Warm-up

**Problem 1** What is the smallest natural number such that its double is a perfect square and its triple is a perfect cube?

**Problem 2** Can the number $a^2 + b^2 + c^2$ be divisible by 5 if neither of the numbers $a$, $b$, and $c$ is divisible by 5? Why or why not?
The number 13 is considered by many as an unlucky number. Equally, Friday is considered by many as an unlucky day. Solve the following problem for a regular, not leap, year.

**Problem 3** • *Does there exist a year without a Friday the 13th?*

- *What is the maximal number of times Friday the 13th can occur in a year?*
Problem 4 Using a compass and a ruler, draw a triangle with the given sides $a$, $b$, and $c$ in the space below.
Problem 5 Using a compass and a ruler, draw the given angle $\alpha$ having the given ray as its side.
Problem 6 Using a compass and a ruler, draw a triangle with the angle $\alpha$ and adjacent sides $b$ and $c$ given below. In this case, the word “adjacent” means that the vertex of $\alpha$ is an endpoint of the sides $b$ and $c$. 
Problem 7 Draw a triangle with the side \( c \) and adjacent angles \( \alpha \) and \( \beta \) given below. In this case, the word “adjacent” means that the endpoints of \( c \) are the vertices of \( \alpha \) and \( \beta \).
Note that solutions to Problems 4, 6, and 7 (nearly) prove the following theorem fundamental for what follows.

**Theorem 1** Two triangles in the Euclidean plane are equal if either of the following holds.

- Their side lengths are pairwise equal.
  \[ |a| = |a'|, \quad |b| = |b'|, \quad |c| = |c'| \]

- They have an angle of equal size, and the lengths of the sides adjacent to the equal angles are pairwise equal.
  \[ \alpha = \alpha', \quad |b| = |b'|, \quad |c| = |c'| \]

- They have a side of equal length, and the adjacent angles are pairwise equal.
  \[ |c| = |c'|, \quad \alpha = \alpha', \quad \beta = \beta' \]

**Question 1** What does it mean that two triangles are equal?

**Question 2** How would you prove the first statement of Theorem 1?
Triangles and circles

**Problem 8** Using a compass and a ruler, construct the shortest possible path from the point $A$ to the straight line $l$ below.

- $A$

---

Use the Pythagoras’ Theorem to prove that the path you have constructed is indeed the shortest.
Problem 9 Let us call $B$ the end of the shortest path opposite to the point $A$ in Problem 8, see the picture below.

Draw the circle centered at $A$ and passing through $B$. Can the circle intersect the straight line $l$ at any point other than $B$? Why or why not?
**Definition 1** A straight line that intersects a circle at one point only is called tangent to the circle at the point.

**Problem 10** Use a compass and a ruler to construct a straight line tangent to the below circle at the given point A.

![Diagram of a circle with a point A on its circumference and a line drawn tangent to the circle at point A]

**Theorem 2** A straight line is tangent to a circle if and only if it is perpendicular to the radius drawn to the tangency point.

**Problem 11** Prove Theorem 2.
Definition 2 A bisectrix, or the angle bisector, is a ray that splits the angle into two equal parts.

Problem 12 Prove that the points of an angle bisector are equidistant from the sides of the angle.
**Theorem 3** Angle bisectors of any triangle in the Euclidean plane have a common point. The point is the center of the circle inscribed in the triangle (tangent to all of its sides).

**Definition 3** The point is called the incenter of the triangle. The corresponding circle is called the incircle.

**Problem 13** Prove Theorem 3
Let us extend two sides of a triangle to rays as below.

Recall that two angles are called supplementary, if they add up to a straight angle. Let $C'$ and $B'$ be the angles pictured above supplementary to $\angle ACB$ and $\angle ABC$ respectively.

**Problem 14** Prove that the angular bisectors of the angles $A$, $B'$ and $C'$ intersect at one point.

The above point is called an *excenter* of the triangle.
Problem 15 Prove that the excenter constructed in Problem 14 is the center of the circle inscribed in the angles $A$, $B'$ and $C''$.

The circle is called an excircle. Below is a picture of a triangle, its incircle and three excircles.

[Diagram of a triangle with its incircle and three excircles]
Problem 16 Consider the triangle $J_AJ_BJ_C$ on the picture above. Do the vertices of the original triangle $ABC$ indeed lie on its sides as drawn on the picture? Why or why not?

Definition 4 A straight line segment bisector is a straight line passing through the middle of the segment. A perpendicular bisector is the bisector forming the right angle with the original line.

Problem 17 Prove that the perpendicular segment bisector is the set of all the points in the plane equidistant from the ends of the segment.
**Theorem 4** Any triangle in the Euclidean plane can be inscribed in a circle.

**Definition 5** The circle is called the circumcircle of the triangle. The center of the circle is called the circumcenter.

**Problem 18** Prove Theorem 4. Use a compass and a ruler to construct the circumcircle for the triangle below.

![Diagram of a triangle with vertices A, B, and C.]
Problem 19 The cities $A$ and $B$ are located on one side of a straight highway. Use a compass and a ruler to construct the shortest possible road connecting $A$ to the highway and then to $B$. 
Problem 20 The cities A and B are separated by a river having parallel straight banks. Using a compass and a ruler, construct the shortest possible highway connecting A to B. The bridge across the river must be perpendicular to the banks.
Homework

Definition 6 A segment of a straight line connecting a vertex of a triangle to the middle of the opposite side is called a median.

Problem 21 Prove that the medians of a triangle intersect at one point. The point divides each median in the ratio 2 : 1 counting from the vertex.
Problem 22 Solve Problem 3 for a leap year.