Cyclic Quadrilaterals

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1 Motivation

Any three non-collinear points always lie on some common circle, but in general it is rare for four points to have that property—unless one is trying to solve an Olympiad problem. Then, the observation that certain quadrilaterals are cyclic often turns out to be the key to the solution. We will survey some facts about cyclic quadrilaterals, and use them to solve actual Olympiad problems.

2 Warm-Up

1. Let \( ABC \) be a triangle. Show that one can always find a circle through all three vertices, and prove that the circle is unique.

2. Find a quadrilateral \( ABCD \) with the property that no circle passes through all four vertices.

3. Let \( ABCD \) be a cyclic quadrilateral. Why is \( \angle ABD = \angle ACD \)?

4. Let \( ABCD \) be a cyclic quadrilateral. Prove that \( \angle A + \angle C = 180^\circ \).

5. Let \( ABCD \) be a cyclic quadrilaterals, and let its diagonals \( AC \) and \( BD \) intersect at \( P \). Show that \( AX \cdot XC = BX \cdot XD \).

3 Recognizing cyclic quadrilaterals

- A quadrilateral \( ABCD \) is cyclic if and only if \( \angle ABD = \angle ACD \).
- A quadrilateral is cyclic if and only if its opposite angles sum to \( 180^\circ \).
- Let \( ABCD \) be a quadrilateral, and let its diagonals \( AC \) and \( BD \) intersect at \( X \). Then it is cyclic if and only if \( AX \cdot XC = BX \cdot XD \).
- A quadrilateral is cyclic if the problem says it is.
- But if the problem doesn’t say a quadrilateral is cyclic, it might still be cyclic.
- And even if the problem doesn’t seem to have any quadrilaterals at all, there might be a cyclic one.

4 Problems

1. (Classical) Let \( ABC \) be a triangle, and let \( D, E, F \) be the feet of the altitudes from \( A, B, C \), respectively. How many sets of 4 cyclic points can you find?

2. (USAMO 1990/5) An acute-angled triangle \( ABC \) is given in the plane. The circle with diameter \( AB \) intersects altitude \( CC' \) and its extension at points \( M \) and \( N \), and the circle with diameter \( AC \) intersects altitude \( BB' \) and its extensions at \( P \) and \( Q \). Prove that the points \( M, N, P, Q \) lie on a common circle.
3. (Simson Line) Let $ABC$ be a triangle, and let $M$ be a point on its circumcircle. Let $D, E, F$ be the feet of the perpendiculors from $M$ to $BC$, $CA$, $AB$. Prove that $D, E, F$ are collinear.

4. (R. Gelca 1997) Let $A, B, C$ be collinear and $M \notin AB$. Prove that $M$ and the circumcenters of $MAB$, $MBC$, and $MAC$ lie on a circle. Note: Inversion, followed by a homothety of ratio 1/2 about $M$, reduces this to the previous problem. You are welcome to try to find an alternative solution, however.

5. (R. Gelca 1997) In a circle, $AB$ and $CD$ are orthogonal diameters. A variable line passing through $C$ intersects $AB$ at $M$ and the circle at $N$. Find the locus of the intersection of the parallel to $CD$ through $M$ with the tangent at $N$.

6. (Half of USAMO 1993/2) Let $ABCD$ be a convex quadrilateral whose diagonals are orthogonal, and let $P$ be the intersection of the diagonals. Let $E, F, G, H$ be the feet of the perpendiculors from $P$ to $AB$, $BC$, $CD$, $DA$.

   (i) Prove that $\angle PGF = \angle ACB$.
   (ii) Prove that $E, F, G, H$ are cyclic.

7. (Full USAMO 1993/2) Let $ABCD$ be a cyclic quadrilateral whose diagonals are orthogonal, and let $P$ be the intersection of the diagonals. Prove that the four points that are symmetric to $P$ with respect to the sides form a cyclic quadrilateral.

8. (R. Gelca 1997) Let $B$ and $C$ be the endpoints, and $A$ the midpoint, of a semicircle. Let $M$ be a point on the line segment $AC$ and $P, Q \in BM$, with $AP \perp BM$ and $CQ \perp BM$. Prove that $BP = PQ + QC$.

9. (R. Gelca 1998) Let $ABCD$ be a cyclic quadrilateral with $AC \perp BD$. Prove that the area of quadrilaterals $AOCD$ and $AOCB$ are equal, where $O$ is the circumcenter.