point of $A$.

2. $\exists$ a sequence $\{a_n\}_{n \geq 1} \subseteq A \setminus \{a\} : d(a_n,a) \xrightarrow{n \to \infty} 0$.

3. Every neighbourhood of $A$ contains infinitely many points from $A \setminus \{a\}$.

Proof. We will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

1 $\Rightarrow$ 2. Let $a \in A'$, then $B_1(a) \cap (A \setminus \{a\}) \neq \emptyset$. Let $a_1 \in B_1(a) \cap (A \setminus \{a\})$. Let $r_1 = \min\{\frac{1}{2},d(a,a_1)\}$. As $a \in A'$, $B_r(a) \cap (A \setminus \{a\}) \neq \emptyset$. In particular, $a_2 \neq \{a,a_1\}$ and $d(a,a_2) < \frac{1}{2}$. Proceeding inductively, one constructs a sequence $\{a_n\}_{n \geq 1} : a_{n+1} \notin \{a,a_1,\ldots,a_n\}$ and $d(a_{n+1},a) < \frac{1}{n+1} \xrightarrow{n \to \infty} 0$.

2 $\Rightarrow$ 3. Fix $r > 0$, then $\exists n_r \in \mathbb{N} : d(a_{n_r},a) < r \forall n \geq n_r$. Then $\{a_n : n \geq n_r\} \subseteq B_r(a)$.

3 $\Rightarrow$ 1. Follows from the definition.

$\square$
Proposition. Let \((X,d)\) be a metric space, \(A,B \subseteq X\).

1. \(c(A) = c \overline{\overline{A}}\)
2. \(cA = \ c(\overline{A})\)
3. If \(A \subseteq B\), then \(\overline{A} \subseteq \overline{B}\)
4. \(\overline{A \cap B} = \overline{A} \cap \overline{B}\)
5. \(\overline{A \cup B} = \overline{A} \cup \overline{B}\)
6. \(\overline{\overline{A}} = \overline{A}\)
7. \(A\) is closed iff \(A = \overline{A}\)
8. \(\overline{A}\) is the smallest closed set that contains \(A\)

Proof. 1. Note that inserting \(cA\) for \(A\) in 1. yields 2. Indeed, \(c(cA) = \overline{\overline{A}} \Rightarrow cA = \overline{\overline{A}}\). One shows similarly that 2. implies 1.

2. \(x \in c\overline{A} \iff x \notin \overline{A} \iff \exists r_x > 0 : B_r(x) \cap A = \emptyset \iff \exists r_x > 0 : B_r(x) \subseteq cA \iff x \in cA\)

3. Let \(x \in \overline{A}\). Fix \(r > 0\). Then \(B_r(x) \cap A \neq \emptyset\). But \(A \subseteq B\) so \(B_r(x) \cap B \neq \emptyset\). By definition, \(x \in \overline{B}\).

4. \(A \cap B \subseteq A \Rightarrow \overline{A \cap B} \subseteq \overline{A} \cap \overline{B} \Rightarrow \overline{A} \cap \overline{B} = \overline{A \cap B} \Rightarrow \overline{A} \cap \overline{B} \subseteq \overline{A} \cap \overline{B}\)

5. \(c(A \cup B) = c(A) \cup cB = cA \cap cB = cA \cap cB = c\overline{A} \cup c\overline{B}\)

6. Since \(A \subseteq \overline{A} \Rightarrow \overline{A} \subseteq \overline{A}\). Let \(x \in \overline{A}\). Fix \(r > 0\). Then \(B_r(x) \cap \overline{A} \neq \emptyset\). Let \(a_r \in B_r(x) \cap \overline{A}\) and \(r_1 = d(x, a_r)\). As \(a_r \in \overline{A}\), we have \(B_{r-r_1}(a_r \cap A \neq \emptyset)\). By the triangle inequality, \(B_{r-r_1}(a_r \subseteq B_r(x)\). Thus \(B_r(x) \cap A \neq \emptyset\). By definition, \(x \in \overline{A}\).

7. \(A\) is closed iff \(cA\) is open iff \(cA = c\overline{A} = c\overline{\overline{A}} \iff A = \overline{A}\).

8. Exercise. \(\overline{A}\) is the smallest closed set that contains \(A\).

Proof. Let \(\emptyset \neq D = \overline{D} : A \subseteq D, A \cap cD \neq \emptyset\). Then \(\overline{A} \cap c(D) \neq \emptyset \Rightarrow \overline{A} \cup cD = \overline{A} \cup cD \neq \emptyset\).

Definition. Let \((X,d)\) be a metric space, \(A \subseteq X\). A point \(a \in cA\) is called an exterior point of \(A\). The exterior of \(A\) is \(\text{Ext}(A) = cA = c\overline{A}\).

Remark. 1. \(\text{Ext}(A)\) is an open set

2. \(\text{Ext}(cA) = \hat{A}\)

3. \(\hat{A} \cup \text{Ext}(A)\) need not be \(X\). Indeed, \(c(\hat{A} \cup \text{Ext}(A)) = c(\hat{A}) \cap c\text{Ext}(A) = c\overline{A} \cap \overline{A}\) is called the frontier of \(A\) and is denoted by \(\text{Fr}(A)\).

