Mathematics 131AH Lecture

Damien Lefebvre

Friday 20^{th} March, 2020

Functions

- Given two non-empty sets A,B a **function** $f : A \to B$ is a way of assigning to each element $a \in A$, a unique element in B, denoted by $f(a)$.
- The set A is called the **domain** of f, the set B is called the **range** of f. If $A' \subseteq A$ then $f(A') = \{f(a) : a \in A'\} \subseteq B$ is called the **image of A' in B under f** and $f(A)$ is called the **image of f**.
- If $f(A) = B$, then f is **surjective**, or onto. If $f(a) = f(a') \Leftrightarrow a = a'$, then f is **injective**, or one-to-one. If f is injective and surjective, then f is **bijective**.
- Two functions $f, g : A \to B$ are **equal** iff $\{(a, f(a)) : a \in A\} = \{(a, g(a)) : a \in A\}$.

Example. $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 2n$ is injective (just divide by 2) but not surjective because it only covers even integers. However, $g : \mathbb{R} \to \mathbb{R}, g(x) = 2x$ is bijective (just plug in $\frac{2x+1}{2}$ to get odd numbers).

Composition

Let $A, B, C \neq \emptyset$ and $f : A \to B$, $g : B \to C$ be functions. The **composition** of g with f is the function $g \circ f : A \to C$ given by $(g \circ f)(a) = g(f(a))$.

Exercise. Let $D \neq \emptyset$, $h : C \to D$ be a function. Show composition is associative.

Proof.
$$(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)$$
□

Remark. Composition need not be commutative. For example, let $f : \mathbb{Z} \to \mathbb{Z}, f(n) = 2n, g : \mathbb{Z} \to \mathbb{Z}, g(n) = n + 1$, then
$$(f \circ g)(n) = f(g(n)) = f(n + 1) = 2(n + 1) \neq (g \circ f)(n) = g(f(n)) = g(2n) = 2n + 1$$

Inverses

Let $f : A \to B$ be bijective. The **inverse** of f is $f^{-1} : B \to A$, defined as follows: if $b \in B$, then $f^{-1}(b) = a \in A$, and a is the unique element in $A : f(a) = b$. In particular, $f^{-1} \circ f = 1_A$ and $f \circ f^{-1} = 1_B$.

Exercise. Let $f : A \to B$ and $g : B \to C$ be bijective. Show $g \circ f$ is also bijective, and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. Assume $(g \circ f)(a) = (g \circ f)(b)$, i.e. $g(f(a)) = g(f(b))$. Since g is injective, we have $f(a) = f(b)$. Since f is injective, we have $a = b$. Thus $g \circ f$ is injective. Since g is surjective, $\forall\ c \in C, \exists\ b \in B : g(b) = c$. And since f is surjective, $\forall\ b \in B, \exists\ a \in A : f(a) = b$. So we have $g(b) = g(f(a)) = c$, so $g \circ f$ is surjective. Furthermore, since $(g \circ f)(a) = c$, we have $((g \circ f)^{-1})(c) = a$. Moreover, $(f^{-1} \circ g^{-1})(c) = f^{-1}(g^{-1}(c)) = f^{-1}(b) = a$. Thus $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. □

Proposition on injective functions

Proposition. *A function $f : B \to C$ is injective iff for any set $A \neq \emptyset$ and any two functions $g, h : A \to B$, we have $f \circ g = f \circ h \implies g = h$.*

Proof. " $\implies$ " Let $a \in A$, then $f(g(a)) = f(h(a)) \implies g(a) = h(a)$ because f is injective.

"$\Leftarrow$" Suppose f isn't injective, i.e. $\exists\, b_1, b_2 \in B : f(b_1) = f(b_2)$ but $b_1 \neq b_2$. Let $A = \{1, 2\}$ and $g, h : A \to B$ be functions defined as

$$g(1) = b_1, \quad g(2) = b_2$$
$$h(1) = h(2) = b_1.$$

Then $g \neq h$, but notice

$$f(g(1)) = f(b_1) = f(h(1))$$
$$f(g(2)) = f(b_2) = f(b_1) = f(h(2))$$

so $f \circ g = f \circ h$, contradiction. □

Proposition on surjective functions

Proposition. *A function $f : A \to B$ is surjective iff for any set $C \neq \emptyset$ and any two functions $g, h : B \to C$, we have $g \circ f = h \circ f \implies g = h$.*

Proof. " $\implies$ " Let $b \in B$, then $\exists\, a \in A : f(a) = b$. Then $(g \circ f)(a) = (h \circ f)(a) \Leftrightarrow g(f(a)) = h(f(a)) \Leftrightarrow g(b) = h(b)$, so $g = h$.

"$\Leftarrow$" Suppose f isn't surjective, then $\exists\, b_0 \in B : b_0 \notin f(A)$. Let $C = \{0, 1\}$ and $g, h : B \to C$ be functions defined as

$$g(b) = 0 \;\forall\, b \in B$$
$$h(b) = \begin{cases} 1 & \text{if } b = b_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \neq h$, but notice

$$(g \circ f)(a) = g(f(a)) = 0 \;\forall\, a \in A$$
$$(h \circ f)(a) = h(f(a)) = 0 \;\forall\, a \in A$$

so $g \circ f = h \circ f$, contradiction. □

Definition. Let $f : A \to B$ be a function, $B' \subseteq B$. The **preimage of B' in A under f** is $f^{-1}(B') = \{a \in A : f(a) \in B'\}$. The preimage of a set exists whether or not f is invertible. In particular, if $B' \cap f(A) = \emptyset$, then $f^{-1}(B') = \emptyset$.

Exercise. Let $f : A \to B$ be a function, $A_1, A_2 \subseteq A$, and $B_1, B_2 \subseteq B$. Then show

1. $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$
2. $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and show f is injective iff the equality holds
3. $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$
4. $f^{-1}(B_1 \cap B_2) \nsubseteq f^{-1}(B_1) \cap f^{-1}(B_2)$

Cardinality

Let A, B be two sets. We say that A and B have the same **cardinality** (or the same cardinal number) if $\exists$ a bijection $f : A \to B$. In this case, we write $A \sim B$.

1. We say A is **finite** if $A = \emptyset$ or $A \sim \{1, \dots, n\}$ for some $n \in \mathbb{N}$. If $A = \emptyset$, then the cardinality of A is 0, i.e. $|A| = 0$. If $A \sim \{1, \dots, n\}$, then the cardinality of A is n, i.e. $|A| = n$.
2. An **infinite** set is a set which is not finite.
3. We say A is **countable** if $A \sim \mathbb{N}$. In this case, $|A| = \aleph_0$.
4. We say A is **at most countable** if A is finite or countable.
5. We say A is **uncountable** if A isn't at most countable.

Theorem. *If A is a finite set and $B \subseteq A$, then B is a finite set.*

Proof. Assume $B \neq \emptyset$ (otherwise it's finite), then $A \neq \emptyset$. As A is finite, $\exists\ n \in \mathbb{N}, f : A \to \{1, \dots, n\}$ bijective. Let $b_i \in B : f(b_i) = min\{f(b) : b \in B \setminus \{b_j : j < i\}\}$. Let $m \in \mathbb{N} : m \leq n, g : B \to \{1, \dots, m\}$ be a function defined as $g(b_i) = i$. Then g is bijective and so B is finite. □

Remark. Let A be a finite set and B a proper subset of A, then $A \not\sim B$. Otherwise, there would exist a bijection between $\{1, \dots, m\}$ and $\{1, \dots, n\}$ with $m \leq n$.

Example.

1. $\mathbb{N} \cup \{0, -1, \dots, -k\} \sim \mathbb{N}$ for any $k \geq 0$

   Proof. Take the bijection $f : \mathbb{N} \cup \{0, -1, \dots, -k\} \to \mathbb{N}$ defined as

   $$f(n) = n + k + 1$$

   □

2. $\mathbb{Z} \sim \mathbb{N}$

   Proof. Take the bijection $f : \mathbb{Z} \to \mathbb{N}$ defined as

   $$f(n) = \begin{cases} 2(n+1) & n \geq 0 \\ -(2n+1) & n < 0. \end{cases}$$

   □

3. $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

   Proof. Take the bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as

   $$f(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + n$$

   (a) We show f is surjective.

   Proof. For $k \geq 1$, let $P(k) : \exists\ (n, m) \in \mathbb{N} \times \mathbb{N} : k = f(n, m)$.

   - Base step: $f(1, 1) = 1 \implies P(1)$ holds.
   - Inductive step: Let $k \geq 1 : P(k)$ holds. We want to show $P(k + 1)$ holds. $\forall\ m \geq 2$, we have

   $$\begin{aligned} k + 1 &= \frac{(n + m - 2)(n + m - 1)}{2} + n + 1 \\ &= \frac{[(n + 1) + (m - 1) - 2][(n + 1) + (m - 1) - 1]}{2} + (n + 1) \\ &= f(n + 1, m - 1). \end{aligned}$$

If $m = 1$, then

$$\begin{aligned}k+1 &= \frac{(n+m-2)(n+m-1)}{2} + n + 1 = \frac{(n+1-2)(n+1-1)}{2} + n + 1 = \frac{(n-1)n}{2} + n + 1 \\ &= \frac{(n-1)n + 2(n+1)}{2} = \frac{n^2 - n + 2n + 2}{2} = \frac{n^2 + n + 2}{2} = \frac{n(n+1)}{2} + 1 = f(1, n+1).\end{aligned}$$

Thus $\forall\ m \geq 1, \exists\ n \in \mathbb{N} : f(n,m) = k+1$, i.e. $P(k+1)$ holds. Thus f is surjective.

□

(b) We show f is injective.

