Problem 18 Find $\sigma^{-1}$ for

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$ 

Step 1:

3 → 4
④ → 1
① → 3
② → 4

Step 2:

① ≡ 1
② ≡ 2
④ ≡ 3
① ≡ 4

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

Problems 17 and 18 exhibit two different non-trivial permutations $\sigma$ that are self-inverse, $\sigma^{-1} = \sigma$. It follows from Problem 4 that there exist only two self-inverse numbers, 1 and $-1$, the latter being the only non-trivial. Unlike numbers, there exist lots of different non-trivial self-inverse permutations.

Problem 19 Find a non-trivial permutation $\sigma$ different from the ones in Problems 17 and 18 such that $\sigma^{-1} = \sigma$.

An example:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Please check using the definition:

$$\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \mathbf{e}.$$
The 15 puzzle was invented by Noyes Palmer Chapman, a postmaster in Canastota, New York, in the mid-1870s. Sam Loyd, a prominent American chess player at the time,\(^1\) has offered $1,000 (about $25,000 of modern day money) for solving the puzzle in the form shown on the picture below.

![15 puzzle grid](image)

Proving that this particular configuration has no solution will be the primary goal of our mini-course.

**Problem 20** Write down the permutation corresponding to the Loyd's puzzle.

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 \times 14
\end{pmatrix}
\]

\(^1\)Ranked 15th in the world.
Problem 21 Let us call $\sigma$ the permutation from Problem 20. Find $\sigma^{-1}$.

$\sigma^{-1} =$

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 15 & 14
\end{pmatrix}
\]

Following the steps described before, it turns out that $\sigma$ is self-inverse.

Homework

Using your copy of the 15 puzzle, attempt to solve the version suggested by Sam Loyd. Try to figure out what goes wrong.