Example. In \(\mathbb{R}^2, d_2\), let

\[
A = \{ x, y, \in \mathbb{R} : \begin{cases} 
d_2(x, y) \leq r & \text{if } y \geq 0 \\
d_2(x, y) < r & \text{otherwise.} 
\end{cases} \}
\]

Then

\[
\hat{A} = B_r(0) \\
\overline{A} = \{ x \in \mathbb{R}^2 : d_2(x, 0) \leq r \} \\
c\overline{A} = \{ x \in \mathbb{R}^2 : d_2(x, 0) \geq r \} \\
\text{Fr}(A) = \{ x \in \mathbb{R}^2 : d_2(x, 0) = r \}
\]
Definition. The boundary of a set \( A \) is \( \text{Bd}(A) = \text{Fr}(A) \cap A \). Notice \( \text{Fr}(A) = \overline{A} \cap \overline{\overline{A}} = \text{Fr}(\complement A) \).

Proposition. The boundary of a set \( A \) contains no non-empty open sets.

Proof. Assume \( O = \hat{O} \subseteq \text{Bd}(A) \), we want to show \( O = \emptyset \). We have
\[
O \subseteq A \cap \text{Fr}(A) = A \cap (\overline{\overline{A}} \cap A) \cap \overline{\overline{A}} = A \cap \overline{A} = A \cap \overline{\overline{A}} = A \cap \overline{\overline{A}}.
\]
Then \( O \subseteq A \implies \hat{O} \subseteq \overline{A} \) but \( O = \hat{O} \) so \( O \subseteq \overline{A} \). Since we also have \( O \subseteq \complement \overline{A} \), we showed \( O \subseteq \complement \overline{A} = \emptyset \).

Definition. Let \( (X,d) \) be a metric space. A set \( A \subseteq X \) is called dense if \( \overline{A} = X \). A set is called nowhere dense if \( \text{Ext}(A) = X \).

Example. \( (X,d) = (\mathbb{R},||) \), \( A = \mathbb{Q} \) is dense as we have proved previously.

Remark. \( A \) is nowhere dense iff \( \emptyset = \complement (\text{Ext}(A)) = \complement \text{Ext}(A) = \overline{A} \).

Definition. A metric space \( (X,d) \) is called separable if it contains a countable dense set.

Example. \( \mathbb{R}^n \) is separable with \( \mathbb{Q}^n \) being a countable dense set.

Subspaces of metric space

Definition. Let \( (X,d) \) be a metric space, \( Y \subseteq X \). In particular, \( (Y,d|_{Y 	imes Y}) \) is a metric space. We say that a set \( A \subseteq Y \) is open in \( Y \) if \( \exists \ D = \hat{D} \subseteq X : A = Y \cap D \). We say that a set \( A \subseteq Y \) is closed in \( Y \) if \( \exists \ F = \mathcal{F} \subseteq X : A = Y \cap F \).

Example. \( (X,d) = (\mathbb{R},||), Y = (0,1] \). The open sets in \( Y \) are of the form \( (a,b) \) with \( 0 \leq a, b \leq 1 \) and \( (a,1] = Y \cap (a,\infty) \) with \( 0 \leq a < 1 \). Some closed sets in \( Y \) are \( Y \cap \{a\} \forall a \in Y ; \{0,\frac{1}{2}\} = [-\frac{1}{2},\frac{1}{2}] \cap Y \).

Remark. If \( A \subseteq Y \) is open in \( Y \), then \( Y \setminus A \) is closed in \( Y \). Indeed, if \( A \) is open in \( Y \), \( \exists \ D = \hat{D} \subseteq X : A = Y \cap D \). Then \( Y \setminus A = Y \cap \complement A = Y \cap \complement D \) and \( \complement D \) is closed.

Lemma. Let \( (X,d) \) be a metric space, \( Y \subseteq X \). Then \( Y \) is open \( \iff \forall A \subseteq Y \) which is open in \( Y \), we have \( A \) is open in \( X \).

Proof. "\( \implies \)" take \( A = Y \).

"\( \impliedby \)" let \( A \subseteq Y \) be open in \( Y \), then \( \exists \ D = \hat{D} \subseteq X : A = Y \cap D \), both are open so \( A \) is open.

Definition. Let \( (X,d) \) be a metric space, \( A \subseteq Y \subseteq X \). The closure of \( A \) in \( Y \) is \( \overline{A}^Y = \overline{Y \cap Y} \).

Complete metric spaces

Definition. Let \( (X,d) \) be a metric space, \( \{x_n\}_{n \geq 1} \subseteq X \). We say \( \{x_n\}_{n \geq 1} \) converges to a point \( x \in X \) if
\[
\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d(x, x_n) < \epsilon \forall n \geq n_\epsilon.
\]
In this case, we say \( x \) is the limit of \( \{x_n\}_{n \geq 1} \) and we write \( \lim_{n \to \infty} x_n = x \) or \( x_n \xrightarrow{n \to \infty} x \).

Exercise. 1. The limit of a convergent sequence \( \{x_n\}_{n \geq 1} \) in \( (X,d) \) is unique.

2. A sequence \( \{x_n\}_{n \geq 1} \) converges to \( x \in X \) iff each of its subsequences converge to \( x \).

Lemma. Let \( (X,d) \) be a metric space, \( \{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \subseteq X : x_n \xrightarrow{n \to \infty} x, y_n \xrightarrow{n \to \infty} y \). Then \( d(x_n, y_n) \xrightarrow{n \to \infty} d(x,y) \).

Proof.
\[
|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \leq d(y_n, y) + d(x_n, x) \xrightarrow{n \to \infty} 0.
\]
Definition. Let \((X, d)\) be a metric space. A sequence \(\{x_n\}_{n \geq 1} \subseteq X\) is called a **Cauchy sequence** if \(\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d(x_n, x_m) < \epsilon \forall n, m \geq n_\epsilon\).

Remark. A sequence in \(\mathbb{R}\) converges iff it is Cauchy, but this is not true in a general metric space.