Proof. Assume $f(n,m) = f(a,b)$, we want to show $(n,m) = (a,b)$. Let $r \in \mathbb{N}$ such that

$$\frac{(n+m-2)(n+m-1)}{2} = \frac{(a+b-2)(a+b-1)}{2} + r.$$

Suppose $r \neq 0$. Let $g : x \mapsto \frac{(x-2)(x-1)}{2}$ and $t \in \mathbb{N}$. Then

$$\begin{aligned}|g(x+t) - g(x)| &= \frac{(x+t-2)(x+t-1) - (x-2)(x-1)}{2} = \frac{(x+t)^2 - 3(x+t) + 2 - (x-2)(x-1)}{2} \\ &= \frac{x^2 + 2tx + t^2 - 3x - 3t + 2 - x^2 - 3x + 2}{2} = \frac{t(t+2x-3}{2} = tx + \frac{t(t-3)}{2}.\end{aligned}$$

Thus $|g(x+t) - g(x)| \geq \max\{t, x\} - 1$. Notice $f(n,m) = f(a,b) \implies r = a - n \implies a = n + r$. Then

$$\begin{aligned}r = g(n+m) - g(a+b) &\geq \max\{a+b, (n+m) - (a+b)\} - 1 \\ &\geq a + b - 1 = (n+r) + b - 1 = r + (n+b-1) \geq r + 1.\end{aligned}$$

This is a contradiction, thus $r = 0$. Then

$$\frac{(n+m-2)(n+m-1)}{2} = \frac{(a+b-2)(a+b-1)}{2}.$$

Then by hypothesis $n = a$ and we have

$$\begin{aligned}a^2 + a(2m-3) + (m-2)(m-1) &= a^2 + a(2b-3) + (b-2)(b-1) \\ 2a(m-b) + m^2 - 3m - b^2 + 3b &= 0 \\ (m-b)(2a+m+b-3) &= 0 \\ m &= b.\end{aligned}$$

□

□

Theorem. *An infinite subset of a countable set is countable.*

Proof. Let A be a countable set, then $A \sim \mathbb{N}$. In particular, $A = \{a_1, \dots\}$. Let $B \subseteq A : B$ is infinite. Consider $S_1 = \{n \in \mathbb{N} : a_n \in B\} \neq \emptyset$. Let $k_1 \in \mathbb{N} : k_1 = \min(S_1)$. Define $g(1) = a_{k_1}$. Proceed inductively. Let $n \in \mathbb{N}$. Assume we have defined $g(1) = a_{k_1}$ and $g(n) = a_{k_n} : g(i) \neq g(j)\ \forall\ 1 \leq i \neq j \leq n$. Let $S_{n+1} = \{n \in \mathbb{N} : a_n \in B \setminus S_n\} \neq \emptyset$. Let $k_{n+1} = \min(S_{n+1}) > k_n$. Let $g(n+1) = a_{k_{n+1}}$.

Exercise. Prove g is bijective.

Proof. Assume $g(n) = g(m)$, i.e. $a_{k_n} = a_{k_m}$, but since $g(i) \neq g(j)\ \forall\ 1 \leq i \neq j \leq n$, we must have $n = m$ and thus g is injective. Let $a_{k_n} \in A$, then $k_n = \min S_n$ where $S_n = \{m \in \mathbb{N} : a_m \in B \setminus S_{n-1}\}$. Thus by definition $\exists\ n \in \mathbb{N} : g(n) = a_{k_n}$ and g is surjective. □

□

Theorem. *An infinite set contains a countable subset.*

Proof. Let A be an infinite set, then $\exists\ a_1 \in A$. Proceed inductively. Assume we found $a_1, \dots, a_n \in A : a_i \neq a_j\ \forall\ 1 \leq i \neq j \leq n$. Consider $A \setminus \{a_1, \dots, a_n\} \neq \emptyset$, otherwise $A \sim \{1, \dots, n\}$. Let $a_{n+1} \in A \setminus \{a_1, \dots, a_n\}$. Clearly $a_{n+1} \neq a_i\ \forall\ 1 \leq i \leq n$. By mathematical induction, A contains a countable set. ☐

Theorem. *A set is infinite iff it is equivalent to one of its proper subsets.*

Proof. " $\Longleftarrow$ " Let A be a set : $A \sim B\ \forall\ B \subsetneq A$. Then A must be infinite.

" $\Longrightarrow$ " Let A be an infinite subset, B a countable subset of $A : B = \{a_1, a_2, \dots\}$. Consider $A \setminus \{a_1\} \subsetneq A$. Let $f : A \to A \setminus \{a_1\}$ be a function defined as

$$f(a) = \begin{cases} a & \text{if } a \in A \setminus B \\ a_{j+1} & \text{if } a = a_j \in B \end{cases}$$

We want to show f is a bijection to show that A is equivalent to its proper subset $A \setminus \{a_1\}$.

Claim. f is injective.

Proof. Let $a, a' \in A : f(a) = f(a')$. We want to show $a = a'$.

Case 1: If $a \in A \setminus B$, then $f(a) = a$ but $f(a') = f(a)$ so $f(a') = a \in A \setminus B \implies a' \notin B \implies f(a') = a'$ but $f(a') = a$ so $a' = a$.

Case 2: If $a = a_j \in B$ then $f(a) = f(a_j) = a_{j+1}$ but $f(a') = f(a)$ so $f(a') = a_{j+1} \in B \implies a' \in B \implies \exists\ i \in \mathbb{N} : a' = a_i$ but then $f(a) = f(a') \implies a_{j+1} = a_{i+1}$ and B is countable so $i = j \implies a = a_i = a_j = a'$. ☐

Claim. f is surjective.

Proof. By definition, $f(A \setminus B) = A \setminus B$, $f(B) = B \setminus \{a_1\} \implies f(A) = f(A \setminus B \cup B) = f(A \setminus B) \cup f(B) = A \setminus B \cup (B \setminus \{a_1\}) = A \setminus \{a_1\}$. ☐

☐

Schröder-Bernstein

Theorem. *Assume $\exists$ two injective functions $f : A \to B$ and $g : B \to A$. Then $A \sim B$.*

Example. $\mathbb{Q} \sim \mathbb{N}$

Proof. Let $f : \mathbb{N} \to \mathbb{Q}$ be a function defined as $f(n) = n$, then f is injective. Let $g : \mathbb{Q} \to \mathbb{N} \times \mathbb{N}$ be a function defined as $g(\frac{m}{n}) = (m, n)$, then g is injective. Since we have proved that $\exists$ bijection $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$, we can find a bijective composition $\mathbb{Q} \to \mathbb{N}$. Using Schröder-Bernstein, this proves $\mathbb{Q} \sim \mathbb{N}$. ☐

Proof. We will decompose A, B into disjoint sets

$$A = A_1 \cup A_2 \cup A_3 : A_i \cap A_j = \emptyset \text{ if } i \neq j$$
$$B = B_1 \cup B_2 \cup B_3 : B_i \cap B_j = \emptyset \text{ if } i \neq j$$

and we will show $A_i \sim B_j\ \forall\ 1 \leq i \leq 3$. Let $a \in A$ and consider

$$S_a = \{a, g^{-1}(a), (f^{-1} \circ g^{-1})(a), (g^{-1} \circ f^{-1} \circ g^{-1})(a), \dots\}.$$

There are three cases. If S_a is finite, let x be its last element.

1. S_a is infinite.
2. S_a terminates in A, i.e. $x = a$ or $x = (f^{-1} \circ g^{-1} \circ \dots \circ g^{-1})(a)$ and $g^{-1}(x) = \emptyset$.
3. S_a terminates in B, i.e. $x = g^{-1}(a)$ or $x = (g^{-1} \circ f^{-1} \circ \dots \circ f^{-1})(a)$ and $f^{-1}(x) = \emptyset$.

Let

$$\begin{aligned} A_1 &= \{a \in A : S_a \text{ is infinite}\} \\ A_2 &= \{a \in A : S_a \text{ ends in } A\} \\ A_3 &= \{a \in A : S_a \text{ ends in } B\} \end{aligned}$$

By construction, $A = A_1 \cup A_2 \cup A_3$ and $A_i \cap A_j = \emptyset \ \forall \ i \neq j$.

For $b \in B$ let $T_b = \{b, f^{-1}(b), (g^{-1} \circ f^{-1})(b), \dots\}$. Similarly, let

$$\begin{aligned} B_1 &= \{b \in B : T_b \text{ is infinite}\} \\ B_2 &= \{b \in B : T_b \text{ ends in } B\} \\ B_3 &= \{b \in B : T_b \text{ ends in } A\} \end{aligned}$$

By construction, $B = B_1 \cup B_2 \cup B_3$ and $B_i \cap B_j = \emptyset \ \forall \ i \neq j$.

Let $f : A_1 \to B_1, f : A_2 \to B_2, g : B_3 \to A_3$ be functions defined as bijections. Let $h : A \to B$ be a function defined as

$$h = \begin{cases} f & \text{on } A_1 \cup A_2 \\ (g|_{B_3})^{-1} & \text{on } A_3, \end{cases}$$

is bijective.

Claim. h is bijective.

□

Theorem. *If A is any set, then A is not equivalent to its power set $P(A) = \{B : B \subseteq A\}$.*

Proof. If $A = \{\emptyset\}$ then $|A| = 0$ but $P(A) = \{\{\emptyset\}\}$ so $|P(A)| = 1$ and so $A \not\sim P(A)$. Assume $A \neq \emptyset$. Suppose towards a contradiction that $A \sim P(A)$. Then $\exists \ f : A \to P(A)$ a surjective function. Consider $B = \{a \in A : a \notin f(a)\} \in P(A)$. As f is surjective, $\exists \ b \in A : f(b) = B$. If $b \in f(b)$, then since $f(b) = B$, we have $b \in B$ and by definition $b \notin f(b)$. If $b \notin f(b)$, then by the definition of B, $b \in B$. But since $f(b) = B$, we have $b \in f(b)$. This is a circular contradiction, thus $A \not\sim P(A)$. □

Remark. b is like the barber who shaved all people who didn't shave themselves. Who shaved the barber?

Theorem. *The interval $[0, 1) \subseteq \mathbb{R}$ has cardinality $2^{\aleph_0}$.*

Proof. Last time we identified $[0, 1)$ with the set of functions

$$F = \{f : \mathbb{N} \to \{0, 1\} : \ \forall \ n \geq 1, \exists \ m > n : f(m) = 0\}.$$

We will show $F \sim 2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0, 1\}\}$. Let $F_0 : F \to 2^{\mathbb{N}}$ be a function defined as $F_0(f) = f$. Then F_0 is injective but not surjective because the image of F_0 doesn't contain functions with finitely many zeroes. □

Define $G : 2^{\mathbb{N}} \to [0, 1)$ via the following procedure: for $f \in 2^{\mathbb{N}}$, define the binary expansion $G(f) = 0.0f(1)0 \cdots = \sum_{n \geq 1} 2^{-2n} f(n)$.

Claim. G is injective.

Proof. Assume $G(f) = G(g)$ for $f, g \in 2^{\mathbb{N}}$. We want to show $f = g$. Consider $A = \{n \geq 1 : f(n) \neq g(n)\}$. If $A = \emptyset \implies f = g$. Assume towards a contradiction that $A \neq \emptyset$. Let $n_0 = \min A$, we have

$$\begin{aligned} 0 &= G(f) - G(g) \\ &= \sum_{n \geq 1} 2^{-2n} f(n) - \sum_{n \geq 1} 2^{-2n} g(n) \\ &= \sum_{n \geq 1} 2^{-2n} \left[f(n) - g(n)\right] \\ -2^{-2n_0} \left[f(n_0) - g(n_0)\right] &= \sum_{n \geq n_0 + 1} 2^{-2n} \left[f(n) - g(n)\right] \\ 2^{-2n_0} &= \left| \sum_{n \geq n_0 + 1} 2^{-2n} \left[f(n) - g(n)\right] \right| \leq \sum_{n \geq n_0 + 1} 2^{-2n} \left[|f(n)| + |g(n)|\right] \\ 2^{-2n_0} &\leq 2 \cdot 2^{-2(n_0+1)} \sum_{k \geq 0} 2^{-2k} \leq 2 \cdot 2^{-2(n_0+1)} \frac{1}{1 - \frac{1}{4}} = \frac{2}{3} 2^{-2n_0} < 2^{-2n_0} \end{aligned}$$

This is a contradiction, proving that G is injective. As $[0,1) \sim F$, G induces an injection from $2^{\mathbb{N}}$ into F. By Schröder-Bernstein, $2^{\mathbb{N}} \sim F \sim [0,1)$. ☐

Metric spaces

Definition. Let X be a non-empty subset. A **metric** on X is a map $d : X \times X \to \mathbb{R}$ that satisfies

1. $d(x,y) \geq 0 \ \forall \ x, y \in X$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x) \ \forall \ x, y \in X$
4. $d(x,y) \leq d(x,z) + d(z,y) \ \forall \ x, y, z \in X$

Then (X,d) is called a **metric space**.

