How to get from point A to point B.

Ex 1

Infinite # of ways
(since we can go around the circle only many times).

Definition Let X be a space.

A path in X from A to B is a function \( \gamma(t) \) such that

\[
\begin{align*}
(1) \quad & \gamma(t) \text{ is continuous (no jumps)} \\
(2) \quad & \gamma(0) = A, \quad \gamma(1) = B.
\end{align*}
\]

Exercise: If there is one path from A to B, then there are infinitely many paths from A to B.

For example, for any number \( n \in \mathbb{N} \), look at the path\( \gamma_n(t) = \gamma(t^n) \)

Also, one can consider
\[
\tilde{\gamma}(t) = \begin{cases} 
\gamma(2t) & 0 \leq t \leq \frac{1}{2} \\
B & \frac{1}{2} \leq t \leq 1
\end{cases}
\]
To study our question: "How many ways are there to get from A to B?"

We need an equivalence relation.
Possible conditions:
- Changing speeds should not make any difference.
  e.g. \( t \rightarrow t' \)
- Only hit each pt once. (This a restriction).
- Two paths can be connected point by point without jumps in the middle.
  e.g. \( \text{Path Diagram} \)

**Definition:** Let \( \gamma(t) \) and \( \beta(t) \) be paths from A to B.

A homotopy from \( \gamma \) to \( \beta \) is a function \( H(s,t) \) such that \( 0 \leq s, t \leq 1 \) and

\[
\begin{align*}
(1) & \quad H(0,t) = \gamma(t) \\
& \quad H(1,t) = \beta(t) \\
(2) & \quad H(s,0) = A \\
& \quad H(s,1) = B.
\end{align*}
\]

**Definition:** Two paths \( \gamma \) & \( \beta \) from A to B are homotopic if there exists a homotopy \( H \) from \( \gamma \) to \( \beta \).

We will write \( [\gamma] = [\beta] \).
Homework Exercise

Let \( \gamma(t) \) be a path from A to B, and let \( \beta(t) = \gamma(t^2) \).

Show that \( \beta \) and \( \gamma \) are homotopic.

1. Stand on A: \( \gamma(t) = A \)

2. Travel once around circle in the counter-clockwise direction
   \[ \gamma(t) = e^{2\pi it} \]
   \[ \mathcal{S} = \{ z \in \mathbb{C} | |z| = 1 \} \]

3. Once around clockwise
   \[ \gamma(t) = e^{-2\pi it} \]

4. Go around \( n \) times
   \[ \gamma(t) = e^{2\pi int} , \quad n \in \mathbb{Z} \]

Q: Are these paths different, i.e. not homotopic?

(1) and (3) are not homotopic.
The Algebraic Structure

We can only multiply two paths if one starts at where the other ends.

Define a path $(\gamma + \beta)(t) = \begin{cases} \gamma(at), & 0 \leq t \leq \frac{1}{2} \\ \beta(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$

We want:

- Associative
  \[
  [(S + \beta) * \gamma] = [S * (\beta * \gamma)]
  \]

- For each $A$, a path $C_A$ from $A$ to $A$ such that
  \[
  \begin{align*}
  [C_A * E] &= [E] \\
  [\gamma * C_A] &= [\gamma].
  \end{align*}
  \]
  $C_A(t) = A$ for all $t$.

- Commutative not applicable in this case.

- Inverse
  \[
  \gamma^{-1}(t) = \gamma(1-t)
  \]

Fact: $[\gamma * \gamma^{-1}] = [C_A]$
Same endpoints, i.e. paths from A to A.

\[ S \xrightarrow{\gamma} B \]

1. \[ (\gamma \ast \beta) \ast \delta = [\gamma \ast (\beta \ast \delta)] \]
2. \[ [\gamma \ast \text{CA}] = [\gamma] \]
3. \[ [\gamma \ast \delta^{-1}] = [\text{CA}] = [\gamma^{-1} \ast \gamma] \]

This is similar to \( \mathbb{R} - \{0\} \) with \( \ast \) replaced by multiplication.

\[ \{ \text{invertible functions} \} \text{ with } \ast \text{ replaced by composition.} \]

We denote the loops @ A up to homotopy by \( \pi_1(S^1, A) \).

Function \( f: \pi_1(S^1, A) \rightarrow \mathbb{Z} \)

\( \gamma \mapsto \# \text{ of times } \gamma \text{ passes } P \) counter-clockwise

\( -\# \text{ of times } \gamma \text{ passes } P \) clockwise.

Properties

\[ f(\gamma \ast \delta) = f(\gamma) + f(\delta) \]
\[ f(\text{CA}) = 0 \]
\[ f(\delta^{-1}) = -f(\delta) \]
\[ f \text{ independent of } P. \]
Fact: \([\gamma] = [\delta] \iff f(\gamma) = f(\delta)\)

So, \(\pi_1(CS, A) = \mathbb{Z}\).