Example. • In \((\mathbb{Q}, ||\cdot||)\), let \(x_1 = 3\) and \(x_{n+1} = \frac{x_n}{2} + \frac{1}{n} \forall n \geq 1\). We showed \(\{x_n\}_{n \geq 1}\) converges to \(\sqrt{2}\).

Consequently, \(\{x_n\}_{n \geq 1} \subseteq \mathbb{Q}\) is Cauchy in \(\mathbb{Q}\). However, it does not converge in \((\mathbb{Q}, ||\cdot||)\).

• In \(((0, 1), ||\cdot||)\), let \(x_n = \frac{1}{n} \forall n \geq 2\). Then \(\{x_n\}_{n \geq 1}\) is Cauchy but not convergent.

Definition. A metric space \((X, d)\) is called **complete** if every Cauchy sequence in \(X\) converges in \((X, d)\).

Exercise. 1. Convergent sequences are Cauchy.

2. A Cauchy sequence with a convergent subsequence is convergent.

Definition. Let \((X, d)\) be a metric space. A sequence \(\{k_n\}_{n \geq 1}\) of subsets of \(X\) is called a **nested sequence of closed balls** if

\[ k_n = K_{r_n}(x_n) = \{x \in X : d(x, x_n) \leq r_n\}, \quad k_{n+1} \subseteq k_n \forall n \geq 1. \]

Remark. In a general metric space, it is not true that

\[ B_r(x) = \{y \in X : d(x, y) < r\} = K_r(x) = \{y \in X : d(x, y) \leq r\} \]

Example. Let \(X = (-\infty, 0] \cup \mathbb{N}\) and \(d(x, y) = |x - y|\). Then

\[ B_1(0) = \{x \in X : |x| < 1\} = (-1, 0] \implies \overline{B_1(0)} = [-1, 0] \]

but

\[ K_1(0) = \{x \in X : |x| \leq 1\} = (-1, 0] \cup \{1\}. \]

Theorem. A metric space \((X, d)\) is complete iff for every nested sequence \(\{k_n\}_{n \geq 1}\) of closed balls with \(\delta(k_n) \xrightarrow{n \to \infty} 0\), we have

\[ \bigcap_{n \geq 1} k_n \neq \emptyset. \]

Proof. • ” \(\implies\) ” Let \(\{k_n\}_{n \geq 1}\) be a nested sequence of closed balls with \(\delta(k_n) \xrightarrow{n \to \infty} 0\). Write \(k_n = K_{r_n}(x_n)\).

Then

\[ \delta(k_n) = 2r_n \implies \lim_{n \to \infty} 2r_n = 0 \iff \lim_{n \to \infty} r_n = 0. \]

Claim. \(\{x_n\}_{n \geq 1}\) is a Cauchy sequence.

Let \(\epsilon > 0\), then \(\exists n_\epsilon \in \mathbb{N} : r_n < \frac{\epsilon}{2} \forall n \geq n_\epsilon\). For \(n, m \geq n_\epsilon\) we have \(k_n, k_m \subseteq k_{n_\epsilon}\) and

\[ d(x_n, x_m) \leq d(x_n, x_{n_\epsilon}) + d(x_{n_\epsilon}, x_m) \leq r_n + r_{n_\epsilon} < \epsilon. \]

As \((X, d)\) is complete, \(\exists x \in X : x_n \xrightarrow{n \to \infty} x\). For \(m \geq 1\), we want to show \(x \in k_m\). Note \(\{x_n\}_{n \geq m} \subseteq k_m\). As \(k_m\) is closed, \(x = \lim_{n \to \infty} x_n \in k_m\). Therefore

\[ x \in \bigcap_{m \geq 1} k_m. \]

• ” \(\iff\) ” Let \(\{x_n\}_{n \geq 1}\) be a Cauchy sequence in \(X\). For every \(n \geq 1, \exists N_n \in \mathbb{N} : d(x_k, x_l) < \frac{1}{2^n} \forall k, l \geq N_n\).

Let \(k_1 = N_1\) and for \(n \geq 1\), let \(k_{n+1} > \max\{k_n, N_{n+1}\}\). In particular, \(d(x_m, x_n) < \frac{1}{2^n} \forall m \geq k_n\). Let

\[ k_n = k_{\frac{1}{2^n}}(x_{k_n}). \]

Claim. \(k_{n+1} \subseteq k_n\)

Let \(y \in k_{n+1} \implies d(y, x_{k_{n+1}}) \leq \frac{1}{2^n} \leq k_{n+1}. \) By the triangle inequality

\[ d(y, x_k) \leq d(y, x_{k_{n+1}}) + d(x_{k_{n+1}}, x_k) \leq \frac{1}{2^n} + \frac{1}{2^{n+1}} = \frac{1}{2^n} \implies y \in k_n. \]

Then

\[ \bigcap_{n \geq 1} k_n \neq \emptyset \iff \exists x \in \bigcap_{n \geq 1} k_n \implies d(x, x_k) \leq \frac{1}{2^n} \forall n \geq 1 \implies x_k \xrightarrow{n \to \infty} x. \]

We have proved previously that a Cauchy sequence with a convergent subsequence converges.
Examples of complete metric spaces

1. \((\mathbb{R}, ||)\)

2. **Lemma.** Let \((X, d_1), (Y, d_2)\) be complete metric spaces. Then \(X \times Y\) with the metric

\[
d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}
\]

is a complete metric space.