Example. 1. The **discrete metric**: if $X \neq \emptyset$, let

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

2. $(\mathbb{R}^n, d_2)$ is a metric space with **Euclidian metric**

$$d_2(x,y) = [\sum_{j=1}^{n} |x_j - y_j|^2]^{\frac{1}{2}}$$

Definition. A metric space (X,d) is called **bounded** if $\exists \ M > 0 : d(x,y) \leq M \ \forall \ x, y \in X$. If (X,d) is not bounded then it is called an **unbounded** metric space.

Lemma. *Let (X,d) be an unbounded metric space. Then $d : X \times X \to \mathbb{R}$ given by $\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a bounded metric on X.*

Proof. Clearly $\tilde{d}(x,y) \leq 1 \ \forall \ x, y \in X$. We only need to show $\tilde{d}$ is a metric. Properties 1,2,3 of the metric are easily verified, we will show property 4. The key observation is that $x \mapsto \frac{x}{1+x} = 1 - \frac{1}{1+x}$ is an increasing function. Thus, since $d(x,y) \leq d(x,z) + d(z,y)$, we get

$$\tilde{d}(x,y) = \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} = \tilde{d}(x,z) + \tilde{d}(z,y).$$

☐

Definition. Let (X,d) be a metric space and $\emptyset \neq A \subseteq X$. Consider $D_A = \{d(x,y) : x, y \in A\} \subseteq \mathbb{R}$. If D_A is bounded, then $\sup D_A = \delta(A)$ is called the **diameter of** A. If D_A is unbounded, we define the diameter of A to be $\delta(A) = \infty$.

Example. Let $(\mathbb{R}^n, d_2), B_R(0) = \{x \in \mathbb{R}^n : d_2(x,0) < R\}$. Then $\delta(B_R(0)) = 2R$.

Definition. Let (X,d) be a metric space and let $\emptyset \neq A, B \subseteq X$. Then the **distance between** A **and** B $d(A,B)$ is defined as $\inf \{d(a,b) : a \in A, b \in B\}$.

Remark. The distance between sets is not a metric, i.e. $d(A,B) = 0 \ \not\Longrightarrow \ A \cap B \neq \emptyset$.

Example. Let $A = (-1,0)$ and $B = (0,1)$. Then $d(A,B) = 0$ but $A \cap B = \emptyset$.

Definition. Let (X,d) be a metric space, $\emptyset \neq A \subseteq X$. For all $x \in X$, the **distance of** x **to A** is $d(x,A) = \inf \{d(x,a) : a \in A\}$.

Remark. $d(x,A) = 0 \ \not\Longrightarrow \ x \in A$

Example. Let $A = (0,1)$ and $x = 0$.

Holder's inequality

Theorem. *Let $1 \leq p \leq \infty$ and q be its dual, that is $\frac{1}{p} + \frac{1}{q} = 1$. Note that if $p = 1$ then $q = \infty$ and vice-versa. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, then*

$$\sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}$$

with the convention that if $r = \infty$ then

$$\left(\sum_{k=1}^{n} |x_k|^r \right)^{\frac{1}{r}} = \max_{1 \leq k \leq n} |x_k|.$$

If $p = q = 2$, then this is called the Cauchy-Schwarz inequality.

Proof. Assume $p = 1$, then

$$\sum_{k=1}^{n} |x_k y_k| \leq \sum_{k=1}^{n} |x_k| \max |y_l| = \max_{1 \leq l \leq n} |y_l| \sum_{k=1}^{n} |x_k|$$

Equality holds iff $|y_k|$ is constant. Similarly, one can prove Holder's inequality if $p = \infty$. Let $1 < p < \infty$. Recall $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = \log x$ is concave, that is

$$f(ta + (1-t)b) \geq tf(a) + (1-t)f(b) \ \forall \ a, b > 0, t \in (0,1)$$

with equality iff $a = b$. This gives

$$\log (ta + (1-t)b) \geq t \log a + (1-t) \log b = \log a^t b^{1-t} \implies a^t b^{1-t} \leq ta + (1-t)b$$

Fix $k \in [1, n]$ and apply the previous inequality with

$$a = \frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p}, \quad b = \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q}, \quad t = \frac{1}{p} \in (0,1)$$

Note $1 - t = \frac{1}{q}$. We get

$$a^{\frac{1}{p}} b^{\frac{1}{q}} = \frac{|x_k|}{\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}} \frac{|y_k|^q}{\left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}} \leq \frac{1}{p} a + \frac{1}{q} b = \frac{1}{p} \frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p} \frac{1}{q} \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q}.$$

Sum over $1 \leq k \leq n$

$$\sum_{k=1}^{n} \frac{|x_k||y_k|}{\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q\right)^{\frac{1}{q}}} \leq \sum_{k=1}^{n} \frac{1}{p} \frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p} \frac{1}{q} \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q} = \frac{1}{p} + \frac{1}{q} = 1$$

$$\implies \sum_{k=1}^{n} |x_k y_k| \leq \left(\sum_{k=1}^{n} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |y_k|^q \right)^{\frac{1}{q}}.$$

We know equality holds iff $a = b$, i.e. $\forall \ 1 \leq k \leq n$

$$\frac{|x_k|^p}{\sum_{k=1}^{n} |x_k|^p} = \frac{|y_k|^q}{\sum_{k=1}^{n} |y_k|^q} \Leftrightarrow |x_k|^p = \frac{\sum_{k=1}^{n} |x_k|^p}{\sum_{k=1}^{n} |y_k|^q} |y_k|^q \Leftrightarrow \exists \ c \in \mathbb{R} : |x_k|^p = c|y_k|^q.$$

□

Remark. The proof extends to sequences of real numbers. More precisely, if $\{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \subseteq \mathbb{R}$, then

$$\sum_{n \geq 1} |x_n y_n| \leq \left(\sum_{k \geq 1} |x_k|^p \right)^{\frac{1}{p}} \left(\sum_{k \geq 1} |y_k|^q \right)^{\frac{1}{q}} \quad \forall \ 1 \leq p, q \leq \infty : \frac{1}{p} + \frac{1}{q} = 1.$$

Minkowski

Corollary. *Let* $1 \le p \le \infty,\ x = (x_1, \dots, x_n),\ y = (y_1, \dots, y_n) \in \mathbb{R}^n$. *Then*

$$\left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}$$

with the convention that for $p = \infty$,

$$\left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} = \max\{|x_k| : 1 \le k \le n\}.$$

Proof. For all $1 \le k \le n$ we have $|x_k + y_k| \le |x_k| + |y_k|$. In particular,

$$\max_{1\le k\le n} |x_k + y_k| \le \max_{1\le k\le n} |x_k| + \max_{1\le k\le n} |y_k| \quad p = \infty$$

$$\sum_{k=1}^{n} \max |x_k + y_k| \le \sum_{k=1}^{n} |x_k| + \sum_{k=1}^{n} |y_k| \quad p = 1.$$

The dual of p is $\frac{p}{p-1}$. Then

$$\sum_{k=1}^{n} |x_k + y_k|^{p-1} |x_k + y_k| \le \sum_{k=1}^{n} |x_k||x_k + y_k|^{p-1} + \sum_{k=1}^{n} |y_k||x_k + y_k|^{p-1}$$

$$\sum_{k=1}^{n} |x_k + y_k|^p \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |x_k + y_k|^{p-1\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |x_k + y_k|^{p-1\frac{p}{p-1}}\right)^{\frac{p-1}{p}}$$

$$\implies \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{1-\frac{p-1}{p}} = \left(\sum_{k=1}^{n} |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |y_k|^p\right)^{\frac{1}{p}}$$

□

Remark. Minkowski for sequences of real numbers becomes

$$\left(\sum_{k\ge1} |x_k + y_k|^p\right)^{\frac{1}{p}} \le \left(\sum_{k\ge1} |x_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k\ge1} |y_k|^p\right)^{\frac{1}{p}}$$

with the obvious modification if $p = \infty$.

Example. Fix $1 \le p \le \infty$ and define $d_p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ via $d_p(x, y) = (\sum_{k=1}^{n} |x_k - y_k|^p)^{\frac{1}{p}}$ with the convention that $d_\infty(x, y) = \max_{1\le k\le n} |x_k - y_k|$. Then $(\mathbb{R}^n, d_p)$ is a metric space, the triangle inequality following from Minkowski.

Topology

Definition. Let (X, d) be a metric space. A **neighbourhood of a point** $a \in X$ is $B_r(a) = \{x \in X : d(a, x) < r\}$ for some $r > 0$.

Example.
1. $(\mathbb{R}^2, d_2),\ B_1(0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < 1\}$ (circle)
2. $(\mathbb{R}^2, d_1),\ B_1(0) = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ (rombus)
3. $(\mathbb{R}^2, d_\infty),\ B_1(0) = \{(x, y) \in \mathbb{R}^2 : \max\{|x|, |y|\} < 1\}$ (square)

Definition. Let (X, d) be a metric space, $\emptyset \ne A \subseteq X$. A point $a \in A$ is called an **interior point of** A if $\exists\, r > 0 : B_r(a) \subseteq A$. The set of all interior points of A is called the **interior of** A and is denoted by $\mathring{A}$. A set A is **open** iff $\mathring{A} = A$.

Example. 1. $\emptyset$

2. X

3. $B_r(a)\ \forall\ a \in X, r > 0$

Exercise. Let (X, d) be a metric space, $\emptyset \neq A, B \subseteq X$. Then

1. If $A \subseteq B$, then $\mathring{A} \subseteq \mathring{B}$

2. $\mathring{A} \cup \mathring{B} \subseteq A \mathring{\cup} B$

3. $\mathring{A} \cap \mathring{B} = A \mathring{\cap} B$

4. $\mathring{\mathring{A}} = \mathring{A}$

5. An infinite union of open sets is open.

Proof. 1. Assume $A \subseteq B$. Let $a \in \mathring{A}$, then $\exists\ r > 0 : B_r(a) \subseteq A \subseteq B$. By definition, $a \in \mathring{B}$ and thus $\mathring{A} \subseteq \mathring{B}$.

