Warm up

In each of the following, we let \( x \) represent the instructor’s favorite number.

- \( 5x + 1 = 5 \) so \( 5x = 4 \), and dividing by 5, \( x = \frac{4}{5} = .8 \).
- \( 10x = 6 + x \) so \( 9x = 6 \) and hence \( x = \frac{6}{9} = .6666 \cdots \).
- \( 100x = 100 + x \), hence \( 99x = 100 \), so \( x = \frac{100}{99} = 1.010101 \cdots \).
- \( 1000x = 37 + x \), so \( 999x = 37 \) or \( x = \frac{37}{999} = .037037037 \cdots \).

Problems

1. Yes there is something to prove. What we proved before is called the converse of this statement. That is if we let \( a \) represent the phrase ”\( x \) is a fraction with denominator having only powers of 2 and 5” and \( b \) represent the phrase ”\( x \) is a decimal which terminates”. We have proven

\[ a \implies b \]

what we are looking to prove now is

\[ b \implies a \]

also called the converse, which is different.

2. (a) \( \frac{89}{100} \)
   (b) \( \frac{35}{100} = \frac{7}{20} \)
   (c) \( \frac{125}{1000} = \frac{1}{80} \)
   (d) \( \frac{1111}{100} = \frac{1111}{25} \)

3. Given an arbitrary terminating decimal \( x \) we can write

\[ x = .a_1a_2\cdots a_k0\cdots \]

then \( 10^k x = a_1a_2\cdots a_k \) an integer.

4. In the previous problem we have shown that \( 10^k x \) is an integer, says \( r \). So \( 10^k x = r \), thus \( x = \frac{r}{10^k} \) and we have the claim.

5. (a) \(.1\overline{1}\)\( \cdots \)
   (b) \(.2\overline{2}\)\( \cdots \)
   (c) \(.3\overline{3}\)\( \cdots \)
   (d) \(.4\overline{4}\)\( \cdots \)
   (e) \(.5\overline{5}\)\( \cdots \)
Well, we saw in (a) that \( \frac{1}{9} = .1111 \cdots \) so \( 1 = 9 \cdot \frac{1}{9} = .9999 \cdots \). Thus we have two decimal representations for the number 1.

6.  (a) 3.3333 \cdots = 3 + .3333 \cdots = 3 + \frac{1}{3} = \frac{10}{3} \\
(b) 17.5555 \cdots = 17 + .5555 \cdots = 17 + \frac{5}{9} = \frac{17 \cdot 9 + 5}{9} \\
(c) 2.08888 \cdots = 2 + .08888 \cdots = 2 + \frac{1}{10} \cdot \frac{8}{9} = \frac{188}{99}

7.  (a) 10x = 2.222 \cdots \\
(b) 10x = x + 2 \\
(c) 9x = 2, so \( x = \frac{2}{9} \) (which we already knew from the previous problem.) \\
(d) this is just telling you to realize that in part (c) what we have shown is that .222 \cdots = \frac{2}{9}.

8. 10x = 6 + x, so we get 9x = 6, or \( x = \frac{6}{9} \).

9.  (a) 10w = 4.666 \cdots and 10w - 4 = .666 \cdots \\
(b) Well, we just showed .666 \cdots = \frac{2}{3} so we have that as the fractional version of 10w - 4. \\
(c) Therefore 10w = 4 + \frac{2}{3} = \frac{14}{3} and so \( w = \frac{14}{99} \).

10. (a) 100y = 36.36363636 \cdots \\
(b) 100y = 36 + y \\
(c) Therefore, we get 99y = 36 or \( y = \frac{36}{99} \). \\
(d) Again, this is just looking at part (c) and realize we have derived .36363636 = \frac{36}{99}.

11. 100y = 5.05050505 = 5 + y. Therefore, we solve this and get 99y = 5, or \( y = \frac{5}{99} \).

12. The idea is to multiply by 10 to the power of (length of period of the decimal). Therefore we should multiply by \( 10^3 \) or 1000. We get 1000y = 148.148148 \cdots, and hence 1000y = y + 148, so \( 999y = 148 \) and \( y = \frac{148}{999} \).

13. Let \( y \) represent the decimal number. Multiply by 1000, and we get 1000y = \( abc.abccab \cdots \), that is \\

\[
1000y = abc + y
\]

solving this equation we get 999y = \( abc \) and hence \( y = \frac{abc}{999} \).

14. (a) Yes, \( \frac{5242362343}{1} = 5242362343 \)

(b) Yes, we saw that every terminating decimal comes from a fraction, so 47.26 comes from a fraction, and so \( -47.26 \) also comes from a fraction (just multiply the numerator by \(-1\)).

(c) Yes, but we haven’t actually proven this yet and its quite hard. Heres the proof: Let \( x \) be an arbitrary eventually periodic decimal, then \\
\[
x = .a_1 \cdots a_kb_1 \cdots b_nb_1 \cdots b_n \cdots
\]

so \( a_1, \cdots, a_k \) are random and the pattern starts at \( b_1, \cdots, b_n \). Multiplying \( x \) by \( 10^k \) we get \\
\[
10^kx = a_1 \cdots a_k + .b_1 \cdots b_nb_1 \cdots b_n \cdots
\]

Now, preforming the same process as in problem 13, but with \( n \) digits instead of three, we see that \\
\[
.b_1 \cdots b_nb_1 \cdots b_n \cdots = \frac{b_1 \cdots b_n}{10^n - 1}
\]
So adding it up we get

\[ 10^k x = a_1 \cdots a_k + \frac{b_1 \cdots b_n}{10^n - 1} \]

and solving this for \( x \) we see that \( x \) is indeed a fraction.

(d) No, you came up with some examples yourself last week of decimals that did not repeat, and we have already proven that every rational has a decimal equivalent which repeats.

**Homework**

1. (a) No, it does not repeat.
   
   (b) Yes! we can bring the 7^2 and 9 down into the big denominator making this into a fraction with integers on the top and bottom.

2. Yes, you can do this! (it sounds a little bit surprising, but the once you see the answer you will say D’oh! like Homer Simpsons)
   
   Let \( a = .010100100 \cdots \) (which we know is not rational by the first homework!) and set \( b = -a \), then \( b \) is also not rational because it still has a decimal which does not repeat. But
   
   \[ a + b = a + (-a) = a - a = 0 \]
   
   and of course, \( 0 = \frac{0}{1} \) is rational.

3. If we denote the number by \( q \), multiply the number by 1000 we get
   
   \[ 1000q = def + .abcabc \cdots \]
   
   we already know the fractional equivalent of this decimal, from the worksheet so we get
   
   \[ 1000q = def + \frac{abc}{999} \]
   
   hence
   
   \[ q = \frac{999(def) + abc}{999 \cdot 1000} \]