**Proof.**

**Exercise.** Show \(d\) is a metric.

We show the metric space is complete. Let \(\{(x_n, y_n)\}_{n \geq 1} \subseteq X \times Y\) be Cauchy, then

\[
\forall \epsilon > 0, \exists n_\epsilon \in \mathbb{N} : d((x_n, y_n), (x_m, y_m)) < \epsilon \ \forall \ n, m \geq n_\epsilon
\]

\[
\implies d_1(x_n, x_m) + d_2(y_n, y_m) < \epsilon^2
\]

\[
\implies \begin{cases} d_1(x_n, x_m) < \epsilon \\ d_2(y_n, y_m) < \epsilon \end{cases}
\]

\[
\implies \begin{cases} \{x_n\}_{n \geq 1} \text{ is Cauchy, let } x = \lim_{n \to \infty} x_n \\ \{y_n\}_{n \geq 1} \text{ is Cauchy, let } y = \lim_{n \to \infty} y_n. \end{cases}
\]

**Claim.** \((x_n, y_n) \xrightarrow{d} n \to \infty (x, y)\)

Let \(\epsilon > 0\), then

\[
\exists n_1(\epsilon) \in \mathbb{N} : d_1(x_n, x) < \frac{\epsilon}{2} \ \forall \ n \geq n_1(\epsilon)
\]

\[
\exists n_2(\epsilon) \in \mathbb{N} : d_2(y_n, y) < \frac{\epsilon}{2} \ \forall \ n \geq n_2(\epsilon)
\]

\[
\implies \text{ for } n \geq \max\{n_1(\epsilon), n_2(\epsilon)\},
\]

\[
d((x_n, y_n), (x, y)) = \sqrt{d_1(x_n, x)^2 + d_2(y_n, y)^2} < \sqrt{\left(\frac{n_1(\epsilon)}{2}\right)^2 + \left(\frac{n_2(\epsilon)}{2}\right)^2} < \epsilon.
\]

\[
\square
\]

3. **Corollary.** \((\mathbb{R}^n, d_2)\) is a complete metric space. Recall

\[
d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}.
\]

**Exercise.** Fix \(1 \leq p \leq \infty\) and \(n \in \mathbb{N}\), show that \((\mathbb{R}^n, d_p)\) is complete.

4. Let

\[
l^2 = \{\{x_n\}_{n \geq 1} \subseteq \mathbb{R} : \sum_{n \geq 1} |x_n|^2 < \infty\}
\]

and \(d_2 : l^2 \times l^2 \to \mathbb{R}\) be defined as

\[
d_2(x, y) = \left(\sum |x_n - y_n|^2\right)^{\frac{1}{2}} \ \forall \ x = \{x_n\}_{n \geq 1}, y = \{y_n\}_{n \geq 1}.
\]

**Claim.** \((l^2, d_2)\) is a complete metric space.
Lemma. Let \((X, d)\) be a metric space. A set \(A \subseteq X\) is dense iff \(\forall \emptyset \neq O = \hat{O} \subseteq X\), we have \(A \cap O \neq \emptyset\).

Proof. 
• " \(\implies\) " Assume towards a contradiction that 
\[
\exists \emptyset \neq O = \hat{O} \subseteq X : A \cap O = \emptyset.
\]

Then we reach a contradiction:
\[
O \subseteq \overset{c}{A} \implies \emptyset \neq O = \hat{O} \subseteq \overset{c}{A} = \overset{c}{(A)} = \overset{c}{X} = \emptyset.
\]

• " \(\iff\) " Assume \(\overline{A} \neq X\). Then \(\overset{c}{A} = \overset{c}{(A)} \neq \emptyset \iff \exists B_r(x) \subseteq \overset{c}{A}\) for some \(r > 0\). In particular, \(\emptyset \neq B_r(x)\) is an open set such that \(B_r(x) \cap A = \emptyset\). This is a contradiction.

Baire property

Definition. We say a metric space \((X, d)\) has the Baire property if for every \(\{A_n\}_{n \geq 1} \subseteq P(X) : A_n = \hat{A}_n, \overline{A} = X\), we have \(\cap_{n \geq 1} A_n = X\). Namely, for each countable collection of open dense sets, their intersection is dense.

Lemma. Let \((X, d)\) be a metric space, the following are equivalent

1. \(X\) has the Baire property
2. If \(\{F_n\} \subseteq P(X) : F_n = \overline{F}_n, \hat{F}_n = \emptyset\), then \(\cup_{n \geq 1} F_n = \emptyset\). Namely, the interior of the union of closed sets with empty interior is empty.

Proof. 
• "1. \(\implies\) 2." Let \(F \subseteq X : F = \overline{F} \text{ and } \hat{F} = \emptyset\). Define \(A_n = \overset{c}{F}_n\), then \(A_n = \hat{A}_n\) and \(\overline{A_n} = \overset{c}{F}_n = \hat{A}_n = \emptyset = X\). As \(X\) has the Baire property, \(\cap_{n \geq 1} A_n = X\). But \(\cap_{n \geq 1} \overline{A_n} = \cap_{n \geq 1} \overset{c}{F}_n = \overset{c}{\cap_{n \geq 1} F_n} = \overset{c}{X} = \emptyset\). Thus \(\cup_{n \geq 1} F_n = \emptyset\).