2. Let $x \in \mathring{A} \cup \mathring{B}$, then exactly one of the following must be true:

   - $x \in \mathring{A} \cap \mathring{B}$, then $\exists\ r_a > 0 : B_{r_a}(x) \in A$ and $\exists\ r_b > 0 : B_{r_b}(x) \subseteq B$. Let $r = \min\{r_a, r_b\}$. Then $B_r(x) \subseteq A \cap B$. But $A \cap B \subseteq A \cup B$ so $B_r(x) \subseteq A \cup B$.
   - $x \notin \mathring{A} \cap \mathring{B}$ but $x \in \mathring{A} \setminus \mathring{B}$, then $\exists\ r_a > 0 : B_{r_a}(x) \subseteq A \setminus B$. But $A \setminus B \subseteq A \cup B$ so $B_{r_a}(x) \subseteq A \cup B$.
   - $x \notin \mathring{A} \cap \mathring{B}$ but $x \in \mathring{B} \setminus \mathring{A}$, then $\exists\ r_b > 0 : B_{r_b}(x) \in B \setminus A$. But $B \setminus A \subseteq A \cup B$ so $B_{r_b}(x) \subseteq A \cup B$.

   In all cases $\exists\ r > 0 : B_r(x) \subseteq A \cup B \iff x \in A \mathring{\cup} B$.

3. Let $x \in \mathring{A} \cap \mathring{B}$, then $\exists\ r_a > 0 : B_{r_a}(x) \subseteq A$ and $\exists\ r_b > 0 : B_{r_b}(x) \subseteq B$. Let $r = \min\{r_a, r_b\}$, then $B_r(x) \subseteq A \cap B \implies x \in A \mathring{\cap} B$.

4.
   - $\mathring{\mathring{A}} \subseteq \mathring{A} \subseteq A$
   - Suppose $A \not\subseteq \mathring{\mathring{A}}$, i.e. $\exists\ a \in A : a \notin \mathring{\mathring{A}}$. Then $\forall\ r_a > 0$, we would have $B_{r_a}(a) \cap \mathring{A} = \emptyset$. Then $\forall\ b \in B_{r_a}(a)$, we would also have $b \notin \mathring{A}$. Thus $\forall\ r_b > 0$, we would have $B_{r_b}(b) \cap A = \emptyset$. And since $b \in B_{r_b}(b)$, we would have $b \notin A$. But notice $a \in B_{r_a}(a)$ so we can pick $b = a$. But we chose a so that $a \in A$, thus we have a contradiction so $A \subseteq \mathring{\mathring{A}}$.

   So $\mathring{\mathring{A}} = A$.

5. Let $U = \cup_{n \geq 1} A_n : A_n = \mathring{A}_n$, then

$$u \in U = \cup_{n\geq 1} A_n \iff \exists\ m \geq 1 : u \in A_m = \mathring{A_m} \iff \exists\ r > 0 : u \in B_r(u) \subseteq \cup_{n\geq 1} A_n \iff u \in \mathring{\cup_{n\geq 1} A_n} = \mathring{U}$$

□

Remark. An infinite intersection of open sets needs not be open. Consider the open set $A_n = (-\frac{1}{n}, \frac{1}{n})$ and its infinite intersection $\cap_{n\geq 1} A_n = \{0\}$ which is not open.

Exercise. Let (X, d) be a metric space and let $A \subseteq X$. Show that $\mathring{A}$ is the largest open set contained in A.

Proof. Let $S \subseteq A$ be an open subset of A, i.e. an arbitrary point $s \in S$ is an interior point. Since S is a subset of A, $s \in A$. Since s is an interior point, $s \in \mathring{A}$, by definition. Since s was arbitrary, $S \subseteq \mathring{A}$. But since S was an arbitrary open set, $\mathring{A}$ must be the largest open set in A. □

Definition. Let (X, d) be a metric space. A set $A \subseteq X$ is **closed** if ${}^cA$ is open.

Example. $\emptyset$, X, ${}^cB_r(x) = \{y \in X : d(x, y) \geq r\}$ are all closed sets.

Proposition. *1. An infinite intersection of closed sets is closed.*

2. A finite union of closed sets is closed.

Proof. 1. Let I be an infinite set and $\{A_i\}_{i\in I}$ a collection of closed sets. Then ${}^c\left(\cap_{i\in I} A_i\right) = \cup_{i\in I}\ {}^cA_i$. Since ${}^cA_i$ is open for all i, we showed that the infinite union is also open.

2. Let $A_1, \dots, A_n$ be closed sets. Then ${}^c\left(\cup_{k=1}^n A_k\right) = \cap_{k=1}^n\ {}^cA_k$. Since ${}^cA_k$ is open for all i, we showed that the finite intersection is also open.

□

Definition. Let (X, d) be a metric space, $A \subseteq X$.

- A point $a \in X$ is called an **adherent point of** A if $\forall\ r > 0$, we have $B_r(a) \cap A \neq \emptyset$.
- The collection of all adherent points of A is called the **closure of** A and is denoted by $\overline{A}$.
- An adherent point of A is called **isolated** if $\exists\ r > 0 : B_r(a) \cap A = \{a\}$.
- If every point in A is isolated, then A is called an **isolated set**.
- An adherent point a of A that is not isolated is called an **accumulation point of** A.
- The collection of accumulation points of A is denoted by $A' = \{a \in X : \forall\ r > 0, B_r(a) \cap (A \setminus \{a\}) \neq \emptyset\}$.

Remark. 1. $A \subseteq \overline{A}$

2. $\overline{A} = A \cup A'$

3. In $(\mathbb{R}, ||)$, $\overline{\mathbb{R} \setminus [-1, 1]} = (-\infty, -1] \cup [1, \infty)$

4. In $(\mathbb{R}^2, d_2)$, $\overline{\mathbb{R}^2 \setminus ([-1, 1] \times \{0\}} = \mathbb{R}^2$

Exercise. Make it rigorous.

Proof. 1. Let $a \in A$, then $\forall\ r > 0, a \in B_r(a) \implies a \in B_r(a) \cap A$ so $B_r(a) \cap A \neq \emptyset \implies a \in \overline{A}$.

2.
   - Let $x \in \overline{A}$, then $\forall\ r > 0, B_r(x) \cap A \neq \emptyset$. Since $x \in B_r(x)$, either $x \in A$ or $\exists\ x \neq y \in B_r(x) \cap A$. Then $x \in A \implies x \in A \cup A'$, and if $\exists\ x \neq y \in B_r(x) \cap A$, then $B_r(x) \cap (A \setminus \{x\}) \neq \emptyset \implies x \in A' \implies x \in A \cup A'$.
   - Let $x \in A \cup A'$, then either $x \in A \subseteq \overline{A}$ or $x \in A' \cap\ {}^cA \implies \forall\ r > 0, B_r(x) \cap (A \setminus \{x\}) \neq \emptyset \implies B_r(x) \cap A \neq \emptyset \implies x \in \overline{A}$.

3. Incomplete

$$
\begin{aligned}
x \in \overline{\mathbb{R} \setminus [-1, 1]} &\iff \forall\ r > 0, B_r(x) \cap \mathbb{R} \setminus [-1, 1] \neq \emptyset \\
&\iff B_r(x) \cap \mathbb{R} \cap\ {}^c[-1, 1] \neq \emptyset \\
&\iff B_r(x) \cap \mathbb{R} \cap ((-\infty, -1) \cup (1, \infty)) \neq \emptyset \\
&\iff B_r(x) \cap ((-\infty, -1) \cup (1, \infty)) \neq \emptyset \\
&\iff (B_r(x) \cap (-\infty, -1)) \cup (B_r(x) \cap (1, \infty)) \neq \emptyset
\end{aligned}
$$

Then either $B_r(x) \cap (-\infty, -1) \neq \emptyset$ or $B_r(x) \cap (1, \infty) \neq \emptyset$ or both. Then $B_r(x) \cap (-\infty, -1) \neq \emptyset \implies x \in \overline{(-\infty, -1)}$

□

Proposition. *Let (X, d) be a metric space, $A \subseteq X$. The following are equivalent:*

1. *The point $a \in X$ is an accumulation point of A.*
2. *$\exists$ a sequence $\{a_n\}_{n\geq 1} \subseteq A \setminus \{a\} : d(a_n, a) \xrightarrow[n\to\infty]{} 0$.*
3. *Every neighbourhood of A contains infinitely many points from $A \setminus \{a\}$.*

Proof. We will show $1 \implies 2 \implies 3 \implies 1$.

$1 \implies 2$. Let $a \in A' \implies B_1(a) \cap (A \setminus \{a\}) \neq \emptyset$. Let $a_1 \in B_1(a) \cap (A \setminus \{a\})$. Let $r_1 = \min\{\frac{1}{2}, d(a, a_1)\}$. As $a \in A', B_r(a) \cap (A \setminus \{a\} \neq \emptyset$. In particular, $a_2 \neq \{a, a_1\}$ and $d(a, a_2) < \frac{1}{2}$. Proceeding inductively, one constructs a sequence $\{a_n\}_{n\geq 1} : a_{n+1} \notin \{a, a_1, \dots, a_n$ and $d(a_{n+1}, a) < \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0$.

$2 \implies 3$. Fix $r > 0$, then $\exists\ n_r \in \mathbb{N} : d(a_n, a) < r\ \forall\ n \geq n_r$. Then $\{a_n : n \geq n_r\} \subseteq B_r(a)$.

$3 \implies 1$. Follows from the definition. □

Proposition. *Let (X, d) be a metric space, $A, B \subseteq X$.*

1. ${}^c(\mathring{A}) = \overline{{}^cA}$
2. $\overset{\circ}{{}^cA} = {}^c(\overline{A})$
3. *If* $A \subseteq B$, *then* $\overline{A} \subseteq \overline{B}$
4. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
5. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
6. $\overline{\overline{A}} = \overline{A}$
7. *A is closed iff* $A = \overline{A}$
8. *$\overline{A}$ is the smallest closed set that contains A*

Proof. 1. Note that inserting ${}^cA$ for A in 1. yields 2. Indeed, ${}^c(\overset{\circ}{{}^cA}) = \overline{A} \implies \overset{\circ}{{}^cA} = {}^c(\overline{A})$. One shows similarly that 2. implies 1.

2. $x \in {}^c\overline{A} \iff x \notin \overline{A} \iff \exists\, r_x > 0 : B_{r_x}(x) \cap A = \emptyset \iff \exists\, r_x > 0 : B_{r_x}(x) \subseteq {}^cA \iff x \in \overset{\circ}{{}^cA}$

3. Let $x \in \overline{A}$. Fix $r > 0$. Then $B_r(x) \cap A \neq \emptyset$. But $A \subseteq B$ so $B_r(x) \cap B \neq \emptyset$. By definition, $x \in \overline{B}$.

