GEOMETRY - FINDING ANGLES II

MATH CIRCLE (INTERMEDIATE) 03/03/2013

Facts about parallel lines:
When two parallel lines are cut by a transversal (see figure; line $m$ is the transversal), the following sets of angles are congruent:

- Alternate interior angles (for example, $\alpha$ and $\beta$)
- Vertical angles (for example, $\gamma$ and $\alpha$)
- Corresponding angles (for example, $\gamma$ and $\beta$)

(1) Let's prove that the sum of the angles in a triangle is always 180 degrees.

(a) Draw a triangle, any triangle. It does not need to be equilateral or even isosceles. Label the vertices as $A$, $B$, and $C$.

(b) Draw on the above diagram a line $l$ that passes through point $C$ and is parallel to segment $AB$.

(c) Consider the three acute angles formed by the intersection of line $l$ with triangle $ABC$. What is the sum of these angles' measures?

(d) Use your result from part (c), and the facts about parallel lines cut by a transversal above, to prove that the sum of the measures of the interior angles of triangle $ABC$ is 180 degrees.
(2) In the picture, $AD = DC$, $AB = AC$, and the measures of angles $ABC$ and $ADC$ are 75 and 50 degrees, respectively. What is the measure of angle $BAD$?

(3) $ABC$ is an isosceles triangle such that $AC = BC$. $CBD$ is an isosceles triangle such that $CB = DB$. $BD$ meets $AC$ at a right angle. If the measure of angle $A$ is 57 degrees, what is the measure of angle $D$?

(4) Some angle measures of the quadrilateral $ABCD$ are shown in the picture to the right. In addition, $AD = BC$. Find the measure of angle $ABC$. 

2
(5) Triangle $ABC$ is equilateral. Point $B$ is the midpoint of segment $AD$. Point $E$ belongs to a circle which has a center at point $D$ and whose radius equals $AB$. What is the measure of angle $BED$ when the distance $CE$ is the greatest possible?

(6) In the convex quadrilateral $ABCD$, $AB = AC$. In addition, $m\angle BAD = 80^0$, $m\angle ABC = 75^0$, and $m\angle ADC = 65^0$. What is the measure of angle $BDC$?

(7) What is the measure of angle $a$ in the figure?
(8) In triangle $ABC$, the bisectors of angles $ABC$ and $ACB$ intersect at point $D$. We know that the measure of angle $BDC$ is 150 degrees. What is the measure of angle $BAC$?

(9) In quadrilateral $ABCD$, $\angle ABC = 110^\circ$, $\angle BCD = 100^\circ$, and angles $BAD$ and $CDA$ are trisected as shown. What is the degree measure of $\angle AFD$?

(10) Triangle $ABC$ is isosceles with base $AC$. Points $P$ and $Q$ are respectively on $CB$ and $AB$ such that $AC = AP = PQ = QB$. Find the measure of $\angle B$. 
(11) The diameter of the circle with center \( O \) is also the base of the isosceles triangle \( ABC \). Points \( K \) and \( L \) are points of intersection of the triangle with the circle. If \( \angle KOL = 100^\circ \), what is the measure of angle \( C \) of triangle \( ABC \)?

(12) In a general triangle \( ADE \) (as shown) lines \( EB \) and \( EC \) are drawn.

(a) Show that \( x + y = m \). (This convenient result is called the Exterior Angle Theorem).

(b) Show that \( x + y + n = a + b + m \).
(13) Given triangle $PQR$ with $RS$ bisecting $\angle R$, $PQ$ extended to $D$, and $CD$ perpendicular to $RS$, show that $m = \frac{p+q}{2}$. 
HINTS:

Problem 2.

*Hint 2.1:* If two sides of a triangle are congruent, the angles opposite those sides are also congruent.

*Hint 2.2:* What is the sum of the measures of the angles of a triangle?

Problem 3.

*Hint 3.1:* See Hint 2.1.

Problem 4.

*Hint 4.1:* See Hint 2.2.

*Hint 4.2:* Hint 2.1 goes the other way, too: if two angles of a triangle are congruent, then the sides opposite those angles are also congruent.

Problem 5.

*Hint 5.1:* Complete the figure by drawing the specified circle. Label point E on the circle and draw segment BE.

*Hint 5.2:* What is the measure of each angle in an equilateral triangle?

*Hint 5.3:* What is the sum of the measures of the angles that form a straight line?

Problem 6.

*Hint 6.1:* What is the sum of the measures of the angles of a quadrilateral?

*Hint 6.2:* See Hint 4.2.

Problem 8.

*Hint 8.1:* Let a be the measure of one of the angles formed by the bisector of angle ABC. Let b be the measure of one of the angles formed by the bisector of angle ACB. What is the value of a + b?

*Hint 8.2:* What is the measure of angle BAC in terms of a and b? Now apply the result of Hint 8.1.

Problem 9.

*Hint 9.1:* See Hint 6.1.

*Hint 9.2:* Use a similar approach to Problem 8 (see Hint 8.1). Can you express the measure of \( \angle AFD \) in terms of \( x \) and \( y \)?

Problem 10.

*Hint 10.1:* Let \( x \) equal the measure of \( \angle B \). What is the measure of angle \( PQA \) in terms of \( x \)? \( APC? \ BCA? \) (Use Hints 2.1 and the result of Problem 1.)

Problem 11.

*Hint 11.1:* Can you find any isosceles triangles in the figure?