• "2. \(\implies\) 1." Let \(A_n = A_n, \overline{A_n} = X\), we want to show \(\cap_{n \geq 1} A_n = X\). Define \(F_n = \overset{c}{A}_n\), then \(F_n = \overline{F}_n\) and \(\overset{c}{F}_n = \overset{c}{A}_n = \overset{c}{\overline{A}_n} = \overset{c}{X} = \emptyset\). Therefore, \(\cup_{n \geq 1} F_n = \emptyset \implies \cap_{n \geq 1} F_n = X\). But \(\cup_{n \geq 1} A_n = \overset{c}{\cap_{n \geq 1} F_n} = \overset{c}{X} = \emptyset \implies \cap_{n \geq 1} A_n = X\).
Baire category theorem 1

**Theorem.** A complete metric space $(X,d)$ has the Baire property.

**Proof.** Let $A_n = A_n$ and $\overline{A_n} = X$, we want to show $\cap_{n=1} \overline{A_n} = X$. Equivalently, it suffices to show that $\forall \emptyset \neq W = W \subseteq X$, we have $W \cap (\cap_{n=1} A_n) \neq \emptyset$. Fix $\emptyset \neq W = W \subseteq X$.

- As $\overline{A_n} = X$, we have the open set $W \cap A_1 \neq \emptyset \implies \exists B_{r_1}(x_1) \subseteq W \cap A_1$ for some $r_1 > 0$. Let $\rho_1 < \min\{r_1, 1\}$, then $K_{\rho_1}(x_1) \subseteq B_{r_1}(x_1) \subseteq W \cap A_1$.

- As $\overline{A_n} = X$, we have the open set $B_{\rho_1}(x_1) \cap A_2 \neq \emptyset \implies \exists B_{r_2}(x_2) \subseteq B_{\rho_1}(x_1) \cap A_2$ for some $r_2 > 0$. Let $\rho_2 < \min\{r_2, \frac{1}{2}\}$, then $K_{\rho_2}(x_2) \subseteq B_{r_2}(x_2) \subseteq K_{\rho_1}(x_1) \cap A_2$.

Proceed inductively to find a sequence of points $\{x_n\}_{n=1} \subseteq X$ and a sequence of radii $\{\rho_n\} \subseteq (0, \infty) : \rho_n < \frac{1}{n}$ and $K_{\rho_n}(x_n) \subseteq K_{\rho_{n+1}}(x_{n+1}) \cap A_n$, $\forall \ n \geq 1$. As $(X,d)$ is complete, we get $\cap_{n=1} K_{\rho_n}(x_n) \neq \emptyset$. Note $\emptyset \neq \cap_{n=1} K_{\rho_n}(x_n) \subseteq W \cap A_1 \cap (\cap_{n=1} A_n) = W \cap (\cap_{n=1} A_n)$.

**Example.** $\emptyset$ is of first category.

**Corollary.** A metric space $(X,d)$ is of first category if $X = \cup_{n=1} A_n$ with $A_n = \overline{A_n}$, then $\cup_{n=1} A_n = X$. Recall $\cup_{n=1} A_n = X \implies \cup_{n=1} (\overline{A_n}) = X$. So $X$ is of first category if $X = \cup_{n=1} A_n$ with $A_n = \overline{A_n}$ and $\overline{A_n} = \emptyset$.

**Definition.** Let $(X,d)$ be a metric space. A set $A \subseteq X$ is said to be of **first (Baire) category** if it can be written as a countable union of closed, nowhere dense sets. If $A$ is not of the first category, then $A$ is said to be of **second (Baire) category**.

**Remark.** A metric space $(X,d)$ is of first category if $X = \cup_{n=1} A_n$ with $A_n = \overline{A_n}$ and $\cup_{n=1} A_n = X$. Recall $\cup_{n=1} A_n = X \implies \cup_{n=1} (\overline{A_n}) = X$. So $X$ is of first category if $X = \cup_{n=1} A_n$ with $A_n = \overline{A_n}$ and $\overline{A_n} = \emptyset$.

**Theorem.** A complete metric space $(X,d)$ is of second category.

**Proof.** We argue by contradiction. Assume $X$ is of first category, i.e. $X = \cup_{n=1} A_n$, $A_n = \overline{A_n}$, $A_n = \emptyset$. By the previous theorem, $X$ complete $\implies X$ has the Baire property $\implies \cup_{n=1} A_n = \emptyset \iff \hat{X} = \emptyset \iff X = \emptyset$, contradiction.

**Corollary.** $\mathbb{R} \setminus \mathbb{Q}$ is of second category.

**Proof.** Assume towards a contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is of first category, then we can write $\mathbb{R} \setminus \mathbb{Q} = \cup_{n=1} A_n$, $A_n = \overline{A_n}$, $A_n = \emptyset$. Notice $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = (\cup_{n=1} A_n) \cup (\cup_{q \in \mathbb{Q}} \{q\})$ with $\{q\} = \overline{\{q\}}$ and $\{q\} = \emptyset$ so $\mathbb{R}$ is of first category, contradiction.

**Remark.** If $(X,d)$ is complete and $A \subseteq X$ is of first category, then $X \setminus A$ is of second category.

**The Banach-Mazur game**

Imagine we have two players $P_1, P_2$ playing the following game. Let $I_0$ be a closed interval.

- $P_1$ gets dealt a subset $A \subseteq I_0$,
- $P_2$ gets dealt a subset $B \subseteq I_0 \setminus A$.

Then

- $P_1$ chooses a closed interval $I_1 \subseteq I_0$,
- $P_2$ chooses a closed interval $I_2 \subseteq I_1$,
- $\ldots$,
- $P_1$ chooses a closed interval $I_{2n+1} \subseteq I_{2n}$,
- $P_2$ chooses a closed interval $I_{2n+2} \subseteq I_{2n+1}$.
Then $P_1$ wins if $(\cap I_n) \cap A \neq \emptyset$, otherwise $P_2$ wins.

**Question.** Can either player ensure a winning strategy by choosing the intervals wisely, no matter how the opponent plays?