4. $A \cap B \subseteq A \implies \overline{A \cap B} \subseteq \overline{A}, A \cap B \subseteq B \implies \overline{A \cap B} \subseteq \overline{B} \implies \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

5. ${}^c(\overline{A \cup B}) = \overset{\circ}{{}^c(A \cup B)} = \overset{\circ}{{}^cA \cap {}^cB} = \overset{\circ}{{}^cA} \cap \overset{\circ}{{}^cB} = {}^c(\overline{A}) \cap {}^c(\overline{B}) = {}^c(\overline{A} \cup \overline{B})$

6. Since $A \subseteq \overline{A} \implies \overline{A} \subseteq \overline{\overline{A}}$. Let's prove $\overline{\overline{A}} \subseteq \overline{A}$. Let $x \in \overline{\overline{A}}$. Fix $r > 0$. Then $B_r(x) \cap \overline{A} \neq \emptyset$. Let $a_r \in B_r(x) \cap \overline{A}$ and $r_1 = d(x, a_r)$. As $a_r \in \overline{A}$, we have $B_{r-r_1}(a_r \cap A \neq \emptyset)$. By the triangle inequality, $B_{r-r_1}(a_r \subseteq B_r(x)$. Thus $B_r(x) \cap A \neq \emptyset$. By definition, $x \in \overline{A}$.

7. A is closed iff ${}^cA$ is open iff ${}^cA = \overset{\circ}{{}^cA} = {}^c(\overline{A}) \Leftrightarrow A = \overline{A}$.

8.

   Exercise. $\overline{A}$ is the smallest closed set that contains A.

   Proof. Let $\emptyset \neq D = \overline{D} : A \subseteq D, \overline{A} \cap {}^cD \neq \emptyset$. Then $\overline{A} \cap {}^c(\overline{D}) \neq \emptyset \implies \overline{A \cup {}^cD} = \overline{A} \cup {}^c\overline{D} \neq \emptyset$. ☐

☐

Definition. Let (X, d) be a metric space, $A \subseteq X$. A point $a \in \overset{\circ}{{}^cA}$ is called an **exterior point of** A. The **exterior of** A is $\mathrm{Ext}(A) = \overset{\circ}{{}^cA} = {}^c(\bar{A})$.

Remark. 1. $\mathrm{Ext}(A)$ is an open set

2. $\mathrm{Ext}({}^cA) = \mathring{A}$

3. $\mathring{A} \cup \mathrm{Ext}(A)$ need not be X. Indeed, ${}^c(\mathring{A} \cup \mathrm{Ext}(A)) = {}^c(\mathring{A}) \cap {}^c\mathrm{Ext}(A) = \overline{{}^cA} \cap \overline{A}$ is called the **frontier of** A and is denoted by $\mathrm{Fr}(A)$.

Example. In $\mathbb{R}^2, d_2$, let

$$A = \{x, y, \in \mathbb{R} : \begin{cases} d_2(x, y) \leq r & \text{if } y \geq 0 \\ d_2(x, y) < r & \text{otherwise.} \end{cases} \}$$

Then

$$\begin{aligned} \mathring{A} &= B_r(0) \\ \overline{A} &= \{x \in \mathbb{R}^2 : d_2(x, 0) \leq r\} \\ \overline{{}^cA} &= \{x \in \mathbb{R}^2 : d_2(x, 0) \geq r\} \\ Fr(A) &= \{x \in \mathbb{R}^2 : d_2(x, 0) = r\} \end{aligned}$$

Definition. The **boundary of** A is $Bd(A) = Fr(A) \cap A$. Notice $Fr(A) = \overline{A} \cap \overline{{}^cA} = Fr({}^cA)$.

Proposition. *The boundary of a set A contains no non-empty open sets.*

Proof. Assume $O = \mathring{O} \subseteq Bd(A)$, we want to show $O = \emptyset$. We have

$$O \subseteq A \cap Fr(A) = A \cap (\overline{A} \cap \overline{{}^cA} = A \cap \overline{A} \cap \overline{{}^cA} = A \cap \overline{{}^cA} = A \cap {}^c(\mathring{A}).$$

Then $O \subseteq A \implies \mathring{O} \subseteq \mathring{A}$ but $O = \mathring{O}$ so $O \subseteq \mathring{A}$. Since we also have $O \subseteq {}^c\mathring{A}$, we showed $O \subseteq \mathring{A} \cap {}^c\mathring{A} = \emptyset$. $\square$

Definition. Let (X, d) be a metric space. A set $A \subseteq X$ is called **dense** if $\overline{A} = X$. A set is called **nowhere dense** if $\overline{Ext(A)} = X$.

Example. $(X, d) = (\mathbb{R}, ||), A = \mathbb{Q}$ is dense as we have proved previously.

Remark. A is nowhere dense iff $\emptyset = {}^c\left(\overline{Ext(A)}\right) = {}^c\mathring{Ext}(A) = \overline{A}$.

Definition. A metric space (X, d) is called **separable** if it contains a countable dense set.

Example. $\mathbb{R}^n$ is separable with $\mathbb{Q}^n$ being a countable dense set.

Subspaces of metric space

Definition. Let (X, d) be a metric space, $Y \subseteq X$. In particular, $(Y, d|_{Y \times Y})$ is a metric space. We say that a set $A \subseteq Y$ is **open in Y** if $\exists\ D = \mathring{D} \subseteq X : A = Y \cap D$. We say that a set $A \subseteq Y$ is **closed in Y** if $\exists\ F = \overline{F} \subseteq X : A = Y \cap F$.

Example. $(X, d) = (\mathbb{R}, ||), Y = (0, 1]$. The open sets in Y are of the form (a, b) with $0 \leq a, b \leq 1$ and $(a, 1] = Y \cap (a, \infty)$ with $0 \leq a < 1$. Some closed sets in Y are $Y; \{a\}\ \forall\ a \in Y; (0, \frac{1}{2}] = [-1, \frac{1}{2}] \cap Y$.

Remark. If $A \subseteq Y$ is open in Y, then $Y \setminus A$ is closed in Y. Indeed, if A is open in $Y, \exists\ D = \mathring{D} \subseteq X : A = Y \cap D$. Then $Y \setminus A = Y \cap {}^cA = Y \cap {}^cD$ and ${}^cD$ is closed.

Lemma. *Let (X, d) be a metric space, $Y \subseteq X$. Then Y is open $\iff$ $\forall\ A \subseteq Y$ which is open in Y, we have A is open in X.*

Proof. " $\Longleftarrow$ " take $A = Y$.
" $\implies$ " let $A \subseteq Y$ be open in Y, then $\exists\ D = \mathring{D} \subseteq X : A = Y \cap D$, both are open so A is open. $\square$

Definition. Let (X, d) be a metric space, $A \subseteq Y \subseteq X$. The **closure of** A **in** Y is $\overline{A}^Y = \overline{Y} \cap Y$.

Complete metric spaces

Definition. Let (X, d) be a metric space, $\{x_n\}_{n\geq 1} \subseteq X$. We say $\{x_n\}_{n\geq 1}$ **converges** to a point $x \in X$ if

$$\forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : d(x, x_n) < \epsilon\ \forall\ n \geq n_\epsilon.$$

In this case, we say x is the limit of $\{x_n\}_{n\geq 1}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \xrightarrow[n\to\infty]{d} x$

Exercise.
1. The limit of a convergent sequence $\{x_n\}_{n\geq 1}$ in (X, d) is unique.
2. A sequence $\{x_n\}_{n\geq 1}$ converges to $x \in X$ iff each of its subsequences converge to x.

Lemma. *Let (X, d) be a metric space, $\{x_n\}_{n\geq 1}, \{y_n\}_{n\geq 1} \subseteq X : x_n \xrightarrow[n\to\infty]{d} x, y_n \xrightarrow[n\to\infty]{d} y$. Then $d(x_n, y_n) \xrightarrow[n\to\infty]{(\mathbb{R}, ||)} d(x, y)$.*

Proof.

$$|d(x_n, y_n) - d(x, y)| \leq |d(x_n, y_n) - d(x_n, y)| + |d(x_n, y) - d(x, y)| \leq d(y_n, y) + d(x_n, x) \xrightarrow[n\to\infty]{} 0.$$

$\square$

Definition. Let (X,d) be a metric space. A sequence $\{x_n\}_{n\geq 1} \subseteq X$ is called a **Cauchy sequence** if $\forall\ \epsilon > 0,\ \exists\ n_\epsilon \in \mathbb{N} : d(x_n, x_m) < \epsilon\ \forall\ n, m \geq n_\epsilon$.

Remark. A sequence in $\mathbb{R}$ converges iff it is Cauchy, but this is not true in a general metric space.

Example.
- In $(\mathbb{Q}, ||)$, let $x_1 = 3$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}\ \forall\ n \geq 1$. We showed $\{x_n\}_{n\geq 1}$ converges to $\sqrt{2}$. Consequently, $\{x_n\}_{n\geq 1} \subseteq \mathbb{Q}$ is Cauchy in $(\mathbb{R}, ||)$. However, it does not converge in $(\mathbb{Q}, ||)$.
- In $((0,1), ||)$, let $x_n = \frac{1}{n}\ \forall\ n \geq 2$. Then $\{x_n\}_{n\geq 1}$ is Cauchy but not convergent.

Definition. A metric space (X,d) is called **complete** if every Cauchy sequence in X converges in (X,d).

Exercise.
1. Convergent sequences are Cauchy.
2. A Cauchy sequence with a convergent subsequence is convergent.

Definition. Let (X,d) be a metric space. A sequence $\{k_n\}_{n\geq 1}$ of subsets of X is called a **nested sequence of closed balls** if

$$k_n = K_{r_n}(x_n) = \{x \in X : d(x, x_n) \leq r_n\}, \quad k_{n+1} \subseteq k_n\ \forall\ n \geq 1.$$

Remark. In a general metric space, it is not true that

$$\overline{B_r(x)} = \overline{\{y \in X : d(x,y) < r\}} = K_r(x) = \{y \in X : d(x,y) \leq r\}$$

Example. Let $X = (-\infty, 0] \cup \mathbb{N}$ and $d(x,y) = |x-y|$. Then

$$B_1(0) = \{x \in X : |x| < 1\} = (-1, 0] \implies \overline{B_1(0)} = [-1, 0]$$

but

$$K_1(0) = \{x \in X : |x| \leq 1\} = (-1, 0] \cup \{1\}.$$

Theorem. *A metric space (X,d) is complete iff for every nested sequence $\{k_n\}_{n\geq 1}$ of closed balls with $\delta(k_n) \xrightarrow[n\to\infty]{} 0$, we have*

$$\bigcap_{n\geq 1} k_n \neq \emptyset.$$

Proof.
- " $\implies$ " Let $\{k_n\}_{n\geq 1}$ be a nested sequence of closed balls with $\delta(k_n) \xrightarrow[n\to\infty]{} 0$. Write $k_n = k_{r_n}(x_n)$. Then

  $$\delta(k_n) = 2r_n \implies \lim_{n\to\infty} 2r_n = 0 \iff \lim_{n\to\infty} r_n = 0.$$

  Claim. $\{x_n\}_{n\geq 1}$ is a Cauchy sequence.

