Invariants II

Los Angeles Math Circle | Advanced Group | 01.19.2014

1. Let’s start off with a problem that you may have seen before and perhaps solved in a slightly different way. We’re now going to look at it using invariants: Prove that you can or cannot cover an $8 \times 8$ chessboard with $3 \times 1$ tiles.

(a) Consider the process of adding $3 \times 1$ tiles. Can you find an invariant quantity?

(b) Finish the proof on your own.

2. From an $8 \times 8$ chessboard, two diagonally opposite tiles are removed. Can the resulting board be covered using dominoes ($2 \times 1$ tiles)? Make sure you use the method of invariants and write down a complete proof.
3. Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

4. At first, a room is empty. Each minute, either one person enters or two people leave.
   
   (a) After exactly 15 minutes, could the room contain exactly 9 people?  
   (If you don’t think it’s possible, you don’t need to prove it. Just explain why you think it’s impossible.)

   (b) After exactly 15 minutes, could the room contain exactly 8 people?  
   (If you don’t think it’s possible, you don’t need to prove it. Just explain why you think it’s impossible.)
(c) After exactly 1000 minutes, could the room contain exactly 200 people?
   (If you don’t think it’s possible, prove it!)

(d) After exactly 3000 minutes, could the room contain exactly 3600 people?
   (If you don’t think it’s possible, prove it!)

5. A special chess piece called an “Angela” moves along a 10x10 (NOT 8x8!) chessboard like a (1, 3) knight. That is, an Angela moves to any adjacent square and then moves three squares in any perpendicular direction. (A ‘usual’ knight would be classified as (1, 2).)

(a) Is it possible for an Angela to go from one corner square to a non-opposite corner square?
   (for example: from square a1 to j1)

(b) How about from any random square to an adjacent square?
6. Polyominoes on chessboards.

(a) Can an 8x8 chessboard be covered without overlapping by fifteen 1x4 polyominoes and the single L-shaped polymino shown in the figure below?
(Hint: You can color the chessboard however you like to best suit the needs of your problem. For this problem, we recommend that you color the chessboard by rows, using the usual two colors. For instance, the top row is all black, and the second row is all white, etc.)

![Diagram of polyominoes on chessboard](image)

(b) Can a 10x10 board be covered without overlapping by the polyominoes shown in the figure below?
(Hint: Use the standard coloring of the chessboard.)

![Diagram of polyominoes on chessboard](image)
(c) Can a 102 x 102 board be covered without overlapping by 1 x 4 polyominoes?
   (Hint: Consider a similar coloring scheme as part (a), except now using four colors instead
   of just black and white. For instance, the top row could be red, the second row green,
   the third row blue...)

7. The number $8^n$ is written on a blackboard. The sum of its digits is calculated, then the sum of
   the result is calculated, and so on, until we get a single digit. What is this digit if $n = 1989$?

8. Prince Ivan has two magic swords. One of these can cut off 21 heads of an evil Dragon. Another sword
   can cut off 4 heads, but after that the dragon grows 1985 new heads. Can Prince Ivan cut off all the
   heads of the 100-headed dragon?
   (Note: If, for instance, the Dragon had three heads, then it is impossible to cut them off with
   either sword.)
9. There are Martian amoebae of three types, A, B, and C, in a test tube. Two amoabae of any two different types can merge into one amoeba of the third type. After several such merges, only one amoeba remains in the test tube. There were initially 20 amoebae of type A, 21 of type B, and 22 of type C. What is the type of the final amoeba?

10. There is a heap of 1001 stones on a table. You are allowed to perform the following operation: you choose one of the heaps containing more than one stone, throw away one stone from that heap and divide it into two smaller (not necessarily equal!) heaps. Is it possible to reach a situation in which all the heaps on the table contain exactly 3 stones?