**Answer.** If $A$ is of first (Baire) category, then $P_2$ has a winning strategy. Indeed, assume $A = \cup_{n \geq 1} A_n$, with $A_n = \overline{A_n}, A = \emptyset$. Then $P_2$ needs only choose $I_{2n} \subseteq I_{2n-1} \setminus A_n \forall n \geq 1$. Then $\cap_{n \geq 1} I_n \subseteq I_0 \setminus A$ and so $P_2$ wins.

**Conjecture.** $P_2$ has a winning strategy $\iff A$ is of first category (proved by Banach).

This gives insight into how "small" a set of first category is, namely, it is a set on which even the first player is bound to lose, unless his opponent fails to take advantage of the situation.

**Theorem.** $P_1$ has a winning strategy iff there is an interval $I_1 \subseteq I_0 : I_0 \cap B$ is of first category.

---

### Connected sets

**Definition.** Let $(X, d)$ be a metric space. We say that two sets $A, B \subseteq X$ are **separated** if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

**Remark.** Any two separated sets are disjoint because $A \cap B \subseteq \overline{A} \cap B = \emptyset$. However, two disjoint sets need not be separated.

**Example.** $A = (0, 1), \overline{A} = [0, 1], B = \{1\}$

**Lemma.** Let $(X, d)$ be a metric space, $A, B \subseteq X : d(A, B) > 0$. Then $A, B$ are separated.

**Proof.** We argue by contradiction. Assume $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$. Since $x \in \overline{A}, d(x, A) = \inf \{ d(x, a) : a \in A \} = 0$. But then $d(B, A) = \inf \{ d(b, A) : b \in B \} = 0$, contradiction. Similarly, one shows that $A \cap \overline{B} = \emptyset$.

**Remark.** There are separated sets $A, B$ for which $d(A, B) = 0$.

**Example.** $A = (0, 1), B = (1, 2) \implies \overline{A} \cap B = [0, 1] \cap (1, 2) = \emptyset, A \cap \overline{B} = (0, 1) \cap [1, 2] = \emptyset, d(A, B) = 0$

**Exercise.** Let $(X, d)$ be a metric space, $A, B \subseteq X$ separated. If $A_1 \subseteq A$ and $B_1 \subseteq B$, then $A_1$ and $B_1$ are separated.

**Proposition.** Let $(X, d)$ be a metric space, then

1. Two closed sets are separated iff they’re disjoint.
2. Two open sets are separated iff they’re disjoint.

**Proof.**

1. $\bullet$ " $\implies$ " This is clear.
   
   $\bullet$ " $\iff$ " Let $\overline{A} = A, \overline{B} = B, A \cap B = \emptyset$. Then $\overline{A} \cap B = A \cap B = \emptyset, A \cap \overline{B} = A \cap B = \emptyset$, so $A, B$ are separated.

2. $\bullet$ " $\implies$ " This is clear.
   
   $\bullet$ " $\iff$ " Let $\hat{A} = A, \hat{B} = B, A \cap B = \emptyset$. We want to show $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Assume towards a contradiction that $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Let $x \in \overline{A} \cap B$, since $x \in \overline{B}, \exists r_0 > 0 : B_{r_0}(x) \subseteq B$. Since $\forall x \in \overline{A}, \forall r > 0, B_{r}(x) \subseteq A$. For any $r > r_0, \emptyset \neq B_{r}(x) \cap A \subseteq A \cap B$, contradiction. This shows $\overline{A} \cap B = \emptyset$, one shows similarly that $A \cap \overline{B} = \emptyset$.

**Proposition.** Let $(X, d)$ be a metric space.

1. If a closed set is the union of two separated sets $A, B$, then $A, B$ are closed.
2. If an open set is the union of two separated sets $A, B$, then $A, B$ are open.

**Proof.**

1. Let $F = \overline{F} : F = A \cup B, \overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then

$$\overline{A} = A \cap \overline{F} = A \cap F = A \cap (A \cup B) = (A \cap A) \cup (A \cap B) = A \cap \emptyset = A.$$ 

Similarly, one proves $B$ is closed.
2. Let \( D = \bar{D} : D = A \cup B, A \cap B = A \cap \bar{B} = \emptyset \). We want to show \( \hat{A} = A, \hat{B} = B \). We know \( \hat{A} \subseteq A \).

Let \( a \in A \subseteq D = \hat{D} \implies \exists r_0 > 0 : B_{r_0}(a) \subseteq D = A \cup B \). As \( A \cap \bar{B} = \emptyset \) and \( a \in A, a \notin B \). Thus \( \exists r_1 > 0 : B_{r_1}(a) \cap B = \emptyset \). Then for \( r < \min \{r_0, r_1\}, B_r(a) \subseteq D \setminus B = A \). So \( a \in A \implies A \subseteq \hat{A} \).

Exercise. Is it true that in a metric space \((X,d), B_r(a) \) cannot be written as the union of separated sets?

Definition. Let \((X,d)\) be a metric space, \( A \subseteq X \).

- We say \( A \) is disconnected if it can be written as the union of two non-empty separated sets.
- If \( A \) is not disconnected, then we say \( A \) is connected.

Theorem. Let \((X,d)\) be a metric space. Then \( X \) is connected iff the only subsets of \( X \) that are clopen are \( \emptyset, X \).

Proof. We argue by contradiction. Assume that \( \exists A \subseteq X : \emptyset \neq A \neq X, A \) clopen. Then \( X \setminus A \neq \emptyset \) is both closed and open. As \( A \) and \( X \setminus A \) are disjoint, they are separated. Then \( X = A \cup (X \setminus A) \) and \( A, X \setminus A \neq \emptyset \) separated implies that \( X \) is disconnected, contradiction.