  Let $\epsilon > 0$, then $\exists\ n_\epsilon \in \mathbb{N} : r_n < \frac{\epsilon}{2}\ \forall\ n \geq n_\epsilon$. For $n, m \geq n_\epsilon$ we have $k_n, k_m \subseteq k_{n_\epsilon}$ and

  $$d(x_n, x_m) \leq d(x_n, x_{n_\epsilon}) + d(x_{n_\epsilon}, x_m) \leq r_{n_\epsilon} + r_{n_\epsilon} < \epsilon.$$

  As (X,d) is complete, $\exists\ x \in X : x_n \xrightarrow[n\to\infty]{d} x$. For $m \geq 1$, we want to show $x \in k_m$. Note $\{x_n\}_{n\geq m} \subseteq k_m$. As k_m is closed, $x = \lim_{n\to\infty} x_n \in k_m$. Therefore

  $$x \in \bigcap_{m\geq 1} k_m.$$

- " $\impliedby$ " Let $\{x_n\}_{n\geq 1}$ be a Cauchy sequence in X. For every $n \geq 1, \exists\ N_n \in \mathbb{N} : d(x_k, x_l) < \frac{1}{2^{n+1}}\ \forall\ k, l \geq N_n$. Let $k_1 = N_1$ and for $n \geq 1$, let $k_{n+1} > \max\{k_n, N_{n+1}\}$. In particular, $d(x_m, x_{k_n}) < \frac{1}{2^{n+1}}\ \forall\ m \geq k_n$. Let $k_n = k_{\frac{1}{2^n}}(x_{k_n})$.
  Claim. $k_{n+1} \subseteq k_n$

  Let $y \in k_{n+1} \implies d(y, x_{k_{n+1}}) \leq \frac{1}{2^{n+1}}$. By the triangle inequality

  $$d(y, x_{k_n}) \leq d(y, x_{k_{n+1}}) + d(x_{k_{n+1}}, x_{k_n}) \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n} \implies y \in k_n.$$

  Then

  $$\bigcap_{n\geq 1} k_n \neq \emptyset \iff \exists\ x \in \bigcap_{n\geq 1} k_n \implies d(x, x_{k_n}) \leq \frac{1}{2^n}\ \forall\ n \geq 1 \implies x_{k_n} \xrightarrow[n\to\infty]{d} x.$$

  We have proved previously that a Cauchy sequence with a convergent subsequence converges.

$\square$

Examples of complete metric spaces

1. $(\mathbb{R}, ||)$

2.

   Lemma. *Let $(X, d_1), (Y, d_2)$ be complete metric spaces. Then $X \times Y$ with the metric*

   $$d((x_1, y_1), (x_2, y_2)) = \sqrt{d_1(x_1, x_2)^2 + d_2(y_1, y_2)^2}$$

   is a complete metric space.

   Proof.

   Exercise. Show d is a metric.

   We show the metric space is complete. Let $\{(x_n, y_n)\}_{n\geq 1} \subseteq X \times Y$ be Cauchy, then

   $$\begin{aligned} &\forall\ \epsilon > 0, \exists\ n_\epsilon \in \mathbb{N} : d((x_n, y_n), (x_m, y_m)) < \epsilon\ \forall\ n, m \geq n_\epsilon \\ &\implies d_1(x_n, x_m)^2 + d_2(y_n, y_m)^2 < \epsilon^2 \\ &\implies \begin{cases} d_1(x_n, x_m) < \epsilon \\ d_2(y_n, y_m) < \epsilon \end{cases} \\ &\implies \begin{cases} \{x_n\}_{n\geq 1} \text{ is Cauchy, let } x = \lim_{n\to\infty} x_n \\ \{y_n\}_{n\geq 1} \text{ is Cauchy, let } y = \lim_{n\to\infty} y_n. \end{cases} \end{aligned}$$

   Claim. $(x_n, y_n) \xrightarrow[n\to\infty]{d} (x, y)$

   Let $\epsilon > 0$, then

   $$\begin{cases} \exists\ n_1(\epsilon) \in \mathbb{N} : d_1(x_n, x) < \frac{\epsilon}{2}\ \forall\ n \geq n_1(\epsilon) \\ \exists\ n_2(\epsilon) \in \mathbb{N} : d_2(y_n, y) < \frac{\epsilon}{2}\ \forall\ n \geq n_2(\epsilon) \end{cases}$$
   $$\implies \text{for } n \geq \max\{n_1(\epsilon), n_2(\epsilon)\},$$
   $$d((x_n, y_n), (x, y)) = \sqrt{d_1(x_n, x)^2 + d_2(y_n, y)^2} < \sqrt{\left(\frac{n}{\epsilon}\right)^2 + \left(\frac{n}{\epsilon}\right)^2} < \epsilon.$$

   □

3.

   Corollary. $(\mathbb{R}^n, d_2)$ *is a complete metric space. Recall*

   $$d_2(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}.$$

   Exercise. Fix $1 \leq p \leq \infty$ and $n \in \mathbb{N}$, show that $(\mathbb{R}^n, d_p)$ is complete.

4. Let

   $$l^2 = \{\{x_n\}_{n\geq 1} \subseteq \mathbb{R} : \sum_{n\geq 1} |x_n|^2 < \infty\}$$

   and $d_2 : l^2 \times l^2 \to \mathbb{R}$ be defined as

   $$d_2(x, y) = \left(\sum |x_n - y_n|^2\right)^{\frac{1}{2}}\ \forall\ x = \{x_n\}_{n\geq 1}, y = \{y_n\}_{n\geq 1}.$$

   Claim. (l^2, d_2) is a complete metric space.

Proof. We will prove completeness. Let $\{x^{(k)}\}$ be a Cauchy sequence. Note $x^{(k)} = \{x_n^{(k)}\}_{n\geq 1} \in l^2$. For $\epsilon > 0$,

$$\exists\ k_\epsilon \in \mathbb{N} : d_2(x^k, x^l) < \epsilon\ \forall\ k,l \geq k_\epsilon \implies \sum_{n\geq 1} |x_n^{(k)} - x_n^{(l)}|^2 < \epsilon^2\ \forall\ k,l \geq k_\epsilon.$$

In particular, for all $n \geq 1$ we have $|x_n^{(k)} - x_n^{(l)}| < \epsilon\ \forall\ k,l \geq k_\epsilon$. Thus $\forall\ n \geq 1, \{x_n^{(k)}\}$ is Cauchy in $\mathbb{R}$. Let $x_n = \lim_{k\to\infty} x_n^{(k)}, x = \{x_n\}_{n\geq 1}$. We will show $x^{(k)} \xrightarrow[k\to\infty]{d_2} x$ and $x \in l^2$ (this follows from the triangle inequality). For $\epsilon > 0$,

$$\exists\ k_\epsilon \in \mathbb{N} : d(x^{(k)}, x^{(l)}) < \epsilon\ \forall\ k,l \geq k_\epsilon.$$

Fix $N \geq 1$ and let $k,l \geq k_\epsilon$. We have

$$\sum_{n\geq 1}^{N} |x_n^{(k)} - x_n^{(l)}|^2 < \epsilon^2$$

Freeze k and let $l \to \infty$ to get

$$\sum_{n\geq 1}^{N} |x_n^{(k)} - x_n|^2 < \epsilon^2$$

Thus $\sigma_N = \sum_{n\geq 1}^{N} |x_n^{(k)} - x_n|^2$ is an increasing sequence bounded above by ϵ^2. Then

$$\lim \sigma_N = \sum_{n\geq 1} |x_n^{(k)} - x_n|^2 \leq \epsilon^2 \implies d(x^{(k)}, x) \leq \epsilon\ \forall\ k \geq k_\epsilon.$$

□

Lemma. *Let (X,d) be a metric space. A set $A \subseteq X$ is dense iff $\forall\ \emptyset \neq O = \mathring{O} \subseteq X$, we have $A \cap O \neq \emptyset$.*

Proof.

- " $\implies$ " Assume towards a contradiction that

$$\exists\ \emptyset \neq O = \mathring{O} \subseteq X : A \cap O = \emptyset.$$

Then we reach a contradiction:

$$O \subseteq\ ^cA \implies \emptyset \neq O = \mathring{O} \subseteq \mathring{^cA} =^c (\overline{A}) =\ ^cX = \emptyset.$$

- " $\impliedby$ " Assume $\overline{A} \neq X$. Then $\mathring{^cA} =^c (\overline{A}) \neq \emptyset \iff \exists\ B_r(x) \subseteq\ ^cA$ for some $r > 0$. In particular, $\emptyset \neq B_r(x)$ is an open set such that $B_r(x) \cap A = \emptyset$. This is a contradiction.

□

Baire property

Definition. We say a metric space (X,d) has the **Baire property** if for every $\{A_n\}_{n\geq 1} \subseteq P(X) : A_n = \mathring{A}_n, \overline{A} = X$, we have $\overline{\cap_{n\geq 1} A_n} = X$. Namely, for each countable collection of open dense sets, their intersection is dense.

Lemma. *Let (X,d) be a metric space, the following are equivalent*

1. *X has the Baire property*
2. *If $\{F_n\} \subseteq P(X) : F_n = \overline{F_n}, \mathring{F}_n = \emptyset$, then $\mathring{\cup_{n\geq 1} F_n} = \emptyset$. Namely, the interior of the union of closed sets with empty interior is empty.*

Proof.

- "1. $\implies$ 2." Let $F \subseteq X : F = \overline{F_n}$ and $\mathring{F}_n = \emptyset$. Define $A_n =\ ^cF_n$, then $A_n = \mathring{A}_n$ and $\overline{A_n} = \overline{^cF_n} =\ ^c\mathring{F}_n =\ ^c\emptyset = X$. As X has the Baire property, $\overline{\cap_{n\geq 1} A_n} = X$. But $\overline{\cap_{n\geq 1} A_n} = \overline{\cap_{n\geq 1}\ ^cF_n} = \overline{^c \cup_{n\geq 1} F_n} =\ ^c(\mathring{\cup_{n\geq 1} F_n}) = X \implies \mathring{\cup_{n\geq 1} F_n} = \emptyset$.
- "2. $\implies$ 1." Let $A_n = \mathring{A}_n, \overline{A_n} = X$, we want to show $\overline{\cap_{n\geq 1} A_n} = X$. Define $F_n =\ ^cA_n$, then $F_n = \overline{F_n}$ and $\mathring{F}_n = \mathring{^cA_n} =\ ^c\overline{A_n} =\ ^cX = \emptyset$. Therefore, $\mathring{\cup_{n\geq 1} F_n} = \emptyset \implies\ ^c\mathring{\cup_{n\geq 1} F_n} = X$. But $\mathring{\cup_{n\geq 1} F_n} = \mathring{\cup_{n\geq 1}\ ^cA_n} =\ ^c\overline{\cap_{n\geq 1}\ A_n} =\ ^c(\overline{\cap_{n\geq 1}\ A_n}) = \emptyset \implies \overline{\cap_{n\geq 1}\ A_n} = X$.

□

Baire category theorem 1

Theorem. *A complete metric space (X, d) has the Baire property.*

Proof. Let $A_n = \mathring{A}_n$ and $\overline{A_n} = X$, we want to show $\overline{\cap_{n\geq 1} A_n} = X$. Equivalently, it suffices to show that $\forall\ \emptyset \neq W = \mathring{W} \subseteq X$, we have $W \cap (\cap_{n\geq 1} A_n) \neq \emptyset$. Fix $\emptyset \neq W = \mathring{W} \subseteq X$.

