METRIC SPACES ON UNUSUAL SETS

MATH CIRCLE (HS1) 10/27/2013

Recall that \((M, d)\) a metric space if \(M\) a set and \(d : M^2 \rightarrow \mathbb{R}\) such that \(d\) is a metric. \(d\) is a metric if it satisfies properties 2, 3, and 4 below. We say that \(d\) is an ultrametric if \(d\) satisfies properties 2, 3, and 4'.

1. \(d\) is symmetric \((d(x, y) = d(y, x))\)
2. \(x = y\) if and only if \(d(x, y) = 0\)
3. \(d\) satisfies the triangle inequality \((d(x, z) \leq d(x, y) + d(y, z))\)
4'. \(d(x, z) \leq \max\{d(x, y), d(y, z)\}\)

Note that property 4' implies property 4, so any ultrametric is a metric.

In particular, recall an example of an ultrametric \(d\) (called the discrete metric) so that \((S, d)\) is a metric space for any \(S\):

\[
d(x, y) = \begin{cases} 
1, & \text{if } x \neq y \\
0, & \text{if } x = y 
\end{cases}
\]

Today we will explore examples of metrics spaces (and non metric spaces) for more unusual sets.

**Words**

Let \(W\) consist of all words of length \(n\). More formally, \(W = \{x_1x_2\cdots x_n|x_i \in \{a, b, c, \ldots, z\}\}\).

1. (Hamming Distance) For two words \(x = x_1 \cdots x_n\) and \(y = y_1 \cdots y_n\) in \(S\), define

\[
d(x, y) = k, \text{ where } k \text{ is the number of times } x_i \neq y_i.
\]

Show that \((W, d)\) is a metric. Hint: Think about how the \(n = 1\) case is relates to the general case.

2. For two words \(x = x_1 \cdots x_n\) and \(y = y_1 \cdots y_n\) in \(S\), define

\[
d(x, y) = \begin{cases} 
0, & \text{if } x = y \\
2^{-k}, & \text{if } x \neq y \text{ and } k \text{ is the smallest integer such that } x_k \neq y_k
\end{cases}
\]

Show that \(d\) is an ultrametric on \(W\).

**Points on a Circle**

Suppose we have a circle \(C\). Let \(P\) consist of all the points on the circle \(C\).

3. Let, for \(P, Q \in C\), \(d(p, q) = \) the length of the smaller arc from \(p\) to \(q\) on the circle \(C\).

Is \(d\) a metric?

4. Let, for \(P, Q \in C\), \(d(p, q) = \) the length of the arc going clockwise from \(p\) to \(q\) on the circle \(C\).

Is \(d\) a metric?
On Finite Sets

Let $S$ consist of all finite sets. Recall that if $A$ is a set, then $|A|$ denotes the number of elements in $A$.

5) Let, for $A, B \in S$, $d(A, B) = ||A| - |B||$. Is $(S, d)$ a metric space?

6) Recall that if $A, B$ are sets, then $A \triangle B$ is the set of things in $A$ or in $B$, but not in both. Let, for $A, B \in S$, $d(A, B) = |A \triangle B|$. Is $(S, d)$ a metric space?

On Lines in the Plane

Let $L$ consist of all lines in a plane.

7) Define, for $\ell_1, \ell_2$ both lines,

$$d(\ell_1, \ell_2) = \begin{cases} s, & \text{if } \ell_1, \ell_2 \text{ are parallel, and } s \text{ is the distance between them} \\ \theta, & \text{OW, and } \theta \in (0^\circ, 90^\circ] \text{ is the angle of intersection between } \ell_1, \ell_2 \end{cases}$$

Show that $d$ is not a metric.

Note: We can modify this to get a metric, see the homework.

For Further Investigation (Homework):

1) Suppose that $(M, d)$ is a metric space. Prove that $(M, d')$ is also a metric space, where

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Hint: Check that

$$d'(x, y) + d'(y, z) = \frac{d(x, y) + d(y, z) + 2 \cdot d(x, y) \cdot d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y) \cdot d(y, z)}.$$

We thus need to show

$$d'(x, z) = \frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y) + d(y, z) + 2 \cdot d(x, y) \cdot d(y, z)}{1 + d(x, y) + d(y, z) + d(x, y) \cdot d(y, z)} = d'(x, y) + d'(y, z),$$

which isn’t hard, just a bit messy!

2) As in the setup for 7), let

$$d(\ell_1, \ell_2) = \begin{cases} s, & \text{if } \ell_1, \ell_2 \text{ are parallel, and } s \text{ is the distance between them} \\ \theta + 1, & \text{OW, and } \theta \in (0^\circ, 90^\circ] \text{ is the angle of intersection between } \ell_1, \ell_2 \end{cases}.$$

Convince yourself that $d$ is a metric.

Hint: i) Look at cases! ii) At some point, an argument similar to that in 1) may be helpful.