Theorem. Let \((X,d)\) be a metric space, \( A \subseteq X \). Then \( A \) is connected iff the only subsets of \( A \) that are both open and closed in \( A \) are \( \emptyset, X \).

Proof. We argue by contradiction. Assume that \( \exists \emptyset \neq B \subseteq A : B \) is both open and closed. Then \( \emptyset \neq A \setminus B \subseteq A \) is both open and closed. Thus \( A = B \cup (A \setminus B) \).

Claim. \( B, A \setminus B \) are separated.

\( B \) is closed in \( A \) so \( B \cap A = B \cap \bar{B} = B \). Then

\[
\widehat{B} \cap (A \setminus B) = (\widehat{B} \cap A) \cap \bar{c}B = B \cap \bar{c}B = \emptyset.
\]

\( A \setminus B \) is closed in \( A \) so \( A \setminus \bar{B} = A \setminus B \). Then

\[
B \cap (A \setminus B) = B \cap A \cap (A \setminus B) = B \cap A \cap B = \emptyset.
\]

Thus \( A \) can be written as the union of two separated sets, contradiction.

Claim. \( B \) is closed in \( A \).

\[
\widehat{B} \cap A = \widehat{B} \cap (B \cup C) = (\widehat{B} \cap B) \cup (\widehat{B} \cap C) = B \cap \emptyset = B.
\]

Similarly, one shows that if \( C \) is closed in \( A \) then \( B = A \setminus C \) is open in \( A \). Thus \( \emptyset \neq B \subseteq A \) is both open and closed in \( A \), contradiction.

Theorem. Let \((X,d)\) be a metric space. The following are equivalent.

1. \( A \) is disconnected.
2. There exists open sets \( D_1, D_2 : A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, A \cap D_2 \neq \emptyset, A \cap D_1 \cap D_2 = \emptyset \).
3. There exists closed sets \( F_1, F_2 : A \subseteq F_1 \cup F_2, A \cap F_1 \neq \emptyset, A \cap F_2 \neq \emptyset, A \cap F_1 \cap F_2 = \emptyset \).

Proof. We show \( 3 \implies 2 \implies 1 \implies 3 \).
• "3 $\implies$ 2" Let $D_1 = cF_1$, $D_2 = cF_2$. Then $D_1, D_2$ are open. Since $A \subseteq F_1 \cup F_2$, we have

$$A \cap D_1 \cap D_2 = A \cap cF_1 \cap cF_2 = A \cap c(F_1 \cup F_2) = \emptyset.$$  

We know $A \cap F_1 \cap F_2 = \emptyset \implies A \subseteq (cF_1 \cap cF_2) = cF_1 \cup cF_2 = D_1 \cup D_2$. Let’s show $A \cap D_1 \neq \emptyset$. Notice $A \cap D_1 = \emptyset \implies A \subseteq D_2 \implies A \subseteq cF_2 \implies F_2 = \emptyset$, contradiction. Similarly, $A \cap D_2 \neq \emptyset$.

• "2 $\implies$ 1" Let $B = A \cap D_1, C = A \cap D_2$. Then $A = B \cup C, B \neq \emptyset, C \neq \emptyset, B \cap C = \emptyset$. Note that if $B$ and $C$ are open in $A$ then $B$ is closed in $A$. $B \neq \emptyset, B \neq A$ since $A = B \cup C, C \neq \emptyset, B \cap C \neq \emptyset$. Thus $A$ is disconnected.

• "1 $\implies$ 3" As $A$ is disconnected, $\exists \emptyset \neq B \subseteq A : B$ is both open and closed in $A$. In particular, $C = A \setminus B$ is both open and closed in $A$ and $\emptyset \neq C \subseteq A$. Let $F_1, F_2$ be closed sets such that $B = A \cap F_1, C = A \cap F_2$. Then $A = B \cup C \subseteq F_1 \cup F_2, A \cap F_1 = B \neq \emptyset, A \cap F_2 = C \neq \emptyset, A \cap F_1 \cap F_2 = B \cap C = \emptyset$.

\[ \square \]

**Proposition.** Let $(X,d)$ be a metric space, $A \subseteq X$ be disconnected. Let $D_1, D_2$ be open sets such that $A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, A \cap D_2 \neq \emptyset, A \cap D_1 \cap D_2 = \emptyset$. If $B \subseteq A$ connected, then $B \subseteq D_1$ or $B \subseteq D_2$.

**Proof.** Assume towards a contradiction, that $B \cap D_1 \neq \emptyset$ and $B \cap D_2 \neq \emptyset$. Then $B \subseteq A \subseteq D_1 \cup D_2$ and $B \cap D_1 \cap D_2 \subseteq A \cap D_1 \cap D_2 = \emptyset$ implies that $B$ is disconnected, contradiction.

A similar argument yields

**Proposition.** Let $(X,d)$ be a metric space, $A \subseteq X$ be disconnected. Let $F_1, F_2$ be closed sets such that $A \subseteq F_1 \cup F_2, A \cap F_1 \neq \emptyset, A \cap F_2 \neq \emptyset, A \cap F_1 \cap F_2 = \emptyset$. If $B \subseteq A$ is connected, then $B \subseteq F_1$ and $B \subseteq F_2$.

**Proposition.** Let $(X,d)$ be a metric space, $A \subseteq X$ be connected. If $A \subseteq B \subseteq \overline{A}$, then $B$ is connected.