- As $\overline{A_1} = X$, we have the open set $W \cap A_1 \neq \emptyset \implies \exists\ B_{r_1}(x_1) \subseteq W \cap A_1$ for some $r_1 > 0$. Let $\rho_1 < \min\{r_1, 1\}$, then $K_{\rho_1}(x_1) \subseteq B_{r_1}(x_1) \subseteq W \cap A_1$.
- As $\overline{A_2} = X$, we have the open set $B_{\rho_1}(x_1) \cap A_2 \neq \emptyset \implies \exists\ B_{r_2}(x_2) \subseteq B_{\rho_1}(x_1) \cap A_2$ for some $r_2 > 0$. Let $\rho_2 < \min\{r_2, \frac{1}{2}\}$, then $K_{\rho_2}(x_2) \subseteq B_{r_2}(x_2) \subseteq K_{\rho_1}(x_1) \cap A_2$.

Proceed inductively to find a sequence of points $\{x_n\}_{n\geq 1} \subseteq X$ and a sequence of radii $\{\rho_n\} \subseteq (0, \infty) : \rho_n < \frac{1}{n}$ and $K_{\rho_{n+1}}(x_{n+1}) \subseteq K_{\rho_n}(x_n) \cap A_{n+1}\ \forall\ n \geq 1$. As (X, d) is complete, we get $\cap_{n\geq 1} K_{\rho_n}(x_n) \neq \emptyset$. Note $\emptyset \neq \cap_{n\geq 1} K_{\rho_n}(x_n) \subseteq W \cap A_1 \cap (\cap_{n\geq 2} A_n) = W \cap (\cap_{n\geq 1} A_n)$. □

Definition. Let (X, d) be a metric space. A set $A \subseteq X$ is said to be of **first (Baire) category** if it can be written as a countable union of closed, nowhere dense sets. If A is not of the first category, then A is said to be of **second (Baire) category**.

Remark. A metric space (X, d) is of first category if $X = \cup_{n\geq 1} A_n$ with $A_n = \overline{A_n}$ and $\overline{Ext(A_n)} = X$. Recall $Ext(A_n) = {}^c\mathring{A}_n \implies X = \overline{Ext(A_n)} = \overline{{}^c\mathring{A}_n} = \overline{{}^c(\overline{A_n})} = {}^c\left(\mathring{\overline{A_n}}\right) \implies \mathring{\overline{A_n}} = \emptyset$. So X is of first category if $X = \cup_{n\geq 1} A_n$ with $A_n = \overline{A_n}$ and $\mathring{\overline{A_n}} = \emptyset$.

Example. $\mathbb{Q}$ is of first category.

Baire category theorem 2

Theorem. *A complete metric space (X, d) is of second category.*

Proof. We argue by contradiction. Assume X is of first category, i.e. $X = \cup_{n\geq 1} A_n, A_n = \overline{A_n}, \mathring{\overline{A_n}} = \emptyset$. By the previous theorem, X complete $\implies$ X has the Baire property $\implies \mathring{\overline{\cup_{n\geq 1} A_n}} = \emptyset \iff \mathring{X} = \emptyset \iff X = \emptyset$, contradiction. □

Corollary. $\mathbb{R} \setminus \mathbb{Q}$ *is of second category.*

Proof. Assume towards a contradiction that $\mathbb{R} \setminus \mathbb{Q}$ is of first category, then we can write $\mathbb{R} \setminus \mathbb{Q} = \cup_{n\geq 1} A_n$ with $A_n = \overline{A_n}$ and $\mathring{A}_n = \emptyset$. Notice $\mathbb{R} = (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Q} = (\cup_{n\geq 1} A_n) \cup (\cup_{q\in\mathbb{Q}} \{q\})$ with $\{q\} = \overline{\{q\}}$ and $\{\mathring{q}\} = \emptyset$ so $\mathbb{R}$ is of first category, contradiction. □

Remark. If (X, d) is complete and $A \subseteq X$ is of first category, then $X \setminus A$ is of second category.

The Banach-Mazur game

Imagine we have two players P_1, P_2 playing the following game. Let I_0 be a closed interval.

- P_1 gets dealt a subset $A \subseteq I_0$,
- P_2 gets dealt a subset $B \subseteq I_0 \setminus A$.

Then

- P_1 chooses a closed interval $I_1 \subseteq I_0$,
- P_2 chooses a closed interval $I_2 \subseteq I_1$,
- $\ldots$,
- P_1 chooses a closed interval $I_{2n+1} \subseteq I_{2n}$,
- P_2 chooses a closed interval $I_{2n+2} \subseteq I_{2n+1}$.

Then P_1 wins if $(\cap I_n) \cap A \neq \emptyset$, otherwise P_2 wins.

Question. Can either player ensure a winning strategy by choosing the intervals wisely, no matter how the opponent plays?

Answer. If A is of first (Baire) category, then P_2 has a winning strategy. Indeed, assume $A = \cup_{n\geq 1} A_n$, with $A_n = \overline{A_n}, \mathring{A} = \emptyset$. Then P_2 needs only choose $I_{2n} \subseteq I_{2n-1} \setminus A_n \ \forall\ n \geq 1$. Then $\cap_{n\geq 1} I_n \subseteq I_0 \setminus A$ and so P_2 wins.

Conjecture. *P_2 has a winning strategy $\iff$ A is of first category (proved by Banach).*

This gives insight into how "small" a set of first category is, namely, it is a set on which even the first player is bound to lose, unless his opponent fails to take advantage of the situation.

Theorem. *P_1 has a winning strategy iff there is an interval $I_1 \subseteq I_0 : I_0 \cap B$ is of first category.*

Connected sets

Definition. Let (X, d) be a metric space. We say that two sets $A, B \subseteq X$ are **separated** if $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$.

Remark. Any two separated sets are disjoint because $A \cap B \subseteq \overline{A} \cap B = \emptyset$. However, two disjoint sets need not be separated.

Example. $A = (0, 1), \overline{A} = [0, 1], B = \{1\}$

Lemma. *Let (X, d) be a metric space, $A, B \subseteq X : d(A, B) > 0$. Then A, B are separated.*

Proof. We argue by contradiction. Assume $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$. Since $x \in \overline{A}, d(x, A) = \inf \{d(x, a) : a \in A\} = 0$. But then $d(B, A) = \inf \{d(b, A) : b \in B\} = 0$, contradiction. Similarly, one shows that $A \cap \overline{B} = \emptyset$. □

Remark. There are separated sets A, B for which $d(A, B) = 0$.

Example. $A = (0, 1), B = (1, 2) \implies \overline{A} \cap B = [0, 1] \cap (1, 2) = \emptyset, A \cap \overline{B} = (0, 1) \cap [1, 2] = \emptyset, d(A, B) = 0$

Exercise. Let (X, d) be a metric space, $A, B \subseteq X$ separated. If $A_1 \subseteq A$ and $B_1 \subseteq B$, then A_1 and B_1 are separated.

Proposition. *Let (X, d) be a metric space, then*

1. *Two closed sets are separated iff they're disjoint.*
2. *Two open sets are separated iff they're disjoint.*

Proof. 1.
- " $\implies$ " This is clear.
- " $\impliedby$ " Let $\overline{A} = A, \overline{B} = B, A \cap B = \emptyset$. Then $\overline{A} \cap B = A \cap B = \emptyset, A \cap \overline{B} = A \cap B = \emptyset$, so A, B are separated.

2.
- " $\implies$ " This is clear.
- " $\impliedby$ " Let $\mathring{A} = A, \mathring{B} = B, A \cap B = \emptyset$. We want to show $\overline{A} \cap B = A \cap \overline{B} = \emptyset$. Assume towards a contradiction that $\overline{A} \cap B \neq \emptyset$. Let $x \in \overline{A} \cap B$, since $x \in B = \mathring{B}, \exists\ r_0 > 0 : B_{r_0}(x) \subseteq B$. Since $x \in \overline{A},\ \forall\ r > 0, B_r(x) \subseteq A$. For any $r >\leq r_0, \emptyset \neq B_r(x) \cap A \subseteq A \cap B$, contradiction. This shows $\overline{A} \cap B = \emptyset$, one shows similarly that $A \cap \overline{B} = \emptyset$.

□

Proposition. *Let (X, d) be a metric space.*

1. *If a closed set is the union of two separated sets A, B, then A, B are closed.*
2. *If an open set is the union of two separated sets A, B, then A, B are open.*

Proof. 1. Let $F = \overline{F} : F = A \cup B, \overline{A} \cap B = A \cap \overline{B} = \emptyset$. Then

$$\overline{A} = \overline{A} \cap \overline{F} = \overline{A} \cap F = \overline{A} \cap (A \cup B) = (\overline{A} \cap A) \cup (\overline{A} \cap B) = A \cap \emptyset = A.$$

Similarly, one proves B is closed.

2. Let $D = \mathring{D} : D = A \cup B, \overline{A} \cap B = A \cap \overline{B} = \emptyset$. We want to show $\mathring{A} = A, \mathring{B} = B$. We know $\mathring{A} \subseteq A$. Let $a \in A \subseteq D = \mathring{D} \implies \exists\, r_0 > 0 : B_r(a) \subseteq D = A \cup B$. As $A \cap \overline{B} = \emptyset$ and $a \in A, a \notin B$. Thus $\exists\, r_1 > 0 : B_{r_1}(a) \cap B = \emptyset$. Then for $r < \min\{r_0, r_1\}, B_r(a) \subseteq D \setminus B = A$. So $a \in \mathring{A} \implies A \subseteq \mathring{A}$.

□

Exercise. Is it true that in a metric space $(X, d), B_r(a)$ cannot be written as the union of separated sets?

Definition. Let (X, d) be a metric space, $A \subseteq X$.

- We say A **is disconnected** if it can be written as the union of two non-empty separated sets.
- If A is not disconnected, then we say A **is connected**.

Theorem. *Let (X, d) be a metric space. Then X is connected iff the only subsets of X that are clopen are $\emptyset, X$.*

Proof.
- " $\implies$ " Assume towards a contradiction that $\exists\, A \subseteq X : \emptyset \neq A \neq X, A$ clopen. Then $X \setminus A \neq \emptyset$ is both closed and open. As A and $X \setminus A$ are disjoint, they are separated. Then $X = A \cup (X \setminus A)$ and $A, X \setminus A \neq \emptyset$ separated implies that X is disconnected, contradiction.
- " $\impliedby$ " Assume towards a contradiction that we can rewrite $A = B \cup C$ with $B \neq \emptyset, C \neq \emptyset$ and $\overline{B} \cap C = B \cap \overline{C} = \emptyset$. As X is open, B, C are open. As X is closed, B, C are closed. Then B, C are both open and closed so B or $C = \emptyset$, contradiction.

