Invariants

Los Angeles Math Circle | Advanced Group | 01.12.2014

Figure 1: Invariants in a clock.

1. Let’s play a game! Consider the following list of numbers (there are six 0’s and five 1’s):

   \[ 0, 0, 0, 0, 0, 0, 1, 1, 1, 1 \]

   (a) Perform the following operation ten times: cross out any two numbers, and...
       - If the two numbers were equal, add a 0 to the end of the list.
       - If the two numbers were not equal, add a 1 to the end of the list.

       SHOW YOUR WORK. What number are you left with?

(b) Let’s play again.

   \[ 0, 0, 0, 0, 0, 1, 1, 1, 1, 1 \]

   Cross off different numbers than you did last time. What number are you left with?

It turns out that this will ALWAYS be the case. Cool, huh?
(c) Now you get to prove it!
   i. What is the sum of all the numbers in the initial list? Let’s call it $T$ (for total). Is $T$
ev even or odd?

   ii. What happens to the value of $T$ if you cross off...
       A. two zeroes?

       B. two ones?

       C. a zero and a one?

   iii. What happens to the parity of $T$ as we continue to cross off numbers?

   iv. What does this tell you the final number remaining must be?

2. Let’s summarize our results in the previous problem. We found that, in the problem situation,
   there was a quantity that did not change, regardless of what numbers we chose to remove. This
   quantity is called an **invariant**. Identify the invariant in Problem 1. Explain.
DEFINITION:

An INVARIANT is a property of a set of objects that does not change when transformations of a certain type are applied to the objects.

3. There are only two letters in the alphabet of the Ao-Ao language: A and O. Moreover, the language satisfies the following conditions:
   - If you delete two neighboring letters AO from any word, then you will get a word with the same meaning.
   - If you insert the combinations OA or AAOO any place in a word, then you will get a word with the same meaning.

   Do the words AOO and OAA have the same meaning?
   Justify your answer! Be sure to specify what is invariant in this problem.

4. A delicious cherry pie is divided into 6 equal slices. One fresh cherry is placed on top of each slice. Jeff wants to play with the cherries on top to try to put them all on his slice. His mom, a clever mathematician, will only allow this under the condition that Jeff can only shift any two cherries to slices bordering those they stand on at the moment. He can do this as many times as he likes. The following steps will help you determine: can Jeff meet his goal?

   (a) Make an illustration of the situation. Number the slices 1 through 6. Find $S$, the sum of the numbers on the slices of pie that contain cherries. Is $S$ odd or even?
(b) Shift any cherry from one slice to a slice bordering it. Now find the value of $S$. What happened to its parity?

(c) Shift another cherry from one slice to a slice bordering it. Now find the value of $S$. What happened to its parity from part (b)?

(d) What is the net effect of parts (b) and (c) on the parity of $S$?

(e) Finish the proof on your own.

5. The numbers 1, 2, 3, ..., 19, 20 are written on a blackboard. It is allowed to erase any two numbers $a$ and $b$ and write the new number $a + b - 1$. What number will be on the blackboard after 19 such operations?
6. There are 6 sparrows sitting on 6 trees, one sparrow on each tree. The trees stand in a single-file line, with 10 meters between any two neighboring trees. If a sparrow flies from one tree to another, then at the same time some other sparrow flies from some tree to another the same distance away, but in the opposite direction. Is it possible for the sparrows to gather on one tree? If so, provide a way. If not, prove it. (Hint: Number the sparrows based on the tree they are in. Can you find an invariant?)

7. In Problem 6, what if there had been 7 sparrows and 7 trees?

8. In an 8x8 table, one of the boxes is colored black, and all the others are white. Prove that one cannot make all the boxes white by recoloring the rows and columns. (“Recoloring” is the operation of changing the color of all the boxes in a row or column.)
9. Solve Problem 8 for a $3 \times 3$ table if initially there is only one black box in the upper left corner of the table.

10. Solve Problem 8 for an $8 \times 8$ table if initially all four corner boxes are black and all the others are white.

11. There are 13 grey, 15 brown, and 17 red chameleons on Chromatic Island. When two chameleons of different colors meet, they both change their color to the third one. Is it possible that after some time, all the chameleons on the island are the same color?