**Proof.** Assume towards a contradiction that $B$ is disconnected. Then $\exists F_1, F_2$ closed subsets of $X : B \subseteq F_1 \cup F_2, B \cap F_1 \neq \emptyset, B \cap F_2 \neq \emptyset, B \cap F_1 \cap F_2 = \emptyset$. Then $A \subseteq B \subseteq F_1 \cup F_2$ implies either $A \subseteq F_1$ or $A \subseteq F_2$. Without loss of generality, assume $A \subseteq F_1$. Then $B \subseteq \overline{A} \subseteq \overline{F_1} = F_1 \implies \emptyset = B \subseteq F_1 \cup F_2 = B \cap F_2 \neq \emptyset$, contradiction.

**Proposition.** Let $(X,d)$ be a metric space, $\{A_i\}_{i \in I}$ be a family of connected subsets of $X$ such that for any $i \neq j$, $A_i$ and $A_j$ are not separated. Then $\bigcup_{i \in I} A_i$ is connected.

**Proof.** Assume towards a contradiction that $\bigcup_{i \in I} A_i$ is disconnected, then $\exists B, C \neq \emptyset : \overline{B} \cap C = B \cap \overline{C} = \emptyset$ and $\bigcup_{i \in I} A_i = B \cup C$. For any $i \in I$, $A_i = (B \cap A_i) \cup (C \cap A_i)$. But $A_i$ is connected while $B \cap A_i$ and $C \cap A_i$ are separated. So either $B \cap A_i = \emptyset$ or $C \cap A_i = \emptyset$. In particular, if $A_i \cap B \neq \emptyset$, then $A \subseteq B$. Then $\bigcup_{i \in I} A_i = B \cup C \implies \exists i_1, i_2 \in I :$

$$A_{i_1} \cap B \neq \emptyset \implies A_{i_1} \subseteq B$$
$$A_{i_2} \cap C \neq \emptyset \implies A_{i_2} \subseteq C$$

But $B, C$ separated $\implies A_{i_1}, A_{i_2}$ separated, contradiction.

**Corollary.** Let $(X,d)$ be a metric space, $\{A_i\}_{i \in I}$ be a family of connected subsets of $X : \bigcap_{i \in I} A_i \neq \emptyset$. Then $\bigcup_{i \in I} A_i$ is connected.

**Theorem.** The only non-empty connected subsets of $\mathbb{R}$ are the intervals. In particular, $\mathbb{R} = (-\infty, \infty)$ is connected so the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset, \mathbb{R}$.

**Proof.** Let’s first show that intervals are connected. Let $I \subseteq \mathbb{R}$ be an interval. Assume towards a contradiction that $I$ is disconnected. Then $\exists \emptyset \neq A \subseteq I : A$ is both open and closed in $I$. Then its complement $\emptyset \neq B = I \setminus A \subseteq I$ is both open and closed in $I$. Let $a_1 \in A, b_1 \in B$.

• Set $c_1 = \frac{a_1 + b_1}{2}$. If $c_1 \in A$, set $a_2 = c_1, b_2 = b_1$. If $c_1 \in B$, set $a_2 = a_1, b_2 = c_1$. In either case, $b_2 - a_2 = \frac{b_1 - a_1}{2}$.

• Set $c_2 = \frac{a_2 + b_2}{2}$. If $c_2 \in A$, set $a_3 = c_2, b_3 = b_2$. If $c_2 \in B$, set $a_3 = a_2, b_3 = c_2$. In either case, $b_3 - a_3 = \frac{b_1 - a_1}{2^2}$.

Proceeding inductively, we construct $\{a_n\} \subseteq A, \{b_n\} \subseteq B$:

- $\{a_n\}$ is non-decreasing and bounded above by $b$ so it converges, let $a = \lim_{n \to \infty} a_n$
- $\{b_n\}$ is non-increasing and bounded below by $a$ so it converges, let $b = \lim_{n \to \infty} b_n$
Thus \( a = b \). But \( a \in \overline{A} \cap I = A \) and \( b \in \overline{B} \cap I = B \) so \( A \cap B \neq \emptyset \), contradiction. Finally, show connected sets are necessarily intervals. Let \( A \subseteq \mathbb{R} \) be connected. Let \( a = \inf A \) (possibly \(-\infty\)). Let \( b = \sup A \) (possibly \(\infty\)).

We have to show that if \( a < c < b \), then \( c \in A \). Assume, towards a contradiction, that \( \exists c \in (a,b) \setminus A \). Set \( D_1 = (-\infty,c), D_2 = (c,\infty) \) open in \( \mathbb{R} \). Then \( A \subseteq D_1 \cup D_2, A \cap D_1 \cap D_2 = \emptyset, A \cap D_1 \neq \emptyset \) (because \( \inf A < c \)), \( A \cap D_2 \neq \emptyset \) (because \( \sup A > c \)). Thus \( A \) is disconnected, contradiction.

\[ b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \xrightarrow{n \to \infty} 0 \]

Lemma. Let \((X,d)\) be a metric space, \( A \subseteq X \). If any pair of points in \( A \) is contained in a connected subset of \( A \), then \( A \) is connected.

**Proof.** Assume towards a contradiction that \( A \) is disconnected. Then \( \exists D_1, D_2 \) open : \( A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, \emptyset \neq A \cap D_1 \cap D_2 = \emptyset \). Let \( a \in A \cap D_1, b \in A \cap D_2 \). Then \( \exists B \subseteq A \) connected : \( \{a,b\} \subseteq B \). Then \( B \subseteq D_1 \cup D_2, B \cap D_1 \neq \emptyset, B \cap D_2 \neq \emptyset, \emptyset = B \cap D_1 \cap D_2 = \emptyset \implies B \) is disconnected, contradiction.

Exercise. Let \((\mathbb{R}^n,d), B_1(0) = \{ x \in \mathbb{R}^n : d(x,0) < 1 \} \). Then \( B_1(0) \) is connected.