□

Theorem. *Let (X, d) be a metric space, $A \subseteq X$. Then A is connected iff the only subsets of A that are both open and closed in A are $\emptyset, X$.*

Proof.
- " $\implies$ " We argue by contradiction. Assume that $\exists\, \emptyset \neq B \subsetneq A : B$ is both open and closed. Then $\emptyset \neq A \setminus B \subsetneq A$ is both open and closed. Thus $A = B \cup (A \setminus B)$.

  Claim. $B, A \setminus B$ are separated.

  B is closed in A so $\overline{B} \cap A = B$. Then
  $$\overline{B} \cap (A \setminus B) = (\overline{B} \cap A) \cap {}^cB = B \cap {}^cB = \emptyset.$$
  $A \setminus B$ is closed in A so $A \cap \overline{A \setminus B} = A \setminus B$. Then
  $$B \cap \overline{(A \setminus B)} = B \cap A \cap \overline{(A \setminus B)} = B \cap A \setminus B = \emptyset.$$
  Thus A can be written as the union of two separated sets, contradiction.
- " $\impliedby$ " Assume towards a contradiction that A is disconnected. Then $\exists\, B \neq \emptyset, C \neq \emptyset$ with $\overline{B} \cap C = B \cap \overline{C} = \emptyset : A = B \cup C$.

  Claim. B is closed in A.
  $$\overline{B} \cap A = \overline{B} \cap (B \cup C) = (\overline{B} \cap B) \cup (\overline{B} \cap C) = B \cap \emptyset = B.$$
  Similarly, one shows that if C is closed in A then $B = A \setminus C$ is open in A. Thus $\emptyset \neq B \subsetneq A$ is both open and closed in A, contradiction.

□

Theorem. *Let (X, d) be a metric space. The following are equivalent.*

1. *A is disconnected.*
2. *There exists open sets $D_1, D_2 : A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, A \cap D_2 \neq \emptyset, A \cap D_1 \cap D_2 = \emptyset$.*
3. *There exists closed sets $F_1, F_2 : A \subseteq F_1 \cup F_2, A \cap F_1 \neq \emptyset, A \cap F_2 \neq \emptyset, A \cap F_1 \cap F_2 = \emptyset$.*

Proof. We show $3 \implies 2 \implies 1 \implies 3$.

- "3 $\implies$ 2" Let $D_1 = {}^cF_1, D_2 = {}^cF_2$. Then D_1, D_2 are open. Since $A \subseteq F_1 \cup F_2$, we have

  $$A \cap D_1 \cap D_2 = A \cap {}^cF_1 \cap {}^cF_2 = A \cap {}^c(F_1 \cup F_2) = \emptyset.$$

  We know $A \cap F_1 \cap F_2 = \emptyset \implies A \subseteq {}^c(F_1 \cap F_2) = {}^cF_1 \cup {}^cF_2 = D_1 \cup D_2$. Let's show $A \cap D_1 \neq \emptyset$. Notice $A \cap D_1 = \emptyset \implies A \subseteq D_2 \implies A \subseteq {}^cF_2 \implies F_2 = \emptyset$, contradiction. Similarly, $A \cap D_2 \neq \emptyset$.

- "2 $\implies$ 1" Let $B = A \cap D_1, C = A \cap D_2$. Then $A = B \cup C, B \neq \emptyset, C \neq \emptyset, B \cap C = \emptyset$. Note that if B and C are open in A then B is closed in A. $B \neq \emptyset, B \neq A$ since $A = B \cup C, C \neq \emptyset, B \cap C \neq \emptyset$. Thus A is disconnected.

- "1 $\implies$ 3" As A is disconnected, $\exists\, \emptyset \neq B \subsetneq A : B$ is both open and closed in A. In particular, $C = A \setminus B$ is both open and closed in A and $\emptyset \neq C \subsetneq A$. Let F_1, F_2 be closed sets such that $B = A \cap F_1, C = A \cap F_2$. Then $A = B \cup C \subseteq F_1 \cup F_2, A \cap F_1 = B \neq \emptyset, A \cap F_2 = C \neq \emptyset, A \cap F_1 \cap F_2 = B \cap C = \emptyset$.

□

Proposition. *Let (X, d) be a metric space, $A \subseteq X$ be disconnected. Let D_1, D_2 be open sets such that $A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, A \cap D_2 \neq \emptyset, A \cap D_1 \cap D_2 = \emptyset$. If $B \subseteq A$ connected, then $B \subseteq D_1$ or $B \subseteq D_2$.*

Proof. Assume towards a contradiction, that $B \cap D_1 \neq \emptyset$ and $B \cap D_2 \neq \emptyset$. Then $B \subseteq A \subseteq D_1 \cup D_2$ and $B \cap D_1 \cap D_2 \subseteq A \cap D_1 \cap D_2 = \emptyset$ implies that B is disconnected, contradiction. □

A similar argument yields

Proposition. *Let (X, d) be a metric space, $A \subseteq X$ be disconnected. Let F_1, F_2 be closed sets such that $A \subseteq F_1 \cup F_2$. Then $A \cap F_1 \neq \emptyset, A \cap F_2 \neq \emptyset, A \cap F_1 \cap F_2 = \emptyset$. If $B \subseteq A$ is connected, then $B \subseteq F_1$ and $B \subseteq F_2$.*

Proposition. *Let (X, d) be a metric space, $A \subseteq X$ be connected. If $A \subsetneq B \subseteq \overline{A}$, then B is connected.*

Proof. Assume towards a contradiction that B is disconnected. Then $\exists\, F_1, F_2$ closed subsets of $X : B \subseteq F_1 \cup F_2, B \cap F_1 \neq \emptyset, B \cap F_2 \neq \emptyset, B \cap F_1 \cap F_2 = \emptyset$. Then $A \subseteq B \subseteq F_1 \cup F_2 \implies$ either $A \subseteq F_1$ or$A \subseteq F_2$. Without loss of generality, assume $A \subseteq F_1$. Then $B \subseteq \overline{A} \subseteq \overline{F_1} = F_1 \implies \emptyset = B \subseteq F_1 \cup F_2 = B \cap F_2 \neq \emptyset$, contradiction. □

Proposition. *Let (X, d) be a metric space, $\{A_i\}_{i \in I}$ be a family of connected subsets of X such that for any $i \neq j$, A_i and A_j are not separated. Then $\cup_{i \in I} A_i$ is connected.*

Proof. Assume towards a contradiction that $\cup_{i \in I} A_i$ is disconnected, then $\exists\, B, C \neq \emptyset : \overline{B} \cap C = B \cap \overline{C} = \emptyset$ and $\cup_{i \in I} A_i = B \cup C$. For any $i \in I$, $A_i = (B \cap A_i) \cup (C \cap A_i)$. But A_i is connected while $B \cap A_i$ and $C \cap A_i$ are separated. So either $B \cap A_i = \emptyset$ or $C \cap A_i = \emptyset$. In particular, if $A_i \cap B \neq \emptyset$, then $A \subseteq B$. Then $\cup_{i \in I} A_i = B \cup C \implies \exists\, i_1, i_2 \in I$:

$$A_{i_1} \cap B \neq \emptyset \implies A_{i_1} \subseteq B$$
$$A_{i_2} \cap C \neq \emptyset \implies A_{i_2} \subseteq C$$

But B, C separated $\implies A_{i_1}, A_{i_2}$ separated, contradiction. □

Corollary. *Let (X, d) be a metric space, $\{A_i\}_{i \in I}$ be a family of connected subsets of $X : \cap_{i \in I} A_i \neq \emptyset$. Then $\cup_{i \in I} A_i$ is connected.*

Theorem. *The only non-empty connected subsets of $\mathbb{R}$ are the intervals. In particular, $\mathbb{R} = (-\infty, \infty)$ is connected so the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset, \mathbb{R}$.*

Proof. Let's first show that intervals are connected. Let $I \subseteq \mathbb{R}$ be an interval. Assume towards a contradiction that I is disconnected. Then $\exists\, \emptyset \neq A \subsetneq I : A$ is both open and closed in I. Then its complement $\emptyset \neq B = I \setminus A \subsetneq I$ is both open and closed in I. Let $a_1 \in A, b_1 \in B$.

- Set $c_1 = \frac{a_1 + b_1}{2}$. If $c_1 \in A$, set $a_2 = c_1, b_2 = b_1$. If $c_1 \in B$, set $a_2 = a_1, b_2 = c_1$. In either case, $b_2 - a_2 = \frac{b_1 - a_1}{2}$.
- Set $c_2 = \frac{a_2 + b_2}{2}$. If $c_2 \in A$, set $a_3 = c_2, b_3 = b_2$. If $c_2 \in B$, set $a_3 = a_2, b_3 = c_2$. In either case, $b_3 - a_3 = \frac{b_1 - a_1}{2^2}$. Proceeding inductively, we construct $\{a_n\} \subseteq A, \{b_n\} \subseteq B$:
  - $\{a_n\}$ is non-decreasing and bounded above by b so it converges, let $a = \lim_{n \to \infty} a_n$
  - $\{b_n\}$ is non-increasing and bounded below by a so it converges, let $b = \lim_{n \to \infty} b_n$

–

$$b_n - a_n = \frac{b_1 - a_1}{2^{n-1}} \xrightarrow[n\to\infty]{} 0$$

Thus $a = b$. But $a \in \overline{A} \cap I = A$ and $b \in \overline{B} \cap I = B$ so $A \cap B \neq \emptyset$, contradiction. Finally, show connected sets are necessarily intervals. Let $A \subseteq \mathbb{R}$ be connected. Let $a = \inf A$ (possibly $-\infty$). Let $b = \sup A$ (possibly ∞). We have to show that if $a < c < b$, then $c \in A$. Assume, towards a contradiction, that $\exists\ c \in (a, b) \setminus A$. Set $D_1 = (-\infty, c), D_2 = (c, \infty)$ open in $\mathbb{R}$. Then $A \subseteq D_1 \cup D_2, A \cap D_1 \cap D_2 = \emptyset, A \cap D_1 \neq \emptyset$ (because $\inf A < c$), $A \cap D_2 \neq \emptyset$ (because $\sup A > c$). Thus A is disconnected, contradiction.

□

Lemma. *Let (X, d) be a metric space, $A \subseteq X$. If any pair of points in A is contained in a connected subset of A, then A is connected.*

Proof. Assume towards a contradiction that A is disconnected. Then $\exists D_1, D_2$ open : $A \subseteq D_1 \cup D_2, A \cap D_1 \neq \emptyset, A \cap D_2 \neq \emptyset, A \cap D_1 \cap D_2 = \emptyset$. Let $a \in A \cap D_1, b \in A \cap D_2$. Then $\exists\ B \subseteq A$ connected : $\{a, b\} \subseteq B$. Then $B \subseteq D_1 \cup D_2, B \cap D_1 \neq \emptyset, B \cap D_2 \neq \emptyset, B \cap D_1 \cap D_2 = \emptyset \implies B$ is disconnected, contradiction. □

Exercise. Let $(\mathbb{R}^n, d), B_1(0) = \{x \in \mathbb{R}^n : d(x, 0) < 1\}$. Then $B_1(0)$ is connected.